MATH 242 Elementary Linear Algebra Spring 2018

Study Sheet for Midterm Exam

- This exam will be closed book.
- This study sheet will not be allowed during the test.
- Books, notes and online resources will not be allowed during the test.
- Electronic devices (calculators, cell phones, tablets, laptops, etc.) will not be allowed during the test.

Topics

1. Solving systems of linear equations
2. Echelon form and reduced echelon form
3. Vectors in $\mathbb{R}^n$
4. Matrix operations
5. Matrix inverses
6. Markov chains
7. Linear maps via matrix multiplication
8. Subspaces of $\mathbb{R}^n$
9. Linear combinations and span in $\mathbb{R}^n$
10. Image and kernel

Tips for Studying for a Quiz or Exam

× **Bad**		Forgetting about the homework and the previous quizzes.
✓ **Good**		Making sure you know how to do all the problems on the homework and previous quizzes; seeking help seeking help from the instructor and the tutors about the problems you do not know how to do.
× **Bad**		Doing all the practice problems from some of the sections, and not having enough time to do practice problems from the rest of the sections.
✓ **Good**		Doing a few practice problems of each type from every sections.
× **Bad**		Studying only by reading the book.
✓ **Good**		Doing a lot of practice problems, and reading the book as needed.
× **Bad**		Studying only by yourself.
✓ **Good**		Trying some practice problems by yourself (or with friends), and then seeking help from the instructor and the tutors about the problems you do not know how to do.
× **Bad**		Doing practice problems while looking everything up in the book.
✓ **Good**		Doing some of the practice problems the way you would do them on the quiz or exam, which is with closed book and no calculator.
× **Bad**		Staying up late (or all night) the night before the exam.
✓ **Good**		Studying hard up through the day before the exam, but getting a good night's sleep the night before the exam.

Ethan's Office Hours

- **Monday:** 4:00-5:30
- **Wednesday:** 2:00-3:30
- **Thursday:** 10:30-12:00
- **Or by appointment**

Tutor

- **Eric Zhang:**
  - Office Hour: Monday, 6:00-7:00, Mathematics Common Room (third floor of Albee)
  - Email: jz2226@bard.edu

Practice Problems from Holt, 2nd ed.

Section 1.1: 23, 25, 27, 29, 31, 33, 35, 37, 43, 45, 49, 57, 65

Section 1.2: 19, 21, 23, 25, 27, 29, 31

Section 1.3: 1, 3, 13, 15, 17, 19

Section 2.1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 47, 49, 51, 53

Section 2.2: 1, 3, 5, 7, 9, 11, 35, 37, 39, 41, 43, 45

Section 3.1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 33, 35, 37

Section 3.2: 1, 3, 5, 7, 9, 11, 13, 43, 45, 47, 49

Section 3.3: 1, 3, 5, 7, 9, 11, 13, 15, 35, 37, 39

Section 3.5: 1, 3, 5, 7, 13, 15, 17, 19, 21, 23 47, 49

Section 4.1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 33, 35, 37, 41, 43

Some Important Concepts and Formulas

1. Solving Systems of Equations in General

1. A **solution** to an equation or system of equations is a set of numerical values for the unknowns that, when plugged into the equation or equations yields true statements.
2. A system of equations is **consistent** if it has at least one solution, and is **inconsistent** if it has none.
3. If a system of equations has one or more solutions, the **solution set** is the collection of all solutions. If a system of equations has no solution, then the solution set is thought of as the empty set.
4. To **solve** a system of equations means to find the solution set.

2. Systems of Linear Equations

1. Let m and n be positive integers. A **system of m linear equations in n unknowns** is a system of equations with unknowns $x_1, x_2, \ldots, x_n$ that can be written in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

   for some $a_{11}, a_{12}, \ldots, a_{mn}$ in $\mathbb{R}$ and $b_1, b_2, \ldots, b_m$ in $\mathbb{R}$.

2. A system of m linear equations in n unknowns is **homogeneous** if it has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

3. Leading Unknown

Let m and n be positive integers. In a system of m linear equations in n unknowns, a **leading unknown** is an unknown that has the first non-zero coefficient (from the left) in at least one of the linear equations.

