MATH 242 Elementary Linear Algebra Spring 2018

Study Sheet for Final Exam

- This exam will be closed book.
- This study sheet will not be allowed during the test.
- Books, notes and online resources will not be allowed during the test.
- Electronic devices (calculators, cell phones, tablets, laptops, etc.) will not be allowed during the test.

Topics

1. Linear independence in $\mathbb{R}^n$
2. Bases of subspaces of $\mathbb{R}^n$
3. Dimension of subspaces of $\mathbb{R}^n$
4. Three important subspaces
5. Determinants
6. Applications of determinants
7. Eigenvalues, eigenvectors and eigenspaces
8. Complex numbers
9. Systems of first order linear differential equations
10. Dot product in $\mathbb{R}^n$
11. Orthogonal sets and orthogonal bases
12. Orthogonal projection
13. Gram-Schmidt
14. Vector spaces
15. Inner products
16. Fourier approximations

Tips for Studying for a Quiz or Exam

×	**Bad**	Forgetting about the homework and the previous quizzes.
✓	**Good**	Making sure you know how to do all the problems on the homework and previous quizzes; seeking help seeking help from the instructor and the tutors about the problems you do not know how to do.
×	**Bad**	Doing all the practice problems from some of the sections, and not having enough time to do practice problems from the rest of the sections.
✓	**Good**	Doing a few practice problems of each type from every sections.
×	**Bad**	Studying only by reading the book.
✓	**Good**	Doing a lot of practice problems, and reading the book as needed.
×	**Bad**	Studying only by yourself.
✓	**Good**	Trying some practice problems by yourself (or with friends), and then seeking help from the instructor and the tutors about the problems you do not know how to do.
×	**Bad**	Doing practice problems while looking everything up in the book.
✓	**Good**	Doing some of the practice problems the way you would do them on the quiz or exam, which is with closed book and no calculator.
×	**Bad**	Staying up late (or all night) the night before the exam.
✓	**Good**	Studying hard up through the day before the exam, but getting a good night's sleep the night before the exam.

Ethan's Office Hours

- **Monday:** 4:00-5:30
- **Wednesday:** 2:00-3:30
- **Thursday:** 10:30-12:00
- **Or by appointment**

Tutor

- **Eric Zhang:**
  - Office Hour: Monday, 6:00-7:00, Mathematics Common Room (third floor of Albee)
  - Email: jz2226@bard.edu

Practice Problems from Holt, 2nd ed.

Section 2.3: 1, 3, 5, 7, 9, 11, 19, 21, 23, 37, 39

Section 4.2: 11, 13, 15, 23, 25, 27, 29, 31

Section 4.3: 5, 7, 13, 15, 17, 19, 21, 23, 25

Section 5.1: Just find the determinant: 11, 13, 15, 17, 19, 21, 23, 25

Do the exercise as written: 27, 29, 31, 33, 35, 37, 39, 61, 63

Section 5.2: 1, 3, 5, 15, 17, 23, 25

Section 5.3: 1, 3, 5, 7, 9, 11, 19, 21, 23, 25, 27, 35

Section 6.1: 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 47

Complex Numbers Handout: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 41, 43, 45

Section 6.4: 1, 3, 5, 7, 9, 11, 13, 15, 17

Section 8.1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 37, 39, 41, 43

Section 8.2: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35

Section 7.1: 13, 15, 21, 23, 25

Section 7.2: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25

Section 7.3: 1, 3, 5, 7, 9, 11, 13, 15

Section 9.1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21

Section 10.1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29

Section 10.2: 15, 17

Some Important Concepts and Formulas

1. Linear Dependence and Linear Independence

Let n be a positive integer. Let $v_1, \dots, v_k$ be in $\mathbb{R}^n$.

1. The vectors $v_1, \dots, v_k$ are **linearly dependent** if there are $a_1, a_2, \dots a_k$ in $\mathbb{R}$ that are not all 0, such that $a_1 v_1 + \dots + a_k v_k = \mathbf{0}$.
2. The vectors $v_1, \dots, v_k$ are **linearly independent** if they are not linearly dependent.

2. Strategy for Proving Linear Independence

Let n be a positive integer. Let $v_1, \dots, v_k$ be in $\mathbb{R}^n$. The standard strategy for showing that $v_1, \dots, v_k$ are linearly independent is as follows:

Suppose that $a_1 v_1 + \dots + a_k v_k = \mathbf{0}$ for some $a_1, a_2, \dots a_k$ in $\mathbb{R}$.

$\vdots$

(argumentation)

$\vdots$

Then $a_1 = 0, \dots, a_n = 0$. Hence $v_1, \dots, v_k$ are linearly independent.

3. Linear Independence and Systems of Linear Equations

Let n be a positive integer. Let $v_1, \dots, v_k$ be in $\mathbb{R}^n$. Let $A = [v_1 \; v_2 \; \cdots \; v_k]$. The following are equivalent.

(a) The vectors $\{v_1, \dots, v_k\}$ are linearly independent.

(b) The system of linear equations $x_1 v_1 + \cdots x_k v_k = \mathbf{0}$ has a unique solution.

(c) The matrix equation $Ax = \mathbf{0}$ has a unique solution.

4. Linear Dependence and Linear Independence in $\mathbb{R}^n$

Let n be a positive integer. Let $v_1, \dots, v_k$ be in $\mathbb{R}^n$.

1. If one of $v_1, \dots, v_k$ is $\mathbf{0}$, then $v_1, \dots, v_k$ are linearly dependent.
2. If $k > n$, then $v_1, \dots, v_k$ are linearly dependent.
3. If $k \leq n$, then $v_1, \dots, v_k$ might be linearly dependent or linearly independent.
4. Suppose $k \leq n$. Let $A = [v_1 \; v_2 \; \cdots \; v_k]$. Suppose that A is row equivalent to B, where B is in echelon form. Then $v_1, \dots, v_k$ is linearly independent if and only if B has a pivot position in every column.

