# The Inverse Function Theorem

This short set of notes includes a complete proof of the Inverse Function Theorem. There will be more notes later covering smooth manifolds, immersions, and submersions.

Our goal is to prove the following theorem:

## The Inverse Function Theorem

> *Let $U \subset \mathbb{R}^n$ be open, let $F\colon U \to \mathbb{R}^n$ be a $C^1$ function, and let $\mathbf{a} \in U$. If $DF_{\mathbf{a}}$ is invertible, then there exist neighborhoods $V$ of $\mathbf{a}$ and $W$ of $F(\mathbf{a})$ such that the restriction $F\colon V \to W$ is a diffeomorphism.*

Here $C^1$ is short for "continuously differentiable". This theorem has three parts, which are largely independent of one another:

1. If $DF_{\mathbf{a}}$ is invertible, then $F$ is locally injective at $\mathbf{a}$.
2. If $DF_{\mathbf{a}}$ is invertible, then $F$ is locally surjective at $\mathbf{a}$.
3. If $F\colon V \to W$ is a $C^1$ homeomorphism and each $DF_{\mathbf{x}}$ is invertible, then $F$ is a diffeomorphism.

The proof of the first and third parts are mostly straightforward. The second part has a bit more substance, and can be proven in several different ways. Our approach is similar to that given in Chapter 9 of Rudin's *Principles of Mathematical Analysis*, though we have tried to be a bit more concrete, as well as more explicit about the use of Newton's method. After reading the treatment here, it might be a good idea to look at Rudin's proof, if only to see a general statement of the Contraction Principle.

# 1. Local Injectivity

**Definition: Locally Injective Function**
Let $F\colon X \to Y$ be a continuous function between topological spaces, and let $a \in X$. We say that $F$ is **locally injective** (or **locally one-to-one**) at $a$ if there exists a neighborhood $U$ of $a$ such that $F|_U$ is injective.

We wish to prove that if $DF_{\mathbf{a}}$ is invertible then $F$ is locally injective in a neighborhood of $\mathbf{a}$. We begin with a few lemmas, both of which will also be helpful for proving local surjectivity. The first is an inequality involving the norm of the inverse of a matrix:

### Lemma 1 Inverse Norm Inequality

*Let $A$ be an invertible $n \times n$ matrix, and let $\mathbf{v} \in \mathbb{R}^n$. Then*

$$\|A\mathbf{v}\| \;\geq\; \frac{\|\mathbf{v}\|}{\|A^{-1}\|}$$

**PROOF** We have $\|\mathbf{v}\| \;=\; \|A^{-1}A\mathbf{v}\| \;\leq\; \|A^{-1}\|\,\|A\mathbf{v}\|$. Dividing through by $\|A^{-1}\|$ yields the desired inequality. ■

We also need some sort of lemma to help us take advantage of continuously differentiable functions. One approach would be to prove that continuously differentiable functions are uniformly differentiable. However, the following lemma is slightly easier, and is more than sufficient for our needs:

### Lemma 2 Linear Approximation Lemma

*Let $U \subset \mathbb{R}^n$, let $F\colon U \to \mathbb{R}^m$ be a $C^1$ function, and let $\mathbf{a} \in U$. Then for every $\epsilon > 0$, there exists a neighborhood $V$ of $\mathbf{a}$ so that*

$$\|F(\mathbf{x}) - F(\mathbf{y}) - DF_{\mathbf{a}}(\mathbf{x} - \mathbf{y})\| \;\leq\; \epsilon\|\mathbf{x} - \mathbf{y}\|$$

*for all $\mathbf{x}, \mathbf{y} \in V$.*

**PROOF** Since $DF$ is continuous at $\mathbf{a}$, there exists a $\delta > 0$ so that

$$\|\mathbf{x}-\mathbf{a}\| < \delta \qquad \Rightarrow \qquad \|DF_{\mathbf{x}} - DF_{\mathbf{a}}\| < \epsilon.$$

Let $V$ be the open $\delta$-ball centered at $\mathbf{a}$. Let $\mathbf{x},\mathbf{y} \in V$ and let $\gamma\colon [0,1] \to V$ be a straight-line path from $\mathbf{y}$ to $\mathbf{x}$. By the Fundamental Theorem of Calculus,

$$F(\mathbf{x}) - F(\mathbf{y}) \;=\; \int_0^1 (F\circ\gamma)'(t)\,dt \;=\; \int_0^1 DF_{\gamma(t)}\big(\gamma'(t)\big)\,dt \;=\; \int_0^1 DF_{\gamma(t)}(\mathbf{x}-\mathbf{y})\,dt.$$