4. Triangular System

Let n be a positive integer. A system of n linear equations in n unknowns is a **triangular system** (and is in **triangular form**) if it has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{33}x_3 + \cdots + a_{2n}x_n &= b_3 \\ &\vdots \\ a_{nn}x_n &= b_n, \end{aligned}$$

for some $a_{11}, a_{12}, \ldots, a_{nn}$ in $\mathbb{R}$ and $b_1, b_2, \ldots, b_n$ in $\mathbb{R}$.

1. In a triangular system of linear equations, there are the same number of equations as unknowns.
2. In a triangular system of linear equations, every unknown is a leading unknown in exactly one row.
3. A triangular system of linear equations has a unique solution.

5. Echelon System

Let m and n be positive integers. A system of m linear equations in n unknowns is an **echelon system** (and is in **echelon form**) if the following two properties hold:

1. Each equation that has all zero coefficients for the unknowns is below every equation with at least one non-zero coefficient for an unknown.
2. The leading unknown of each equation with at least one non-zero coefficient lies to the right of the leading unknown of the preceding equation (if there is a preceding equation).

6. Free Unknown

Let m and n be positive integers. In an echelon system of m linear equations in n unknowns, a **free unknown** is an unknown that is not a leading unknown in any equation.

7. Systematic Solution of a System of Linear Equations: Preliminary

1. Let m and n be positive integers. An $m \times n$ **matrix** with entries in $\mathbb{R}$ is a rectangular array of real numbers with m rows and n columns.

2. Given a system of m linear equations in n unknowns of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

the **coefficient matrix** of the system of linear equations is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

Let $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$. The **augmented coefficient matrix** of the system of linear equations is

$$\left[A \mid b\right] = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right].$$

3. The augmented coefficient contains all the information needed to solve the system of linear equations.

8. Systematic Solution of a System of Linear Equations: Part 1 – Gaussian Elimination

To solve a system of linear equations, start with the following steps.

1. Form the augmented coefficient matrix $\left[A \mid b\right]$.

2. The first part of the process, called **Gaussian elimination** (or **Gauss-Jordan elimination** if the steps in parentheses are used), occurs in the the augmented coefficient matrix.

   1. Find the first column of the augmented coefficient matrix (reading from the left) that has a non-zero entry.

   2. If the top entry in this column is zero, place a non-zero element there by interchanging appropriate rows.

   3. Make this non-zero entry into 1 by multiplying the row containing the entry by an appropriate number.

   4. Make all the entries below this non-zero entry into zero by adding appropriate multiples of the row containing this non-zero entry to these other rows.

   5. (For Gauss-Jordan elimination, make all the entries above this non-zero entry into zero by adding appropriate multiples of the top row to these other rows.)

   6. The top row is now complete, and will not be modified further. Continue the above process to the rows below the top row, one row at a time, starting from the top.

   7. Keep going until partially reduced echelon form (or reduced echelon form for Gauss-Jordan elimination) is achieved.

3. If there is a row in the matrix in partially reduced echelon form (or reduced echelon form) that has all zeros except for the last entry, which is not zero, then there is no solution. Otherwise there is a solution (one or infinitely many, to be determined).

4. If there is a solution, convert the matrix in partially reduced echelon form (or reduced echelon form) back into equations.

5. The unknowns in the new equations that correspond to leading entries are **leading unknowns**. The other unknowns are **free unknowns**.

6. If there are no free unknowns, then there is one solution. If there are free unknowns, there are infinitely many solutions, with as many parameters as there are free unknowns.

9. Systematic Solution of a System of Linear Equations: Part 2 – Back Substitution

To solve a system of linear equations, finish with the following steps.

1. The next part of the process, called **back substitution**, occurs with these new equations.
   1. Set each free unknown equal to a parameter (each free unknown is a different parameter).
   2. Solve the final non-zero equation for its lead unknown (in terms of the parameters, if there are any in that equation).
   3. Substitute that solution into the equation above it, and solve for its lead unknown.
   4. Keep going upwards until all free unknowns have been found.
2. The type of solution is evaluated as follows.
   1. If there is a row with all zeros except the last entry is not zero, then there is no solution.
   2. If there is a solution, and there are some free unknowns, then there are infinitely many solutions.
   3. If there is a solution, and every unknown is a leading unknown, then there is a unique solution.