5. Bases of Subspaces of $\mathbb{R}^n$

Let n be a positive integer. Let W be a subspace of $\mathbb{R}^n$. Let $v_1, \ldots, v_k$ be in W.

1. The vectors $v_1, \ldots, v_k$ are a **basis** for W if they are linearly independent and they span W.
2. If $v_1, \ldots, v_k$ are a basis for W, then every vector v in W can be expressed uniquely as a linear combination of $v_1, \ldots, v_k$.
3. If every vector v in W can be expressed uniquely as a linear combination of $v_1, \ldots, v_k$, then $v_1, \ldots, v_k$ are a basis for W.
4. The subspace W has a basis, and all bases of W have the same number of vectors.
5. Any linearly independent set in W can be expanded to be a basis of W.
6. Any set that spans W can be reduced to be a basis of W.

6. Bases of Subspaces of $\mathbb{R}^n$ via Row Equivalence

Let n be a positive integer. Let $v_1, \ldots, v_k$ be in $\mathbb{R}^n$. Let $T = \text{span}\{v_1, \ldots, v_k\}$.

1. Let $A = [v_1\ v_2\ \cdots\ v_k]$. Suppose that A is row equivalent to $B = [u_1\ u_2\ \cdots\ u_k]$, where B is in echelon form or reduced row echelon form.
2. The columns in B with pivot positions are linearly independent, and are a basis for T.
3. The columns of A that correspond to the columns of B with pivot positions are linearly independent, and are a basis for T.
4. If a column u_i of B does not have a pivot position and is expressed as a linear combination of columns in B with pivot positions, then the column v_i of A is expressed as the same linear combination of the corresponding columns in A with pivot positions.

7. Dimension of Subspaces of $\mathbb{R}^n$

Let n be a positive integer. Let W be a subspace of $\mathbb{R}^n$. The **dimension** of W, denoted $\dim W$, is the number of vectors in any basis for W.

8. Dimension of Subspaces of Subspaces of $\mathbb{R}^n$

Let n be a positive integer. Let U and W be subspaces of $\mathbb{R}^n$. Suppose that U is contained in W.

1. $\dim U \leq \dim W$.
2. If $\dim U = \dim W$, then $U = W$.

9. Properties of Linearly Independent and Spanning Sets in Subspaces of $\mathbb{R}^n$

Let n be a positive integer. Let W be a subspace of $\mathbb{R}^n$. Let $m = \dim W$. Let $v_1, \ldots, v_k$ be in W.

1. If $v_1, \ldots, v_k$ span W, then $k \geq m$.
2. If $v_1, \ldots, v_k$ span W and $k = m$, then $v_1, \ldots, v_n$ are a basis for W.
3. If $v_1, \ldots, v_k$ are linearly independent, then $k \leq m$.
4. If $v_1, \ldots, v_k$ are linearly independent and $k = m$, then $v_1, \ldots, v_n$ are a basis for W.
5. If $v_1, \ldots, v_k$ is spans W, then it contains a basis for W.
6. If $v_1, \ldots, v_k$ is linearly independent, then it can be extended to a basis for W.

10. Properties of Linearly Independent and Spanning Sets in $\mathbb{R}^n$

Let n be a positive integer. Let $v_1, \ldots, v_k$ be in $\mathbb{R}^n$.

1. If $v_1, \ldots, v_k$ span $\mathbb{R}^n$, then $k \geq n$.
2. If $v_1, \ldots, v_k$ span $\mathbb{R}^n$ and $k = n$, then $v_1, \ldots, v_n$ are a basis for $\mathbb{R}^n$.
3. If $v_1, \ldots, v_k$ are linearly independent, then $k \leq n$.
4. If $v_1, \ldots, v_k$ are linearly independent and $k = n$, then $v_1, \ldots, v_n$ are a basis for $\mathbb{R}^n$.
5. If $v_1, \ldots, v_k$ is spans $\mathbb{R}^n$, then it contains a basis for $\mathbb{R}^n$.
6. If $v_1, \ldots, v_k$ is linearly independent, then it can be extended to a basis for $\mathbb{R}^n$.

11. Three Important Spaces

Let m and n be positive integers. Let A be an $m \times n$ matrix.

1. The **column space** of A, denoted $\text{col}(A)$, is the subspace of $\mathbb{R}^m$ that is the span of the columns of A.
2. The **row space** of A, denoted $\text{row}(A)$, is the subspace of $\mathbb{R}^n$ that is the span of the rows of A.
3. The **null space** of A, denoted $\text{null}(A)$, is the subspace of $\mathbb{R}^n$ that consists of all vectors v in $\mathbb{R}^n$ such that $Av = \mathbf{0}$.

12. Two Important Spaces and Linear Maps

Let m and n be positive integers. Let A be an $m \times n$ matrix.

1. $\text{col}(A) = \text{im}\, \mathsf{L}_A$.
2. $\text{null}(A) = \ker \mathsf{L}_A$.

13. Three Important Spaces: Row Equivalence

Let m and n be positive integers. Let A be an $m \times n$ matrix. Suppose that A is row equivalent to B. Then the column space, row space and null space for A and B are the same.

14. Three Important Spaces: Bases

Let m and n be positive integers. Let A be an $m \times n$ matrix. Suppose that A is row equivalent to B, where B is in echelon form.

1. The columns of B with pivot positions are a basis of the column space of A.
2. The non-zero rows of B are a basis for the row space of A.
3. A basis for the solution set of $Bv = \mathbf{0}$ is a basis for the null space of A.

