Quadratic Forms

A quadratic form on $\mathbb{R}^2$ is a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ of the form

$$f(x, y) = ax^2 + bxy + cy^2$$

where $a$, $b$, and $c$ are constants. Such functions can be thought of as two-variable analogues of quadratic functions like $f(x) = ax^2$.

**EXAMPLE 1** Consider the quadratic form

$$f(x, y) = x^2 + y^2.$$  

The graph of this function is shown in Figure 1. As you can see, the graph is something like a three-dimensional version of a parabola. This shape is called a **paraboloid**, and can be described as the surface obtained by rotating a parabola around its central axis. Note that the level curves for this function are concentric circles centered at the origin.

**EXAMPLE 2** The graph of the quadratic form

$$f(x, y) = 4x^2 + y^2$$

is shown in Figure 2. This shape is similar to a paraboloid, but it has been compressed in the $x$-direction by a factor of two. The result is called an **elliptic paraboloid** since its level curves are ellipses.

The intersection of this elliptic paraboloid with any vertical plane through the origin is a parabola, but the shape of the parabola varies depending on the plane. For example, the intersection of the graph with the $yz$-plane is the parabola $z = y^2$, while the intersection with the $xz$-plane is the steeper (or thinner) parabola $z = 4x^2$.

**EXAMPLE 3** The graph of the quadratic form

$$f(x, y) = x^2 - y^2.$$  

is shown in Figure 3. This surface is known as a **saddle surface**, and its level curves are a family of hyperbolas in the plane.

Like an elliptic paraboloid, the intersection of the saddle surface with any vertical plane through the origin is a parabola. But unlike the elliptic paraboloid, these parabolas open in opposite directions. In this example, the intersection of the saddle with the $xz$-plane is the upward-facing parabola $z = x^2$, while the intersection of the saddle with the $yz$-plane is the downward-facing parabola $z = -y^2$.

In general, a quadratic form is said to be **elliptic** if its level curves are ellipses (or circles), and **hyperbolic** if its level curves are hyperbolas. The graph of an elliptic quadratic form is an elliptic paraboloid—which may open either upwards or downwards—while the graph of a hyperbolic quadratic form is a saddle surface.

**Quadratic Forms and Matrices**

There is a nice way of representing any quadratic form using a matrix. Given a quadratic form

$$f(x, y) = ax^2 + bxy + cy^2$$

...
we can rewrite the formula as

\[ f(x, y) = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \]

The matrix in the middle is a symmetric 2 × 2 matrix, and is called the **matrix for the quadratic form**. If \( A \) is the matrix for a quadratic form, then the formula for the form can be written

\[ f(x) = x^T A x \]

where \( x \) denotes the vector \((x, y)\).

We can use the matrix to classify a quadratic form as elliptic or hyperbolic. The simplest case is the diagonal case:

### CLASSIFICATION OF QUADRATIC FORMS: DIAGONAL CASE

Let \( f: \mathbb{R}^2 \to \mathbb{R} \) be a quadratic form associated to a 2 × 2 diagonal matrix \[
\begin{bmatrix}
a & 0 \\
0 & c
\end{bmatrix}.
\]

1. If \( a > 0 \) and \( c > 0 \), then the graph of \( f \) is an upward-facing paraboloid.
2. If \( a < 0 \) and \( c < 0 \), then the graph of \( f \) is a downward-facing paraboloid.
3. If \( a \) and \( c \) have opposite signs, then the graph of \( f \) is a saddle surface.

For the general case, recall that a matrix \( A \) is **diagonalizable** if there exists an invertible matrix \( P \) so that \( P^{-1} A P \) is a diagonal matrix. In this case, the diagonal entries of the diagonal matrix are precisely the eigenvalues of \( A \). We will need the following theorem, whose proof can be found in any linear algebra textbook.

### SPECTRAL THEOREM

Any symmetric matrix \( A \) is diagonalizable. Moreover:

1. The eigenvalues of \( A \) are real numbers, and
2. Eigenvectors corresponding to different eigenvalues are orthogonal.

Based on our classification in the diagonal case, it is not hard to guess the general classification theorem.

### CLASSIFICATION OF QUADRATIC FORMS: GENERAL CASE

Let \( f: \mathbb{R}^2 \to \mathbb{R} \) be a quadratic form, and let \( \lambda \) and \( \mu \) be the eigenvalues of the associated matrix.

1. If \( \lambda > 0 \) and \( \mu > 0 \), then the graph of \( f \) is an upward-facing paraboloid.
2. If \( \lambda < 0 \) and \( \mu < 0 \), then the graph of \( f \) is a downward-facing paraboloid.
3. If \( \lambda \) and \( \mu \) have opposite signs, then the graph of \( f \) is a saddle surface.