10. Vectors in $\mathbb{R}^n$

Let n be a positive integer.

1. The set $\mathbb{R}^n$ is the set of all column vectors of length n with entries in $\mathbb{R}$.
2. Let v in $\mathbb{R}^n$. Then $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$.
3. The **zero vector** in $\mathbb{R}^n$ is the vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.
4. The **length** (also known as **norm**) of a vector $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, denoted $\|v\|$, is defined by

$$\|v\| = \sqrt{(v_1)^2 + (v_2)^2 + \cdots + (v_n)^2}.$$

11. Vectors in $\mathbb{R}^n$: Properties of Length

Let n be a positive integer. Let v be in $\mathbb{R}^n$, and let c be a real number.

1. $\|v\| \geq 0$.
2. $\|v\| = 0$ if and only if $v = 0$.
3. $\|cv\| = |c| \cdot \|v\|$.

12. Vectors in $\mathbb{R}^n$: Addition and Scalar Multiplication

Let n be a positive integer. Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ be in $\mathbb{R}^n$, and let c be a real number.

1. $x + y = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$.

2. $x - y = \begin{bmatrix} x_1-y_1 \\ x_2-y_2 \\ \vdots \\ x_n-y_n \end{bmatrix}$.

3. $-x = \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}$.

4. $cx = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$.

13. Vectors in $\mathbb{R}^n$: Properties of Addition and Scalar Multiplication

Let n be a positive integer. Let u, v and w be in $\mathbb{R}^n$, and let s and t be a real number.

1. $u + v = v + u$ (Commutative Law).

2. $u + (v + w) = (u + v) + w$ (Associative Law).

3. $u + \mathbf{0} = u$ and $\mathbf{0} + u = u$ (Identity Law).

4. $u + (-u) = \mathbf{0}$ and $(-u) + u = \mathbf{0}$ (Inverses Law).

5. $s(u + v) = su + sv$ (Distributive Law).

6. $(s + t)u = su + tu$ (Distributive Law).

7. $s(tu) = (st)u$.

8. $1u = u$.

14. Vectors in $\mathbb{R}^n$: Unit Vectors

Let n be a positive integer. If v is a vector in $\mathbb{R}^n$, and if $v \neq \mathbf{0}$, the **unit vector** in the direction of v is $u = \dfrac{1}{\|v\|} v$.

15. Matrices

Let m and n be positive integers. Let A be an $m \times n$ matrix.

1. The matrix A can be written $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$.

2. The **transpose** of A is the $n \times m$ matrix $A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$.

3. The matrix A is **symmetric** if $A^T = A$.

4. The $m \times n$ **zero matrix**, denoted O, is the matrix with all zero entries.

5. The $n \times n$ **identity matrix**, denoted I_n, is the matrix $I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$.

16. Matrices: Addition and Scalar Multiplication

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$, and let c be a real number.

1. $A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$.

2. $A - B = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \cdots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & \cdots & a_{2n}-b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & \cdots & a_{mn}-b_{mn} \end{bmatrix}$.

3. $-A = \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{m1} & -a_{m2} & \cdots & -a_{mn} \end{bmatrix}$.

4. $cA = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$.

17. Matrices: Multiplication

Row times Column

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots a_n b_n.$$

General If A is an $m \times p$ matrix and B is a $p \times n$ matrix, then AB is an $m \times n$ matrix obtained by multiplying each row in A by each column in B.

18. Matrices and Systems of Linear Equations

Let m and n be positive integers. A system of m linear equations in n unknowns can be written in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Define an $m \times n$ matrix A, an $m \times 1$ matrix B and and $n \times 1$ matrix X by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Then the system of linear equations is equivalent to the matrix equation $AX = B$. Solving for the vector X is equivalent to solving for the unknowns $x_1, \dots, x_n$.

19. Matrices: Properties of Addition and Scalar Multiplication

Let m and n be positive integers. Let A, B and C be $m \times n$ matrices, and let s and t be real numbers. Let O be the $m \times n$ zero matrix.