15. Three Important Spaces: Dimension

Let m and n be positive integers. Let A be an $m \times n$ matrix.

1. The **column rank** of A, denoted $\operatorname{columnrank} A$, is the dimension of the column space of A.
2. The **row rank** of A, denoted $\operatorname{rowrank} A$, is the dimension of the row space of A.
3. The **nullity** of A, denoted $\operatorname{nullity} A$, is the dimension of the null space of A.

16. Three Important Spaces: Dimension via Row Equivalence

Let m and n be positive integers. Let A be an $m \times n$ matrix. Suppose that A is row equivalent to B, where B is in echelon form.

1. The column rank of A equals the number of columns of B with pivot positions.
2. The row rank of A equals the number of non-zero rows of B.
3. The nullity of A equals the number of free variables of B.

17. Column Rank Equals Row Rank

Let m and n be positive integers. Let A be an $m \times n$ matrix. Then the column rank of A equals the row rank of A.

18. Rank of a Matrix

Let m and n be positive integers. Let A be an $m \times n$ matrix. The **rank** of A, denoted $\operatorname{rank} A$, is the column rank of A and row rank of A.

19. Rank-Nullity Theorem for Matrices

Let m and n be positive integers. Let A be an $m \times n$ matrix. Then

$$\operatorname{rank} A + \operatorname{nullity} A = n.$$

20. Rank-Nullity Theorem for Linear Maps Given by Matrix Multiplication

Let m and n be positive integers. Let A be an $m \times n$ matrix. Then

$$\dim(\operatorname{im} \mathsf{L}_A) + \dim(\ker \mathsf{L}_A) = \dim(\mathbb{R}^n).$$

21. Determinants of 2×2 Matrices

Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be a 2×2 matrix. The **determinant** of A, denoted $\det A$, is defined by

$$\det A = \det \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = ad - bc.$$

22. Minors and Cofactors

Let n be a positive integer. Let A be an $n \times n$ matrix. Let i and j be positive integers such that $1 \le i \le n$ and $1 \le j \le n$.

1. The ij^{th} **minor** of A, denoted M_{ij}, is the determinant of the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the i^{th} row and j^{th} column of A.
2. The ij^{th} **cofactor** of A, denoted C_{ij}, is $(-1)^{i+j} M_{ij}$.
3. The **cofactor matrix** of A, denoted $\operatorname{cof} A$, is the matrix $\left[C_{ij}\right]$.
4. The **adjoint** of A, denoted $\operatorname{adj} A$, is the matrix $(\operatorname{cof} A)^T$.

23. Determinants of $n \times n$ Matrices

Let n be a positive integer. Let A be an $n \times n$ matrix. The **determinant** of A, denoted $\det A$, can be obtained by expanding along any row or any column and using minors or cofactors. The same result will be obtained no matter which row or column is chosen.

1. Let i be a positive integer such that $1 \le i \le n$. Then expanding along the i^{th} row yields

$$\begin{aligned}\det A &= (-1)^{i+1} a_{i1} M_{i1} + (-1)^{i+2} a_{i2} M_{i2} + \cdots + (-1)^{i+n} a_{in} M_{in} \\ &= a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}.\end{aligned}$$

2. Let j be a positive integer such that $1 \le j \le n$. Then expanding along the j^{th} column yields

$$\begin{aligned}\det A &= (-1)^{1+j} a_{1j} M_{1j} + (-1)^{2+j} a_{2j} M_{2j} + \cdots + (-1)^{n+j} a_{nj} M_{nj} \\ &= a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}.\end{aligned}$$

24. Properties of Determinants

Let n be a positive integer. Let A, B and C be $n \times n$ matrices, and let k be a real number.

1. If B is obtained from A by multiplying a row (or column) of by k, then $\det B = k \det A$.
2. If B is obtained from A by interchanging two rows (or two columns), then $\det B = -\det A$.
3. If B is obtained from A by adding a multiple of one row (or column, respectively) to another row (or column, respectively), then $\det B = \det A$.
4. If A is a triangular matrix, then $\det A$ is the product of the diagonal elements of A.
5. If two rows (or two columns) of A are equal, then $\det A = 0$.
6. If A, B and C are identical except for one row (or one column), and if that row (or column) in C is the sum of the corresponding rows (or columns) in A and B, then $\det C = \det A + \det B$.
7. $\det(A^T) = \det A$.
8. $\det(AB) = \det A \cdot \det B$.

25. Determinants: Area

Let v_1 and v_2 be vectors in $\mathbb{R}^2$. Let $A = [v_1\ v_2]$. Then the area of the parallelogram formed by v_1 and v_2 is $|\det A|$.

26. Determinants: Volume

Let v_1, v_2 and v_3 be vectors in $\mathbb{R}^3$. Let $A = [v_1\ v_2\ v_3]$. Then the area of the parallelepiped formed by v_1, v_2 and v_3 is $|\det A|$.

27. Determinants and Inverse Matrices

Let n be a positive integer. Let A be an $n \times n$ matrix.

1. The matrix A is invertible if and only if $\det A \neq 0$.
2. If A is invertible, then
$$\det(A^{-1}) = \frac{1}{\det A}.$$
3. If A is invertible, then
$$A^{-1} = \frac{1}{\det A}(\operatorname{cof} A)^T = \frac{1}{\det A} \operatorname{adj} A.$$

28. Determinants and Solutions of Systems of Linear Equations

Determinants are useful for solutions of systems of linear equations when the number of equations is the same as the number of unknowns.

