MATH 245 Intermediate Calculus Fall 2018
Study Sheet for Exam #2

GUIDELINES THAT WILL BE ON THE EXAM

EXAM GUIDELINES

In order for this exam to be an honest and accurate reflection of your understanding of the material, you are asked to adhere to the following guidelines:

- The exam is closed book.
- The study sheet is not allowed during the exam.
- Books, notes and online resources are not allowed during the exam.
- Electronic devices (calculators, cell phones, tablets, laptops, etc.) are not allowed during the exam.
- For the duration of the exam, you may not discuss the exam, or related material, with anyone other than the course instructor.
- Giving help to others taking this exam is as much a violation of these guidelines as receiving help.
- Late exams will be allowed only if you discuss it with the course instructor before hand, or if an emergency occurs.
- **Violation of these guidelines will result, at minimum, in a score of zero on this exam.**
- **There will be no opportunity to retake this exam.**

Further comments:

- Write your solutions carefully and clearly.
- Show all your work. You will receive partial credit for work you show, but you will not receive credit for what you do not write down. In particular, correct answers with no work will not receive credit.

TOPICS

1. Ordinary differential equations—basics
2. Separable ODEs
3. Applications of separable ODEs
4. First order linear ODEs
5. Euler's method
6. Complex numbers
7. Second order homogeneous linear ODEs with constant coefficients
8. Second order non-homogeneous linear ODEs with constant coefficients, undetermined coefficients
9. Springs

TIPS FOR STUDYING FOR THE EXAM

×	**Bad**	Forgetting about the homework and the previous quizzes.
✓	**Good**	Making sure you know how to do all the problems on the homework and previous quizzes; seeking help seeking help from the instructor and the tutors about the problems you do not know how to do.
×	**Bad**	Doing all the practice problems from some of the sections, and not having enough time to do practice problems from the rest of the sections.
✓	**Good**	Doing a few practice problems of each type from every sections.
×	**Bad**	Studying only by reading the book.
✓	**Good**	Doing a lot of practice problems, and reading the book as needed.
×	**Bad**	Studying only by yourself.
✓	**Good**	Trying some practice problems by yourself (or with friends), and then seeking help from the instructor and the tutors about the problems you do not know how to do.
×	**Bad**	Doing practice problems while looking everything up in the book.
✓	**Good**	Doing some of the practice problems the way you would do them on the quiz or exam, which is with closed book and no calculator.
×	**Bad**	Staying up late (or all night) the night before the exam.
✓	**Good**	Studying hard up through the day before the exam, but getting a good night's sleep the night before the exam.

Ethan's Office Hours

- **Monday:** 4:30-6:00
- **Tuesday:** 5:00-6:00
- **Wednesday:** 2:00-3:30
- **Or by appointment**

Tutor

- Weronica Nguyen
- **Office hours**: Wednesday: 6:00-7:00, Mathematics Common Room (third floor of Albee)
- **Email to Make an Appointment**: tn3599 "at" bard "dot" edu.

PRACTICE PROBLEMS FROM STEWART, CALCULUS CONCEPTS AND CONTEXTS, 4TH ED.

Section 7.2: 19ac, 21, 23

Section 7.3: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 43

Section 7.4: 1, 3, 5, 9, 11, 13, 15

Appendix I: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 41, 43, 45

First Order Linear Handout: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 27

Homogeneous Second Order Linear Handout: 1, 3, 5, 7, 9, 11, 13, 17, 19, 21, 23

Non-Homogeneous Second Order Linear Handout: 1, 3, 5, 7, 9, 13, 15, 17

Applications of Second Order ODEs Handout: 1, 3

SOME IMPORTANT CONCEPTS AND FORMULAS

1. Ordinary Differential Equations

1. An **ordinary differential equation**, abbreviated ODE, is an equation that involves the derivatives of single variable functions, all of which have the same variable.
2. A **partial differential equation**, abbreviated PDE, is an equation that involves partial derivatives of multi-variable functions.
3. A differential equation has **order** n if the highest derivative in the differential equation is an n^{th} derivative.

