These problems must be written up in \LaTeX{}, and are due on Thursday, March 17.

**Rules:** This is a midterm. You must solve the problems entirely on your own, and you should not discuss the problems with anyone but me. You may use the following sources:

1. *A Primer of Lebesgue Integration* by H. S. Bear.
5. *The Real Numbers and Real Analysis* by Ethan Bloch.

You should not consult any other textbooks when working on the problems, and you should not look anything up on the internet.

1. (a) Let $f: [0, 1] \to [0, 1]$ be a measurable function. Prove that there exists a sequence $\{E_n\}$ of measurable subsets of $[0, 1]$ such that

$$f(x) = \sum_{n=1}^{\infty} \frac{\chi_{E_n}(x)}{2^n}$$

for all $x \in [0, 1]$.

(b) Let $E \subset [0, 1]$ be a measurable set, and let $\epsilon > 0$. Prove that there exists a continuous function $g: [0, 1] \to [0, 1]$ so that

$$m(\{x : g(x) \neq \chi_E(x)\}) < \epsilon.$$

*Hint:* Use Urysohn’s lemma (see Munkres, §33).

(c) Let $f: [0, 1] \to [0, 1]$ be a measurable function, and let $\epsilon > 0$. Use parts (a) and (b) to prove that there exists a continuous function $g: [0, 1] \to [0, 1]$ so that

$$m(\{x : f(x) \neq g(x)\}) < \epsilon.$$

2. Use the Dominated Convergence Theorem to prove that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \exp \left[ -2|x| \left( 1 + \frac{\tan^{-1}(nx)}{\pi} \right) \right] dx = \frac{4}{3}.$$
3. Let \( f : [0, 1] \to [0, \infty) \) be a bounded measurable function. Let
\[
\|f\|_p = \left( \int_{[0,1]} f^p \right)^{1/p}
\]
for all \( p \geq 1 \), and define
\[
\|f\|_\infty = \sup\{\alpha \in \mathbb{R} \mid m(E_\alpha) > 0\},
\]
where \( E_\alpha = \{x \in \mathbb{R} \mid f(x) \geq \alpha\} \).
(a) Prove that \( \|f\|_p \leq \|f\|_\infty \) for all \( p \geq 1 \).
(b) Prove that \( \|f\|_p \to \|f\|_\infty \) as \( p \to \infty \). (Hint: Let \( \alpha < \|f\|_\infty \), and prove that \( \|f\|_p > \alpha \) for sufficiently large values of \( p \).)

4. Prove that there exists a nonconstant function \( f : \mathbb{R} \to \mathbb{R} \) so that \( f(0) = f(1) = 0 \) and
\[
f\left( \frac{x+y}{2} \right) = \frac{f(x) + f(y)}{2}
\]
for all \( x, y \in \mathbb{R} \).