Let n be a positive integer. Let A be an $n \times n$ matrix, let b be in $\mathbb{R}^n$, and let $v = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

1. If $\det A \neq 0$, then $Av = b$ has a unique solution, which is $v = A^{-1}b$.
2. If $\det A \neq 0$, then $Av = \mathbf{0}$ has a unique solution, which is $v = \mathbf{0}$.
3. If $\det A = 0$, then $Av = b$ either has no solution or it has infinitely many solutions.
4. If $\det A = 0$, then $Av = \mathbf{0}$ has infinitely many solutions.

29. Cramer's Rule

Let n be a positive integer. Let A be an $n \times n$ matrix, let b be in $\mathbb{R}^n$, and let $v = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

1. For each i in $1, \ldots, n$, let A_i be the matrix obtained by replacing the i^{th} column of A with b.
2. Suppose that A is invertible. Then the system of linear equations $Av = b$ has a unique solution, which is given by
$$x_i = \frac{\det A_i}{\det A}$$
for each i in $1, \ldots, n$.

30. Eigenvalues and Eigenvectors

Let n be a positive integer. Let A be an $n \times n$ matrix.

1. Let λ be a real number. The number λ is an **eigenvalue** of A if there is a non-zero vector v in $\mathbb{R}^n$ such that $Av = \lambda v$; such a vector v is an **eigenvector** of A corresponding to λ.
2. The **characteristic polynomial** of A is the polynomial $\det(A - \lambda I_n)$.
3. The eigenvalues of A, if there are any, are the roots of the characteristic polynomial. For each eigenvalue, the corresponding eigenvectors can be found by substituting the eigenvalue λ into the equation $(A - \lambda I_n)v = \mathbf{0}$, and finding non-zero solutions for v.

31. Eigenvalues of Triangular Matrices

Let n be a positive integer. Let A be an $n \times n$ matrix. If A is a triangular matrix, then the the eigenvalues of A are the diagonal elements of A.

32. Unifying Theorem with Eigenvalues

Let n be a positive integer. Let A be an $n \times n$ matrix. Then $\det A \neq 0$ if and only if $\lambda = 0$ is not an eigenvalue of A.

33. Eigenspaces

Let n be a positive integer. Let A be an $n \times n$ matrix. Let λ be an eigenvalue of A.

1. The **eigenspace** of A associated with λ, denoted E_λ, is the set of all eigenvectors of A corresponding to λ together with $\mathbf{0}$.
2. The eigenspace E_λ is the solution set of the equation $(A - \lambda I_n)v = \mathbf{0}$.
3. The eigenspace E_λ is a subspace of $\mathbb{R}^n$.
4. The dimension of E_λ is less than or equal to the multiplicity of λ in the characteristic polynomial of A.

34. The Complex Numbers

1. The number i is the number such that $i^2 = -1$.
2. A **complex number** is any number of the form $a + bi$, where a, b in $\mathbb{R}$.
3. The set of all complex numbers is denoted $\mathbb{C}$.
4. Let $z = a + bi$. The **real part** of z is a, and the **imaginary part** of z is b.
5. The **modulus** of z, denoted $\|z\|$, is defined by $\|z\| = \sqrt{a^2 + b^2}$.
6. The **complex conjugate** of z, denoted $\bar{z}$, is defined by $\bar{z} = a - bi$.

35. The Complex Numbers: Basic Operations

Let $z = a + bi$ and $w = c + di$ be complex numbers, and let t be a real number.

1. $z + w = (a + c) + (b + d)i$.
2. $z - w = (a - c) + (b - d)i$.
3. $zw = (ab - bd) + (ad + bc)i$.
4. $\frac{z}{w} = \frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ab+bd)+(bc-ad)i}{c^2+d^2}$.
5. $tz = ta + tbi$.

36. The Complex Numbers: Roots of Polynomials

Let $p(x)$ be a polynomial with real or complex coefficients. Suppose that $p(x)$ has degree n.

1. (**Fundamental Theorem of Algebra**) The polynomial $p(x)$ can be factored into linear factors over the complex numbers, which means that $p(x)$ can be factored as

$$p(x) = c(x - r_1)(x - r_2)\cdots(x - r_n),$$

for some complex numbers $c, r_1, r_2, \ldots, r_n$ (these numbers might or might not be real numbers, and there might be repeats, corresponding to multiplicity higher than 1).

2. If $p(x)$ has real coefficients, and if r is a root of $p(x)$, the $\bar{r}$ is also a root of $p(x)$.

37. The Complex Numbers: Polar Form

Let $z = a + bi$ be a complex number. Suppose that (r, θ) are the polar coordinates of the point (a, b) in $\mathbb{R}^2$.

1. $a = r\cos\theta$ and $b = r\sin\theta$.
2. $r = \|z\| = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}$.
3. The **argument** of z is the angle θ.
4. The **polar form** of z is $z = r(\cos\theta + i\sin\theta)$.

38. The Complex Numbers: Multiplication and Division in Polar Form

Let $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$ be complex numbers, and let n be a positive integer.

1. $$zw = rs(\cos(\theta + \phi) + i\sin(\theta + \phi)).$$

2. $$\frac{z}{w} = \frac{r}{s}(\cos(\theta - \phi) + i\sin(\theta - \phi)).$$

3. $$z^n = r^n(\cos n\theta + i\sin n\theta).$$

4. The number z has n distinct n^{th} roots, which are given by

$$w_k = r^{\frac{1}{n}}(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\left(\frac{\theta + 2k\pi}{n}\right)$$

for each positive integer k such that $0 \le k \le n - 1$.

39. Euler's Formula

Let $a + bi$ be a complex number.

$$e^{a+bi} = e^a(\cos b + i \sin b).$$

40. Eigenvalues: Real and Complex

Let n be a positive integer. Let A be an $n \times n$ matrix with real entries.