But $\|DF_{\gamma(t)} - DF_{\mathbf{a}}\| < \epsilon$, so

$$\|DF_{\gamma(t)}(\mathbf{x}-\mathbf{y}) - DF_{\mathbf{a}}(\mathbf{x}-\mathbf{y})\| \;\le\; \epsilon\|\mathbf{x}-\mathbf{y}\|.$$

for all $t \in [0,1]$. We conclude that

$$\begin{aligned}
\|F(\mathbf{x}) - F(\mathbf{y}) - DF_{\mathbf{a}}(\mathbf{x}-\mathbf{y})\| \;&=\; \left\|\int_0^1 DF_{\gamma(t)}(\mathbf{x}-\mathbf{y})\,dt \;-\; DF_{\mathbf{a}}(\mathbf{x}-\mathbf{y})\right\| \\
&=\; \left\|\int_0^1 (DF_{\gamma(t)} - DF_{\mathbf{a}})(\mathbf{x}-\mathbf{y})\,dt\right\| \\
&\le\; \int_0^1 \|(DF_{\gamma(t)} - DF_{\mathbf{a}})(\mathbf{x}-\mathbf{y})\|\,dt \\
&\le\; \epsilon\|\mathbf{x}-\mathbf{y}\|.
\end{aligned}$$

■

## Theorem 3 Inverse Function Theorem — Injectivity

> *Let $U \subset \mathbb{R}^n$ be open, let $F\colon U \to \mathbb{R}^n$ be a $C^1$ function, and let $\mathbf{a} \in U$. If $DF_{\mathbf{a}}$ is invertible, then $F$ is locally injective at $\mathbf{a}$.*

**PROOF** By the previous Lemma, there exists a neighborhood $V$ of $\mathbf{a}$ so that

$$\|F(\mathbf{x}) - F(\mathbf{y}) - DF_{\mathbf{a}}(\mathbf{x}-\mathbf{y})\| \;\le\; \frac{\|\mathbf{x}-\mathbf{y}\|}{2\|DF_{\mathbf{a}}^{-1}\|}$$

for all $\mathbf{x},\mathbf{y} \in V$. Since $\|DF_{\mathbf{a}}(\mathbf{x}-\mathbf{y})\| \ge \|\mathbf{x}-\mathbf{y}\|/\|DF_{\mathbf{a}}^{-1}\|$, it follows that

$$\|F(\mathbf{x}) - F(\mathbf{y})\| \;\ge\; \frac{\|\mathbf{x}-\mathbf{y}\|}{2\|DF_{\mathbf{a}}^{-1}\|}$$

for all $\mathbf{x},\mathbf{y} \in V$, so $F$ is injective on $V$. ■

## 2. Local Surjectivity

**Definition: Locally Surjective Function**
Let $F: X \to Y$ be a continuous function between topological spaces, and let $a \in X$. We say that $F$ is **locally surjective** if for every neigborhood $V$ of $\mathbf{a}$, the image $F(V)$ contains a neighborhood of $F(\mathbf{a})$.

If a function $F$ is locally surjective at every point, then the image of any open set under $F$ must be open. Such a function is called an **open map**.

Proving that a function is locally surjective is considerably harder than proving that it is locally injective, since we must actually find a point in the domain that maps to a given point in the range. In an essential way, this involves solving a nonlinear equation in multiple dimensions.

Solving equations requires some kind of numerical method, and almost any such method can be used for this proof. Possibilities include:

1. A bisection method, using $(n-1)$-dimensional homotopy groups to keep track of which cubes contain a root.

2. A differential equations method, where we use a differential equation to define a path that moves continuously toward to the root.

3. An iterative method that produces a sequence of better and better approximations to the root.

We will use the last approach here. In particular, we will use a variant of **Newton's method** to find the root to a given equation.

We begin by recalling Newton's method in one dimension. Suppose we wish to solve the equation $f(x) = c$, where $f: \mathbb{R} \to \mathbb{R}$ is a function and $c$ is a constant. Using Newton's method, we first compute the linear approximation for $f$ based at some point $x_0$:

$$f(x) \;\approx\; f(x_0) + f'(x_0)\,(x - x_0).$$

Next, we set the linear function on the right equal to $c$, and solve for $x$:

$$x \;\approx\; x_0 + \frac{c - f(x_0)}{f'(x_0)}.$$

This does not give the exact solution, but it usually gives an approximate solution that is much better than $x_0$. If we iterate this process, we obtain a sequence $\{x_n\}$ of approximations defined by

$$x_{n+1} \;=\; x_n + \frac{c - f(x_n)}{f'(x_n)}.$$

In most situations, the sequence $\{x_n\}$ will converge to a solution to the equation.