1. $A + B = B + A$ (Commutative Law).

2. $A + (B + C) = (A + B) + C$ (Associative Law).

3. $A + O = A$ and $A + O = A$ (Identity Law).

4. $A + (-A) = O$ and $(-A) + A = O$ (Inverses Law).

5. $s(A + B) = sA + sB$ (Distributive Law).

6. $(s + t)A = sA + tA$ (Distributive Law).

7. $s(tA) = (st)A$.

8. $1A = A$.

20. Matrices: Properties of Multiplication

Let m, n, p and q be positive integers. Let A and P be $m \times n$ matrices, let B and Q be $n \times p$ matrices and let C be a $p \times q$ matrix.

1. $A(BC) = (AB)C$.

2. $AI_n = A$ and $I_m A = A$.

3. $A(B + Q) = AB + AQ$ and $(A + P)B = AB + PB$.

21. Matrices: Properties of Powers

Let n be a positive integer. Let A be an $n \times n$ matrix, and let r and s be non-negative integers.

1. $A^r A^s = A^{r+s}$.
2. $(A^r)^s = A^{rs}$.

22. Matrices: Properties of the Transpose

Let m and n be positive integers. Let A and B be $m \times n$ matrices, and let s be a real number.

1. $(A + B)^T = A^T + B^T$.
2. $(sA)^T = sA^T$.
3. $(A^T)^T = A$.
4. $(I_n)^T = I_n$.
5. $(AB)^T = B^T A^T$.

23. Matrices: Inverses

Let n be a positive integer. Let A be an $n \times n$ matrix. The matrix A is **invertible** (also called **nonsingular**) if there is some $n \times n$ matrix C such that $CA = I_n$ and $AC = I_n$. Such a matrix C, if it exists, is unique, and is called the **inverse** of A, and is denoted A^{-1}.

24. Matrices: Inverses of 2×2 Matrices

Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$.

1. The matrix A is invertible if and only if $ad - bc \neq 0$.
2. If A is invertible, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

25. Matrices: Properties of Inverses

Let n be a positive integer. Let A and B be $n \times n$ matrices.

1. If A is invertible, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.
2. If A and B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
3. If A is invertible, and if p is a non-negative integer, then A^p is invertible, and $(A^p)^{-1} = (A^{-1})^p$.
4. A is invertible if and only if A^T is invertible; if A is invertible, then $(A^T)^{-1} = (A^{-1})^T$.

26. Finding Matrix Inverses

Let n be a positive integer, and let A be an $n \times n$ matrix.

1. The matrix A is invertible if and only if A is row equivalent to I_n.
2. Suppose that A is invertible. The augmented matrix $[A|I_n]$ is row equivalent to $[I_n|A^{-1}]$. To find A^{-1}, start with $[A|I_n]$, and perform elementary row operations on it until the left side is I_n.

27. Elementary Matrices

An **elementary matrix** is the result of doing a single elementary row operation to the identity matrix.

28. Elementary Matrix Multiplication

Let m and n be positive integers. Let E by the $m \times m$ elementary matrix obtained by a some elementary row operation. Let A be an $m \times n$ matrix. Then EA is the same as doing that elementary row operation to A.

29. Application of Invertible Matrices

Let n be a positive integer, and let A be an $n \times n$ matrix. The following are equivalent.

(a) A is invertible.

(b) A is row equivalent to I_n.

(c) $Ax = \mathbf{0}$ has only the trivial (all zero) solution.

(d) For every vector b in $\mathbb{R}^n$, the system $Ax = b$ has a unique solution.

(e) For every vector b in $\mathbb{R}^n$, the system $Ax = b$ is consistent.

30. Probability Vector and Stochastic Matrix

1. A vector is a **probability vector** (also called a **stochastic vector**) if none of the values in the vector is negative, and the sum of the values in the vector is 1.
2. A matrix is a **stochastic matrix** if none of the values in the matrix is negative, and the sum of the values of each column in the matrix is 1.

31. Stochastic Matrices: Properties

Let n be a positive integer. Let A and B be $n \times n$ stochastic matrices.