1. The characteristic polynomial of A can be factored into linear factors over the complex numbers.
2. The matrix A has n complex eigenvalues counting multiplicities.
3. If λ is a complex eigenvalue of A, then $\bar{\lambda}$ is also an eigenvalue of A.
4. If λ is a complex eigenvalue of A, and if v is an eigenvector corresponding to λ, then $\bar{v}$ is an eigenvector corresponding to the eigenvalue $\bar{\lambda}$.

41. Systems of First Order Linear Differential Equations: Homogeneous Equations with Constant Coefficients

A system of first order homogeneous linear ordinary differential equations with constant coefficients can be written in the form

$$\begin{aligned} x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) \\ x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) \\ &\vdots \\ x_n'(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t), \end{aligned}$$

for some real numbers $a_{11}, a_{12}, \ldots, a_{nn}$.

1. Define the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

   Then the system of system of first order linear differential equations is equivalent to the matrix equation

$$x'(t) = Ax(t).$$

   Observe that the matrix A has entries that are numbers, not functions.

2. For convenience it is possible to write this matrix equation as $x' = Ax$, with the understanding that x is a function of t.

42. Systems of First Order Linear Differential Equations: Homogeneous Equations with Constant Coefficients via Eigenvalues

Let $x' = Ax$ be a system of homogeneous first order linear ordinary differential equations with constant coefficients.

1. If r is an eigenvalue of A with corresponding eigenvector v, then $x(t) = ve^{rt}$ is a solution of the system of ordinary differential equations $x' = Ax$.
2. If the matrix A is $n \times n$, and if A has n distinct real eigenvalues, then there are n linearly independent solutions, which together yield the general solution of the system of ordinary differential equations.

43. Dot Product

Let n be a positive integer. Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ be in $\mathbb{R}^n$. The **dot product** of x and y is defined by

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

44. Properties of the Dot Product

Let n be a positive integer. Let x, y and z be in $\mathbb{R}^n$, and let c be in $\mathbb{R}$.

1. $x \cdot y = y \cdot x$ (Symmetry Law).
2. $x \cdot (y + z) = x \cdot y + x \cdot z$ (Distributive Law).
3. $(cx) \cdot y = c(x \cdot y)$ (Homogeneity Law).
4. $x \cdot x \geq 0$, and $x \cdot x = 0$ if and only if $x = \mathbf{0}$ (Positive Definite Law).
5. $\|x\| = \sqrt{x \cdot x}$.
6. $\mathbf{0} \cdot x = 0 = x \cdot \mathbf{0}$.
7. $|x \cdot y| \leq \|x\| \cdot \|y\|$ (Cauchy-Schwarz Inequality).
8. $\|x + y\| \leq \|x\| + \|y\|$ (Triangle Inequality).

45. Polarization Identity in $\mathbb{R}^n$

Let n be a positive integer. Let x and y be in $\mathbb{R}^n$. Then

$$x \cdot y = \frac{1}{2}(\|x\|^2 + \|y\|^2 - \|x - y\|^2) \quad \text{(Polarization Identity)}.$$

46. Geometry of the Dot Product

Let n be a positive integer. Let x and y be in $\mathbb{R}^n$.

1. Let θ be the angle between x and y. Then $x \cdot y = \|x\|\|y\| \cos\theta$.
2. The vectors x and y are **orthogonal** if $x \cdot y = 0$.

47. Pythagorean Theorem in $\mathbb{R}^n$

Let n be a positive integer. Let x and y be in $\mathbb{R}^n$. Then

$$\|x\|^2 + \|y\|^2 = \|x - y\|^2 \quad \text{if and only if} \quad x \cdot y = 0.$$

48. Orthogonal Sets in $\mathbb{R}^n$

Let n be a positive integer. Let $v_1, \ldots, v_k$ be in $\mathbb{R}^n$.

1. The vectors $v_1, \ldots, v_k$ are **orthogonal** if $v_i \cdot v_j = 0$ for all values of i and j such that $i \neq j$.
2. The vectors $v_1, \ldots, v_k$ are **orthonormal** if they are orthogonal and if $\|v_i\| = 1$ for all i.

49. Orthogonality and Linear Independence

Let n be a positive integer.

1. Let $v_1, \ldots, v_k$ be in $\mathbb{R}^n$. If $v_1, \ldots, v_k$ are orthogonal, then they are linearly independent.
2. Let $v_1, \ldots, v_n$ be in $\mathbb{R}^n$. If $v_1, \ldots, v_n$ are orthogonal, then they are a basis for $\mathbb{R}^n$.

50. Orthogonal Basis in Subspaces of $\mathbb{R}^n$

Let n be a positive integer. Let W be a subspace of $\mathbb{R}^n$. Let $v_1, \ldots, v_k$ be in W.

1. The vectors $v_1, \ldots, v_k$ are an **orthogonal basis** of W if they are orthogonal and they are a basis for W.
2. The vectors $v_1, \ldots, v_k$ are an **orthogonal basis** of W if they are orthonormal and they are a basis for W.

51. Orthogonal Basis in $\mathbb{R}^n$

Let n be a positive integer. Let $v_1, \ldots, v_n$ be in $\mathbb{R}^n$.

1. The vectors $v_1, \ldots, v_n$ are an **orthogonal basis** of $\mathbb{R}^n$ if they are orthogonal and they are a basis for $\mathbb{R}^n$.
2. The vectors $v_1, \ldots, v_n$ are an **orthogonal basis** of $\mathbb{R}^n$ if they are orthonormal and they are a basis for $\mathbb{R}^n$.

52. Using Orthogonal Bases in Subspaces of $\mathbb{R}^n$

Let n be a positive integer. Let W be a subspace of $\mathbb{R}^n$. Let $v_1, \ldots, v_k$ be a basis for W. Let y be in W.