Our proof of the following theorem uses a variant of this method:

## **Theorem 4** Inverse Function Theorem — Surjectivity

*Let $U \subset \mathbb{R}^n$ be open, and let $F\colon U \to \mathbb{R}^n$ be a $C^1$ function, and let $\mathbf{a} \in U$. If $DF_{\mathbf{a}}$ is invertible, then $F$ is locally surjective at $\mathbf{a}$.*

**PROOF** Let $\sigma = \|DF_{\mathbf{a}}^{-1}\|$. Since $F$ is continuously differentiable, by Lemma 2 there exists a $\delta > 0$ so that

$$\|F(\mathbf{x}) - F(\mathbf{y}) - DF_{\mathbf{a}}(\mathbf{x}-\mathbf{y})\| \;\leq\; \frac{\|\mathbf{x}-\mathbf{y}\|}{3\sigma}.$$

for all points $\mathbf{x}, \mathbf{y}$ in the $\delta$-ball centered at $\mathbf{a}$. Let $V$ denote this ball, and let $W$ be the open ball of radius $\delta/3\sigma$ centered at $F(\mathbf{a})$. We claim that $F(V) \supset W$.

Let $\mathbf{c}$ be any point in $W$, and let $\mathbf{x}_0 = \mathbf{a}$. Since $\mathbf{c}$ lies in $W$, we know that

$$\|\mathbf{c} - F(\mathbf{x}_0)\| \;\leq\; \frac{\delta}{3\sigma}.$$

Consider the point

$$\mathbf{x}_1 \;=\; \mathbf{x}_0 + DF_{\mathbf{a}}^{-1}\big(\mathbf{c} - F(\mathbf{x}_0)\big).$$

We have

$$\|\mathbf{x}_1 - \mathbf{x}_0\| \;=\; \big\|DF_{\mathbf{a}}^{-1}\big(\mathbf{c} - F(\mathbf{x}_0)\big)\big\| \;\leq\; \sigma\|\mathbf{c} - F(\mathbf{x}_0)\| \;=\; \frac{\delta}{3},$$

so $\mathbf{x}_1$ lies in $V$. Furthermore, since $DF_{\mathbf{a}}(\mathbf{x}_1 - \mathbf{x}_0) = \mathbf{c} - F(\mathbf{x}_0)$, we have

$$\|F(\mathbf{x}_1) - \mathbf{c}\| \;=\; \|F(\mathbf{x}_1) - F(\mathbf{x}_0) - DF_{\mathbf{a}}(\mathbf{x}_1 - \mathbf{x}_0)\| \;\leq\; \frac{\|\mathbf{x}_1 - \mathbf{x}_0\|}{3\sigma} \;\leq\; \frac{\delta}{9\sigma},$$

so $F(\mathbf{x}_1)$ is better approximation to $\mathbf{c}$ than $F(\mathbf{x}_0)$.

Iterating this process, we define a sequence $\{\mathbf{x}_n\}$ of points by

$$\mathbf{x}_{n+1} \;=\; \mathbf{x}_n + DF_{\mathbf{a}}^{-1}\big(\mathbf{c} - F(\mathbf{x}_n)\big).$$

Using the same argument as above, we can prove inductively that

$$\|\mathbf{x}_n - \mathbf{x}_{n-1}\| \;\leq\; \frac{\delta}{3^n} \qquad \text{and} \qquad \|F(\mathbf{x}_n) - \mathbf{c}\| \;\leq\; \frac{\delta}{3^{n+1}\sigma}$$

for each $n$. In particular, $\{\mathbf{x}_n\}$ is a Cauchy sequence, so it converges to some limit $\mathbf{x}$. Since $F$ is continuous and $F(\mathbf{x}_n) \to \mathbf{c}$, we know that $F(\mathbf{x}) = \mathbf{c}$. Furthermore, since

$$\|\mathbf{x} - \mathbf{a}\| \;=\; \lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{a}\| \;\leq\; \sum_{n=1}^{\infty} \|\mathbf{x}_n - \mathbf{x}_{n-1}\| \;\leq\; \sum_{n=1}^{\infty} \frac{\delta}{3^n} \;=\; \frac{\delta}{2} \;<\; \delta,$$

the point $\mathbf{x}$ lies in $V$. ■

## 3. Differentiability of the Inverse

At this point, we have completed most of the proof of the Inverse Function Theorem. All that remains is the following:

### Theorem 5 Differentiability of the Inverse

*Let $U, V \subset \mathbb{R}^n$ be open, and let $F\colon U \to V$ be a $C^1$ homeomorphism. If $F$ has no critical points, then $F^{-1}$ is differentiable.*

**PROOF** Let $\mathbf{a} \in U$, and let $\mathbf{b} = F(\mathbf{a})$. We wish to show that $F^{-1}$ is differentiable at $\mathbf{b}$, with $D(F^{-1})_{\mathbf{b}} = DF_{\mathbf{a}}^{-1}$.