1. If v is a probability vector, then Av is a probability vector.
2. AB is a stochastic matrix.
3. If p is a non-negative integer, then A^p is stochastic matrix.

32. Markov Chain

Let n be a positive integer. Let A be an $n \times n$ stochastic matrix. Let $\mathbf{x}_0$ be a probability vector in $\mathbb{R}^n$.

1. The **Markov chain** generated by A and $\mathbf{x}_0$ is the sequence of vectors

$$\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots ,$$

   where

$$\begin{aligned} \mathbf{x}_1 &= A\mathbf{x}_0 \\ \mathbf{x}_2 &= A\mathbf{x}_1 = A^2\mathbf{x}_0 \\ \mathbf{x}_3 &= A\mathbf{x}_2 = A^3\mathbf{x}_0 \\ &\vdots \; . \end{aligned}$$

2. The vector $\mathbf{x}_0$ is called the **initial state vector** of the Markov chain, and the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$ are called **state vectors** of the Markov chain.
3. If the vectors $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$ converge to a vector $\mathbf{x}$ in $\mathbb{R}^n$, meaning that the vectors in the sequence get closer and closer to $\mathbf{x}$, then the vector $\mathbf{x}$ is called the **steady-state vector** of the Markov chain.
4. Not all Markov chains have steady-state vectors.

33. Regular Stochastic Matrices

Let n be a positive integer. Let A be an $n \times n$ stochastic matrix. The stochastic matrix A is **regular** if there is some positive integer k such that A^k has all positive entries.

34. Regular Stochastic Matrices: Properties

Let n be a positive integer. Let A be an $n \times n$ regular stochastic matrix.

1. For any initial state vector $\mathbf{x}_0$, the Markov chain $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to a steady-state vector.
2. For all initial state vectors, the Markov chains converge to the same steady-state vector.
3. Let $\mathbf{x}$ be the steady-state vector resulting from all initial state vectors. Then $A\mathbf{x} = \mathbf{x}$.
4. The sequence of matrices $A, A^2, A^3, \ldots$ converges to the matrix $[\mathbf{x}\ \mathbf{x}\ \cdots\ \mathbf{x}]$.

35. Regular Stochastic Matrices: Finding the Steady-State Vector

Let n be a positive integer. Let A be an $n \times n$ regular stochastic matrix. Find the steady-state vector $\mathbf{x}$ as follows.

1. Find the general solution of the system of linear equations $(A - I_n)v = \mathbf{0}$, which will have a parameter s.
2. Add up the entries in the general solution and set the sum equal to 1, and then solve for s
3. Use that value of s to find specific values for the entries in the general solution. These specific values form a probability vector.

36. Linear Maps $\mathbb{R}^n \to \mathbb{R}^m$

Let n and m be positive integers. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function. The function f is a **linear map** if it satisfies the following two properties. Let v and w be in $\mathbb{R}^n$ and let c be a real number.

1. $f(v + w) = f(v) + f(w)$.
2. $f(cv) = cf(v)$.

37. Linear Maps $\mathbb{R}^n \to \mathbb{R}^m$: Properties

Let n and m be positive integers. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Let v be in $\mathbb{R}^n$.

1. $f(\mathbf{0}) = \mathbf{0}$.
2. $f(-v) = -f(v)$.

38. Functions $\mathbb{R}^n \to \mathbb{R}^m$ Given by Matrix Multiplication

Let n and m be positive integers. Let A be an $m \times n$ matrix. The **function induced** by A is the function $\mathsf{L}_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $\mathsf{L}_A(v) = Av$ for all v in $\mathbb{R}^n$.

39. Functions $\mathbb{R}^n \to \mathbb{R}^m$ Given by Matrix Multiplication are Linear

Let n and m be positive integers. Let A be an $m \times n$ matrix. The function L_A is a linear map.

40. Linear Maps $\mathbb{R}^n \to \mathbb{R}^m$ are Given by Matrix Multiplication

Let n and m be positive integers. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$. Then $f = \mathsf{L}_A$, where A is the $m \times n$ matrix given by $A = [f(e_1)\ f(e_2)\ \ldots\ f(e_n)]$.