1. Suppose that $v_1, \ldots, v_k$ is an orthogonal basis for W. Then

$$y = \frac{y \cdot v_1}{\|v_1\|^2} v_1 + \frac{y \cdot v_2}{\|v_2\|^2} v_2 + \cdots + \frac{y \cdot v_k}{\|v_k\|^2} v_k.$$

2. Suppose that $v_1, \ldots, v_k$ is an orthonormal basis for W. Then

$$y = (y \cdot v_1)v_1 + (y \cdot v_2)v_2 + \cdots + (y \cdot v_k)v_k.$$

53. Using Orthogonal Bases in $\mathbb{R}^n$

Let n be a positive integer. Let $v_1, \ldots, v_n$ be a basis for $\mathbb{R}^n$. Let y be in $\mathbb{R}^n$.

1. Suppose that $v_1, \ldots, v_k$ are orthogonal. Then

$$y = \frac{y \cdot v_1}{\|v_1\|^2} v_1 + \frac{y \cdot v_2}{\|v_2\|^2} v_2 + \cdots + \frac{y \cdot v_n}{\|v_n\|^2} v_n.$$

2. Suppose that $v_1, \ldots, v_k$ are orthonormal. Then

$$y = (y \cdot v_1)v_1 + (y \cdot v_2)v_2 + \cdots + (y \cdot v_n)v_n.$$

54. Orthogonal Projection onto a Line

Let n be a positive integer. Let u and v be vectors in $\mathbb{R}^n$. Suppose that $v \neq \mathbf{0}$.

1. The **projection** of the vector u onto the line spanned by v, denoted $\text{proj}_v u$, is given by

$$\text{proj}_v u = \frac{u \cdot v}{\|v\|^2} v.$$

2. The vector $u - \text{proj}_v u$ is orthogonal to v.

3. The vector $\text{proj}_v u$ is the vector in the line spanned by v that is closest to u.

55. Orthogonal Projection onto a Subspace

Let n be a positive integer. Let W be a subspace of $\mathbb{R}^n$. Let $v_1, \ldots, v_k$ be an orthogonal basis for W. Let u be in $\mathbb{R}^n$.

1. The **projection** of the vector u onto the subspace W, denoted $\text{proj}_W u$, is given by

$$\text{proj}_W u = \frac{u \cdot v_1}{\|v_1\|^2} v_1 + \frac{u \cdot v_2}{\|v_2\|^2} v_2 + \cdots + \frac{u \cdot v_k}{\|v_k\|^2} v_k.$$

2. The vector $u - \text{proj}_W u$ is orthogonal to every vector in W.

3. The vector $\text{proj}_W u$ is the vector in W that is closest to u.

56. Gram-Schmidt in $\mathbb{R}^n$

Let n be a positive integer. Let $w_1, \ldots, w_k$ be in $\mathbb{R}^n$. Suppose that $w_1, \ldots, w_k$ are linearly independent.

Let $v_1, \ldots, v_k$ in $\mathbb{R}^n$ be defined by

$$\begin{aligned}
v_1 &= w_1 \\
v_2 &= w_2 - \frac{w_2 \cdot v_1}{\|v_1\|^2} v_1 \\
v_3 &= w_3 - \frac{w_3 \cdot v_1}{\|v_1\|^2} v_1 - \frac{w_3 \cdot v_2}{\|v_2\|^2} v_2 \\
&\vdots \\
v_k &= w_k - \frac{w_k \cdot v_1}{\|v_1\|^2} v_1 - \frac{w_k \cdot v_2}{\|v_2\|^2} v_2 - \cdots - \frac{w_k \cdot v_{k-1}}{\|v_{k-1}\|^2} v_{k-1}.
\end{aligned}$$

1. The vectors $v_1, \ldots, v_k$ are orthogonal.

2. None of the vectors $v_1, \ldots, v_k$ is $\mathbf{0}$.

3. $\text{span}\{v_1, \ldots, v_k\} = \text{span}\{w_1, \ldots, w_k\}$.

57. Vector Spaces

A **vector space** is a set V together with an operation called addition, and an operation called multiplication by scalars (which are real numbers), which satisfy the following eight properties. Let u, v and w be in V, and let s and t be in $\mathbb{R}$.

1. $u + v = v + u$ (Commutative Law).
2. $u + (v + w) = (u + v) + w$ (Associative Law).
3. $u + \mathbf{0} = u$ and $\mathbf{0} + u = u$ (Identity Law).
4. $u + (-u) = \mathbf{0}$ and $(-u) + u = \mathbf{0}$ (Inverses Law).
5. $s(u + v) = su + sv$ (Distributive Law).
6. $(s + t)u = su + tu$ (Distributive Law).
7. $s(tu) = (st)u$.
8. $1u = u$.

58. Vector Spaces: Subspaces

Let V be a vector space. Let W be a subset of V.

1. The subset W is **closed under addition** if u, v in W implies $u + v$ in W.
2. The subset W is **closed under scalar multiplication** if v in W and s in $\mathbb{R}$ imply sv in W.
3. The subset W is a **subspace** of V if
   (a) 0 is in W;
   (b) W is closed under addition;
   (c) W is closed under scalar multiplication.

59. Vector Spaces: Linear Combinations

Let V be a vector space. Let $v_1, \ldots, v_k$ be in V. A **linear combination** of vectors of $v_1, \ldots, v_k$ is any vector of the form

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$$

for some $a_1, a_2, \ldots, a_k$ in $\mathbb{R}$.

60. Vector Spaces: Span

Let V be a vector space. Let $v_1, \ldots, v_k$ be in V.

