These problems must be written up in LaTeX, and are due next Friday, April 1.

1. Let $V$ and $W$ be normed vector spaces, and let $T: V \to W$ be a linear function. Prove that $T$ is continuous if and only if there exists an $M > 0$ such that $\|T(v)\| \leq M\|v\|$ for all $v \in V$.

2. Let $V$ be a Hilbert space, and let $\{u_n\}$ be an orthonormal sequence of vectors in $V$. Let $S$ be the space of all finite linear combinations of the $u_n$'s, and let

$$T = \left\{ \sum_{n=1}^{\infty} a_n u_n \left| \{a_n\} \in \ell^2 \right. \right\}.$$

(a) Prove that $T$ is the closure of $S$ in $V$. (Hint: Use the completeness of $\ell^2$.)

(b) Deduce that $\{u_n\}$ is a Hilbert basis for $V$ if and only if $S$ is dense in $V$.

3. Let $(X,\mu)$ be a measure space, let $f,g,h: X \to [0,\infty)$ be measurable functions, and let $p,q,r \in (1,\infty)$ so that $1/p + 1/q + 1/r = 1$. Prove that

$$\int_X fgh \leq \left( \int_X f^p \right)^{1/p} \left( \int_X g^q \right)^{1/q} \left( \int_X h^r \right)^{1/r}$$