41. Subspaces of $\mathbb{R}^n$

Let n be a positive integer. Let W be a non-empty subset of $\mathbb{R}^n$.

1. The subset W is **closed under addition** if u, v in W implies $u + v$ in W.

2. The subset W is **closed under scalar multiplication** if v in W and s in $\mathbb{R}$ imply sv in W.

3. The subset W is a **subspace** of $\mathbb{R}^n$ if

(a) 0 is in W;

(b) W is closed under addition;

(c) W is closed under scalar multiplication.

42. Linear Combinations

Let n be a positive integer. Let $v_1, \ldots, v_k$ be in $\mathbb{R}^n$. A **linear combination** of vectors of $v_1, \ldots, v_k$ is any vector of the form

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$$

for some $a_1, a_2, \ldots, a_k$ in $\mathbb{R}$.

43. Span

Let n be a positive integer. Let $v_1, \ldots, v_k$ be in $\mathbb{R}^n$.

1. The **span** of $v_1, \ldots, v_k$, denoted $\text{span}\{v_1, \ldots, v_k\}$, is the set of all linear combinations of the vectors $v_1, \ldots, v_k$.

2. $\{v_1, \ldots, v_k\} \subseteq \text{span}\{v_1, \ldots, v_k\}$.

3. The subset $\text{span}\{v_1, \ldots, v_k\}$ is a subspace of $\mathbb{R}^n$.

44. Span and Systems of Linear Equations

Let n be a positive integer. Let $v_1, \ldots, v_k$ and w be in $\mathbb{R}^n$. Let $A = [v_1\ v_2\ \cdots\ v_k]$. The following are equivalent.

(a) The vector w is in $\text{span}\{v_1, \ldots, v_k\}$.

(b) The system of linear equations $x_1 v_1 + \cdots x_k v_k = w$ has a solution.

(c) The matrix equation $Ax = w$ has a solution.

45. Spanning $\mathbb{R}^n$

Let n be a positive integer. Let $v_1, \ldots, v_k$ be in $\mathbb{R}^n$.

1. The vectors $v_1, \ldots, v_k$ **span** $\mathbb{R}^n$ if $\operatorname{span}\{v_1, \ldots, v_k\} = \mathbb{R}^n$.
2. If $k < n$, then $v_1, \ldots, v_k$ do not span $\mathbb{R}^n$.
3. If $k \geq n$, then $v_1, \ldots, v_k$ might or might not span $\mathbb{R}^n$.
4. Suppose $k \geq n$. Let $A = [v_1 \; v_2 \; \cdots \; v_k]$. Suppose that A is row equivalent to B, where B is in echelon form. Then $v_1, \ldots, v_k$ span $\mathbb{R}^n$ if and only if B has a pivot position in every row.

46. Image (also called Range) of a Linear Map

Let m and n be positive integers. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. The **image** of f (also called the **range** of f), denoted $\operatorname{im} f$, is the set of all vectors w in $\mathbb{R}^m$ such that $w = f(v)$ for some v in $\mathbb{R}^n$.

47. Image of a Linear Map Given by Matrix Multiplication

Let m and n be positive integers. Let A be an $m \times n$ matrix.

1. $\operatorname{im} \mathsf{L}_A$ is the set of vectors w in $\mathbb{R}^m$ for which $Ax = w$ has a solution.
2. If $A = [v_1 \; v_2 \; \cdots \; v_k]$, then $\operatorname{im} \mathsf{L}_A = \operatorname{span}\{v_1, \ldots, v_k\}$
3. $\operatorname{im} \mathsf{L}_A$ is a subspace of $\mathbb{R}^m$.

48. Kernel of a Linear Map

Let m and n be positive integers. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. The **kernel** of f, denoted $\ker f$, is the set of all vectors v in $\mathbb{R}^n$ such that $f(v) = \mathbf{0}$.

49. Kernel of a Linear Map Given by Matrix Multiplication

Let m and n be positive integers. Let A be an $m \times n$ matrix.

1. $\ker \mathsf{L}_A$ is the set of all solutions of $Ax = \mathbf{0}$.
2. $\ker \mathsf{L}_A$ is a subspace of $\mathbb{R}^n$.