1. The **span** of $v_1, \ldots, v_k$, denoted $\text{span}\{v_1, \ldots, v_k\}$, is the set of all linear combinations of the vectors $v_1, \ldots, v_k$.
2. $\{v_1, \ldots, v_k\} \subseteq \text{span}\{v_1, \ldots, v_k\}$.
3. The subset $\text{span}\{v_1, \ldots, v_k\}$ is a subspace of V.

61. Vector Spaces: Linear Dependence and Linear Independence

Let V be a vector space. Let $v_1, \ldots, v_k$ be in V.

1. The vectors $v_1, \ldots, v_k$ are **linearly dependent** if there are $a_1, a_2, \ldots a_k$ in $\mathbb{R}$ that are not all 0, such that $a_1 v_1 + \ldots + a_k v_k = \mathbf{0}$.
2. The vectors $v_1, \ldots, v_k$ are **linearly independent** if they are not linearly dependent.

62. Vector Spaces: Bases

Let V be a vector space. Let S be a set of vectors in V.

1. The set of vectors S is a **basis** for V if they are linearly independent and they span V.
2. If S is a basis for V, then every vector v in V can be expressed uniquely as a linear combination of a finite collection of vectors in S.
3. If every vector v in V can be expressed uniquely as a linear combination of vectors in S, then S is a basis for V.
4. If V has a finite basis, then all bases of V are finite, and all bases of V have the same number of vectors.

63. Vector Spaces: Dimension

Let V be a vector space.

1. If V has a finite basis, then V is The **finite-dimensional**. If V does not have a finite basis, then V is **infinite-dimensional**.
2. If V is finite-dimensional, the **dimension** of V, denoted $\dim V$, is the number of vectors in any basis for V.

64. Vector Spaces: Dimension of Subspaces

Let V be a vector space. Let W be a subspace of V. Suppose V is finite-dimensional

1. W is finite-dimensional.
2. $\dim W \le \dim V$.
3. If $\dim W = \dim V$, then $W = V$.

65. Vector Spaces: Finding Bases when the Dimension is Known

Let V be a vector space. Suppose that V is finite-dimensional. Let $n = \dim V$. Let S be a set of vectors in V.

1. If S has fewer than n vectors, then S does not span V, and hence is not a basis for V.
2. If S has more than n vectors, then S is not linearly independent, and hence is not a basis for V.
3. If S has n vectors than it spans V if and only if it is linearly independent, and hence to prove that S is a basis requires proving only one of spanning and linear independence. *Warning: that only works when it is known already that* $\dim V = n$.

66. Linear Maps

Let V and W be vector spaces. Let $f : V \to W$ be a function. The function f is a **linear map** if it satisfies the following two properties. Let v and w be in V and let c be a real number.

1. $f(v + w) = f(v) + f(w)$.
2. $f(cv) = cf(v)$.

67. Vector Spaces: Image (also called Range) of a Linear Map

Let V and W be vector spaces. Let $f : V \to W$ be a linear map. The **image** of f (also called the **range** of f), denoted $\operatorname{im} f$, is the set of all vectors w in W such that $w = f(v)$ for some v in V.

68. Vector Spaces: Properties of Image of Linear Maps

Let V and W be vector spaces. Let $f : V \to W$ be a linear map.

1. If B is a basis for V, then $\operatorname{im} f = \operatorname{span} f(B)$
2. $\operatorname{im} f$ is a subspace of W.

69. Vector Spaces: Kernel of a Linear Map

Let V and W be vector spaces. Let $f : V \to W$ be a linear map. The **kernel** of f, denoted $\ker f$, is the set of all vectors v in V such that $f(v) = \mathbf{0}$.

70. Vector Spaces: Properties of Kernel of Linear Maps

Let V and W be vector spaces. Let $f\colon V \to W$ be a linear map. Then $\ker f$ is a subspace of V.

71. Vector Spaces Rank Nullity

Let V and W be vector spaces. Let $f\colon V \to W$ be a linear map. Then

$$\dim \operatorname{im} f + \dim \ker f = \dim V.$$

72. Inner Products

1. Let V be a vector space. An **inner product** on V is a operation, denoted $\langle v, w\rangle$, that assigns a real number to every pair of vectors v and w in V, and which satisfy the following four properties. Let x, y and z be in V, and let c be a real number.
   (1) $\langle x, y\rangle = \langle y, x\rangle$ (Symmetry Law).
   (2) $\langle x, (y + z)\rangle = \langle x, y\rangle + \langle x, z\rangle$ (Distributive Law).
   (3) $\langle cx, y\rangle = c\langle x, y\rangle$ (Homogeneity Law).
   (4) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ if and only if $x = \mathbf{0}$ (Positive Definite Law).
2. An **inner product space** is vector space together with an inner product on it.

73. Inner Products: Norm

Let V be an inner product space. Let x be in V. The **norm** of x, denoted $\|x\|$, is defined by

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

74. Inner Products: Orthogonality

Let V be an inner product space. Let x and y be in V. The vectors x and y are **orthogonal** if $\langle x, y\rangle = 0$.

75. Properties of Inner Products

Let V be an inner product space. Let x and y be in V, and let c be in $\mathbb{R}$.

1. $\langle \mathbf{0}, x\rangle = 0 = \langle x, \mathbf{0}\rangle$.
2. $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$ (Cauchy-Schwarz Inequality).
3. $\|x + y\| \leq \|x\| + \|y\|$ (Triangle Inequality).

76. Inner Products: Orthogonal Sets

Let V be an inner product space. Let $v_1, \ldots, v_k$ be in V.

1. The vectors $v_1, \ldots, v_k$ are **orthogonal** if $v_i \cdot v_j = 0$ for all values of i and j such that $i \neq j$.
2. The vectors $v_1, \ldots, v_k$ are **orthonormal** if they are orthogonal and if $\|v_i\| = 1$ for all i.