Let $\epsilon > 0$, and let $\sigma = \|DF_{\mathbf{a}}^{-1}\|$. Since $F$ is differentiable at $\mathbf{a}$, there exists a $\delta > 0$ so that

$$\|\mathbf{y} - \mathbf{b} - DF_{\mathbf{a}}(\mathbf{x} - \mathbf{a})\| \;\leq\; \frac{\epsilon}{\sigma(\epsilon + \sigma)}\|\mathbf{x} - \mathbf{a}\|,$$

whenever $\|\mathbf{x} - \mathbf{a}\| < \delta$, where $\mathbf{y} = F(\mathbf{x})$. Note then that

$$\frac{\epsilon}{\sigma(\epsilon + \sigma)}\|\mathbf{x} - \mathbf{a}\| \;\geq\; \|DF_{\mathbf{a}}(\mathbf{x} - \mathbf{a})\| - \|\mathbf{y} - \mathbf{b}\| \;\geq\; \frac{1}{\sigma}\|\mathbf{x} - \mathbf{a}\| - \|\mathbf{y} - \mathbf{b}\|$$

so

$$\|\mathbf{y} - \mathbf{b}\| \;\geq\; \left(\frac{1}{\sigma} - \frac{\epsilon}{\sigma(\epsilon + \sigma)}\right)\|\mathbf{x} - \mathbf{a}\| \;=\; \frac{1}{\epsilon + \sigma}\|\mathbf{x} - \mathbf{a}\|.$$

Now, since $F^{-1}$ is continuous at $\mathbf{b}$, there exists a $\delta' > 0$ so that

$$\|\mathbf{y} - \mathbf{b}\| \;<\; \delta' \qquad \Rightarrow \qquad \|\mathbf{x} - \mathbf{a}\| \;<\; \delta.$$

Assuming $\|\mathbf{y} - \mathbf{b}\| < \delta'$, we conclude that

$$\begin{aligned} \|\mathbf{x} - \mathbf{a} - DF_{\mathbf{a}}^{-1}(\mathbf{y} - \mathbf{b})\| \;&=\; \left\|-DF_{\mathbf{a}}^{-1}\big(\mathbf{y} - \mathbf{b} - DF_{\mathbf{a}}(\mathbf{x} - \mathbf{a})\big)\right\| \\ &\leq\; \sigma\,\|\mathbf{y} - \mathbf{b} - DF_{\mathbf{a}}(\mathbf{x} - \mathbf{a})\| \;\leq\; \frac{\epsilon}{\epsilon + \sigma}\|\mathbf{x} - \mathbf{a}\| \;\leq\; \epsilon\|\mathbf{y} - \mathbf{b}\|. \end{aligned}$$

■

We are now ready to put all of the ingredients together:

### The Inverse Function Theorem

*Let $U \subset \mathbb{R}^n$ be open, let $F\colon U \to \mathbb{R}^n$ be a $C^1$ function, and let $\mathbf{a} \in U$. If $DF_{\mathbf{a}}$ is invertible, then there exist neighborhoods $V$ of $\mathbf{a}$ and $W$ of $F(\mathbf{a})$ such that the restriction $F\colon V \to W$ is a diffeomorphism.*

**PROOF** By Theorem 3, $F$ is locally injective at $\mathbf{a}$, so there exists a neighborhood $U'$ of $\mathbf{a}$ so that $F$ is injective on $U'$. Since $DF$ is continuous, there exists a neighborhood $V \subset U'$ of $\mathbf{a}$ so that $DF_{\mathbf{x}}$ is invertible for all $\mathbf{x} \in V$. Then $F$ is locally onto at each point of $V$ by Theorem 4, so the restriction $F|_V$ is an open map. Let $W = F(V)$. Then $F\colon V \to W$ is a continuous bijection that maps open sets to open sets, so $F$ is a homeomorphism. Since $DF_{\mathbf{x}}$ is invertible for each $\mathbf{x} \in V$, it follows from Theorem 5 that this map is a diffeomorphism. ■