1.2 Guess and Check

For a differential equation that involves both y' and y, we cannot simply integrate to obtain a solution. In this case, the most basic method for solving the equation is guess and check.

EXAMPLE 1

Find the general solution to the equation

$$y' = y.$$

SOLUTION This equation is much more interesting than the previous two. In words, it says: *find a function that doesn't change when you take its derivative.*

One solution to this equation is the exponential function

$$y = e^x$$

This works because the derivative of e^x is just e^x. But how can we find the general solution? We could try adding a constant, as in

$$y = e^x + C,$$

but this doesn't work: the derivative of $e^x + C$ is just e^x, not $e^x + C$. We need to find functions that don't change when you take their derivative.

The trouble is just that we're putting C in the wrong place. If we multiply by a constant instead, we always get a solution:

$$\boxed{y = Ce^x}$$

Note that the constant function $y = 0$ is a solution, corresponding to $C = 0$.

This works since the derivative of Ce^x is Ce^x for any constant C. Though it may not be obvious, it turns out that *every* solution to the equation $y' = y$ has this form, so $y = Ce^x$ is the general solution to the given equation.

In the last example, we started by finding a **particular solution** to the dfferential equation, and then we figured out how to insert a constant C into the formula to get the general solution. We often use these two steps when guessing the general solution to a differential equation.

EXAMPLE 2

Find a general solution to the following differential equation.

$$y' = -y^2$$

SOLUTION Where do we even begin? How can we possibly find a formula for y that satisfies this equation?

When we have no idea how to solve a problem, the most basic method to try is **guess and check**. We have no idea what formula might work here, but we might be able to find out if we make some guesses, and then check whether they work.

If we're just guessing formulas at random, it probably makes sense to start with some very simple formulas:

$$y = \sin x, \qquad y = e^x, \qquad y = \sqrt{x}, \qquad y = \frac{1}{x}, \qquad y = x^2, \qquad y = \ln x.$$

Do any of these work? Check them for yourself before continuing.

It turns out that $y = 1/x$ is the right guess. If $y = 1/x$, then $y' = -1/x^2$, and the equation becomes

$$-\frac{1}{x^2} = -\left(\frac{1}{x}\right)^2.$$

So $y = 1/x$ is a particular solution to this differential equation.

Later on, we will learn a method called *separation of variables* that allows us to solve this equation without any guessing.

Now, what about the general solution? We need to figure out how to include an arbitrary constant C. Again, we just have to guess:

$$y = \frac{1}{x} + C, \qquad y = \frac{C}{x}, \qquad y = \frac{1}{Cx}, \qquad y = \frac{1}{x+C}, \qquad y = \frac{1}{x^C}.$$

Do any of these work? Yes—it is easy to check that

$$\boxed{y = \frac{1}{x+C}}$$

is always a solution.

But is it the general solution? Presumably it is, since it includes an arbitrary constant, but it's hard to be sure about such things. We would need to somehow know that *every* solution to the given equation has this form.

In fact, this is not quite the general solution. It turns out that every solution has the above form, with the exception of the constant function $y = 0$. This is also a solution, but it does not correspond to any value for C.

Intuitively, $y = 0$ is the solution that you get when $C = \infty$, i.e. when you take the limit as $C \to \infty$.

Sometimes it helps to make a sequence of educated guesses.

EXAMPLE 3

Find a particular solution to the following differential equation.

$$xy' + 2y = 14x^5$$

SOLUTION Again we should just start by guessing formulas for y. However, the right side of this equation gives us a clue: maybe we should try something involving x^5. If we just try $y = x^5$, we get

$$x(5x^4) + 2(x^5) = 14x^5,$$

which isn't quit right, since the left side is just $7x^5$.

Our guess of x^5 came very close, but our left side was off by a factor of 2. Perhaps it would work to insert a 2 somewhere in our guess? Indeed, if $y = 2x^5$, then the left side will work out correctly:

$$x(10x^4) + 2(2x^5) = 14x^5.$$

Thus $\boxed{y = 2x^5}$ is one solution to this differential equation.

The general solution here is more complicated and would be hard to guess. It turns out to be $y = 2x^5 + Cx^{-2}$, where C is an arbitrary constant. The solution we found corresponds to $C = 0$.

A Closer Look Making Up Differential Equations

Although our goal is to understand how to solve differential equations, you can learn a lot by trying to make up differential equations that have a certain solution. For example, suppose we want a differential equation that has

$$y = x^3$$

as a solution. The simplest possibility is

$$y' = 3x^2.$$

However, any differential equation that holds when you plug in $y = x^3$ and $y' = 3x^2$ will work. For example, since $x(3x^2) = 3x^3$, the equation

$$xy' = 3y$$

has $y = x^3$ as a solution. Some other differential equations with $y = x^3$ as a solution include

$$(y')^3 = 27y^2, \qquad xy' + 4y = 7x^3, \qquad \text{and} \qquad yy' = 3x^5.$$

On your own, you could try making some differential equations that have $y = x^2$ as a solution, or perhaps $y = \sin x$.

EXERCISES

1–2 ■ Use guess and check to find the general solution to the given differential equation.

1. $y' + y \tan x = 0$

2. $(y')^2 = 4y$

3–5 ■ Use guess and check to find a particular solution to the given differential equation.

3. $y' + y = 9e^{2x}$

4. $yy' = 4e^{8x}$

5. $x^2y' + e^y = 2x$