77. Inner Products: Orthogonal Basis

Let V be an inner product space. Let $v_1, \ldots, v_k$ be in V.

1. The vectors $v_1, \ldots, v_k$ are an **orthogonal basis** if they are orthogonal and they are a basis for V.
2. The vectors $v_1, \ldots, v_k$ are an **orthogonal basis** if they are orthonormal and they are a basis for V.

78. Inner Products: Orthogonal Projection onto a Line

Let V be an inner product space. Let u and v be vectors in V. Suppose that $v \neq \mathbf{0}$.

1. The **projection** of the vector u onto the line spanned by the vector v, denoted $\operatorname{proj}_v u$, is given by
$$\operatorname{proj}_v u = \frac{\langle u, v \rangle}{\|v\|^2} v.$$
2. The vector $u - \operatorname{proj}_v u$ is orthogonal to v.
3. The vector $\operatorname{proj}_v u$ is the vector in the subspace spanned by v that is closest to u.

79. Inner Products: Orthogonal Projection onto a Subspace

Let V be an inner product space. Let u and $v_1, \ldots, v_k$ be vectors in V. Suppose that $v_1, \ldots, v_k$ are orthogonal, and that none of the vectors $v_1, \ldots, v_k$ is $\mathbf{0}$. Let $S = \operatorname{span}\{v_1, \ldots, v_k\}$

1. The **projection** of the vector u onto the subspace S, denoted $\operatorname{proj}_S u$, is given by
$$\operatorname{proj}_S u = \frac{\langle u, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle u, v_2 \rangle}{\|v_2\|^2} v_2 + \cdots + \frac{\langle u, v_k \rangle}{\|v_k\|^2} v_k.$$
2. The vector $u - \operatorname{proj}_S u$ is orthogonal to every vector in S.
3. The vector $\operatorname{proj}_S u$ is the vector in S that is closest to u.

80. Inner Products: Using Orthogonal sets in $\mathbb{R}^n$

Let V be an inner product space. Let $v_1, \ldots, v_k$ be in V. Let y be in span$\{v_1, \ldots, v_k\}$.

1. Suppose that $v_1, \ldots, v_k$ are orthogonal, and that none of $v_1, \ldots, v_k$ is $\mathbf{0}$. Then

$$y = \frac{\langle y, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle y, v_2 \rangle}{\|v_2\|^2} v_2 + \cdots + \frac{\langle y, v_k \rangle}{\|v_k\|^2} v_k.$$

2. Suppose that $v_1, \ldots, v_k$ are orthonormal. Then

$$y = \langle y, v_1 \rangle v_1 + \langle y, v_2 \rangle v_2 + \cdots + \langle y, v_k \rangle v_k.$$

81. Inner Products: Orthogonal Basis

Let V be an inner product space. Let $v_1, \ldots, v_k$ be in V.

1. The vectors $v_1, \ldots, v_k$ are an **orthogonal basis** if they are orthogonal and they are a basis for V.

2. The vectors $v_1, \ldots, v_k$ are an **orthogonal basis** if they are orthonormal and they are a basis for V.

82. Inner Products: Using Orthogonal Bases

Let V be an inner product space. Let $v_1, \ldots, v_n$ be a basis for V. Let y be in V.

1. Suppose that $v_1, \ldots, v_k$ are orthogonal, and that none of $v_1, \ldots, v_k$ is $\mathbf{0}$. Then

$$y = \frac{\langle y, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle y, v_2 \rangle}{\|v_2\|^2} v_2 + \cdots + \frac{\langle y, v_n \rangle}{\|v_n\|^2} v_n.$$

2. Suppose that $v_1, \ldots, v_k$ are orthonormal. Then

$$y = \langle y, v_1 \rangle v_1 + \langle y, v_2 \rangle v_2 + \cdots + \langle y, v_n \rangle v_n.$$

83. Inner Products: Gram-Schmidt

Let V be an inner product space. Let $w_1, \ldots, w_k$ be in V. Suppose that $w_1, \ldots, w_k$ are linearly independent.

Let $v_1, \ldots, v_k$ in V be defined by

$$\begin{aligned}
v_1 &= w_1 \\
v_2 &= w_2 - \frac{\langle w_2, v_1\rangle}{|v_1|^2} v_1 \\
v_3 &= w_3 - \frac{\langle w_3, v_1\rangle}{|v_1|^2} v_1 - \frac{\langle w_3, v_2\rangle}{|v_2|^2} v_2 \\
&\vdots \\
v_k &= w_k - \frac{\langle w_k, v_1\rangle}{|v_1|^2} v_1 - \frac{\langle w_k, v_2\rangle}{|v_2|^2} v_2 - \cdots - \frac{\langle w_k, v_{k-1}\rangle}{|v_{k-1}|^2} v_{k-1}.
\end{aligned}$$

1. The vectors $v_1, \ldots, v_k$ are orthogonal.

2. None of the vectors $v_1, \ldots, v_k$ is $\mathbf{0}$.

3. $\operatorname{span}\{v_1, \ldots, v_k\} = \operatorname{span}\{w_1, \ldots, w_k\}$.

84. Fourier Approximations

Let f be a function that is integrable on the interval $[-\pi, \pi]$. Let n be a positive integer. The n^{th} **Fourier approximation** of f, denoted f_n, is the function

$$f_n(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots + a_n \cos nx + b_n \sin nx,$$

where

$$\begin{aligned}
a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \\
a_1 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos x\, dx \quad \text{and} \quad b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin x\, dx \\
a_2 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos 2x\, dx \quad \text{and} \quad b_2 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin 2x\, dx \\
&\vdots \\
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.
\end{aligned}$$