By the way, there is a nice trick for finding the eigenvalues of any 2 × 2 matrix. Recall the following facts:
TRACE AND DETERMINANT

For any $n \times n$ matrix $A$:

- The trace of $A$ is the sum of its eigenvalues (counting multiplicities).
- The determinant of $A$ is the product of its eigenvalues (counting multiplicities).

For a $2 \times 2$ matrix, these facts are sufficient to reconstruct the eigenvalues.

**EXAMPLE 4** Is the graph of $f(x, y) = 3x^2 + 4xy + 6y^2$ an upward-facing paraboloid, a downward-facing paraboloid, or a saddle surface?

**SOLUTION** This quadratic form can be written

$$f(x, y) = 3x^2 + 4xy + 6y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The matrix has a trace of 9 and a determinant of 14, so its two eigenvalues have a sum of 9 and a product of 14. It follows that the two eigenvalues are 2 and 7. Since both of these are positive, the graph is an upward-facing paraboloid.

This technique can also be used to classify a conic section as an ellipse or hyperbola.

**EXAMPLE 5** Is the conic section $3x^2 + 4xy = 2$ an ellipse or a hyperbola?

**SOLUTION** Consider the quadratic form

$$f(x, y) = 3x^2 + 4xy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The matrix has a trace of 3 and a determinant of $-4$, so its two eigenvalues are $-1$ and 4. Since these have opposite signs, $f$ is hyperbolic, so this level curve must be a hyperbola.

**Quadratic Forms on $\mathbb{R}^n$**

The idea of quadratic forms can be generalized to any number of variables. For example, a quadratic form on $\mathbb{R}^3$ is a function $f : \mathbb{R}^3 \to \mathbb{R}$ of the form

$$f(x, y, z) = ax^2 + by^2 + cz^2 + pxy + qxz + ryz$$

where $a$, $b$, $c$, $p$, $q$, and $r$ are constants. Such a form can be described by a $3 \times 3$ symmetric matrix:

$$f(x, y, z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} a & p/2 & q/2 \\ p/2 & b & r/2 \\ q/2 & r/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The level surfaces of a quadratic form in $\mathbb{R}^3$ are **quadric surfaces** (see pg. 93–95 in the textbook). As with a quadratic form in $\mathbb{R}^2$, we can use the eigenvalues of the matrix to classify quadratic forms.
CLASSIFICATION OF QUADRATIC FORMS ON $\mathbb{R}^3$

Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a quadratic form, and let $\lambda$, $\mu$, and $\nu$ be the eigenvalues of the associated $3 \times 3$ symmetric matrix.

1. If $\lambda$, $\mu$, and $\nu$ are all positive, then $f(x, y, z) \geq 0$ for all $(x, y, z) \in \mathbb{R}^3$, the origin is a local minimum for $f$, and the level surfaces for $f$ are ellipsoids centered at the origin.

2. If $\lambda$, $\mu$, and $\nu$ are all negative, then $f(x, y, z) \leq 0$ for all $(x, y, z) \in \mathbb{R}^3$, the origin is a local maximum for $f$, and the level surfaces for $f$ are ellipsoids centered at the origin.

3. If $\lambda$, $\mu$, and $\nu$ are all nonzero but have different signs, then $f(x, y, z)$ takes both positive and negative values, and the level surfaces for $f$ are hyperboloids.

In general, a quadratic form $f: \mathbb{R}^n \to \mathbb{R}$ is said to be **positive definite** if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$. As you might imagine, a quadratic form is positive definite if and only if all of the eigenvalues of the associated matrix are positive.

Similarly, a quadratic form is **negative definite** if $f(x) \leq 0$ for all $x \in \mathbb{R}^n$. This occurs when all of the eigenvalues of the associated matrix are negative.

There is actually a simple test for whether a given symmetric matrix is positive definite. We shall state it only for the $3 \times 3$ case.

**SYLVESTER’S CRITERION**

Let $f: \mathbb{R}^3 \to \mathbb{R}$ be the quadratic form

$$f(x) = x^T \begin{bmatrix} a & u & v \\ u & b & w \\ v & w & c \end{bmatrix} x.$$  

Then $f$ is positive definite if and only if

$$a > 0 \quad \text{and} \quad \begin{vmatrix} a & u \\ u & b \end{vmatrix} > 0 \quad \text{and} \quad \begin{vmatrix} a & u & v \\ u & b & w \\ v & w & c \end{vmatrix} > 0.$$  

The three quantities in this criterion are known as **leading principal minors**. A similar criterion works for quadratic forms on $\mathbb{R}^n$, except that there are $n$ leading principal minors, namely the determinants of the $n$ square submatrices in the upper-left corner.

This criterion can also be used to test for negative definiteness. Specifically, a quadratic form $f$ is negative definite if and only if $-f$ is positive definite.