2. General Solutions, Particular Solutions and Initial Values

1. A **particular solution** of an ODE is a solution that does not involve any constants.
2. A **general solution** of an ODE is a solution with constants.
3. The general solution of an n^{th} order ODE will usually have n constants.
4. A **singular solution** of an ODE is a particular solution that cannot be obtained from the general solution by using any possible numerical values of the constants in the general solution.
5. An **equilibrium solution** of an ODE is a solution that is a constant function.
6. **Initial conditions** of an ODE are a collections of conditions given by values of the function and/or its derivatives at specific inputs.
7. An **initial value problem** is an ODE together with initial conditions.

3. Separable Ordinary Differential Equations

1. A **separable** ODE is an ODE that can be brought into the form $\frac{dy}{dx} = g(x)h(y)$, for some functions $g(x)$ and $h(y)$.
2. A separable ODE can be solved by bringing all instances of one variable to the left side of the equality, and all instances of the other variable to the right side of the equality, and then integrating both sides.

4. Applications of Separable Differential Equations

Exponential Growth of a Population

Let $P(t)$ be a population at time t.

$$\frac{dP}{dt} = kP,$$

where $k > 0$.

Radioactive Decay

Let $x(t)$ be the amount of a radioactive material at time t.

$$\frac{dx}{dt} = kx,$$

where $k < 0$.

Newton's Law of Cooling

Let $T(t)$ be the temperature of an object at time t, and let A be the ambient temperature.

$$\frac{dT}{dt} = -k(T - A),$$

where $k > 0$.

5. First Order Linear Differential Equations

1. A **first order linear differential equation** is an ODE of the form

$$\frac{dy}{dx} + P(x)y = Q(x),$$

for some functions $P(x)$ and $Q(x)$.

2. To solve a first order linear ODE, multiply it by the **integrating factor**

$$\rho(x) = e^{\int P(x)\,dx},$$

and obtain

$$\frac{d}{dx}[y\rho(x)] = Q(x)\rho(x),$$

which can be solved by integrating both sides of the equation.

6. Euler's Method

We are given a first order ordinary differential equation $y' = F(x, y)$ and an initial condition $y(x_0) = y_0$. **Euler's method**, which approximates the value of the solution to the ordinary differential equation, is as follows. Choose a **step size** h, which is a positive number.

1. Form a sequence of number $x_1, x_2, x_3, \ldots$ by letting

$$x_n = x_{n-1} + h$$

   for all natural numbers n.

2. Form a sequence of number $y_1, y_2, y_3, \ldots$ by letting

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

   for all natural numbers n.

3. If $y(x)$ is the solution to the ordinary differential equation, then y_n is an approximation to $y(x_n)$, where the approximation is better if h and/or n are small.

7. The Complex Numbers

1. The number i is the number such that $i^2 = -1$.
2. A **complex number** is any number of the form $a + bi$, where a, b in $\mathbb{R}$.
3. The set of all complex numbers is denoted $\mathbb{C}$.
4. Let $z = a + bi$. The **real part** of z is a, and the **imaginary part** of z is b.
5. The **modulus** of z, denoted $\|z\|$, is defined by $\|z\| = \sqrt{a^2 + b^2}$.
6. The **complex conjugate** of z, denoted $\bar{z}$, is defined by $\bar{z} = a - bi$.

8. The Complex Numbers: Basic Operations

Let $z = a + bi$ and $w = c + di$ be complex numbers, and let t be a real number.

1. $z + w = (a + c) + (b + d)i$.
2. $z - w = (a - c) + (b - d)i$.
3. $zw = (ab - bd) + (ad + bc)i$.
4. $\frac{z}{w} = \frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ab+bd)+(bc-ad)i}{c^2+d^2}$.
5. $tz = ta + tbi$.

9. The Complex Numbers: Roots of Polynomials

Let $p(x)$ be a polynomial with real or complex coefficients. Suppose that $p(x)$ has degree n.

1. (**Fundamental Theorem of Algebra**) The polynomial $p(x)$ can be factored into linear factors over the complex numbers, which means that $p(x)$ can be factored as

$$p(x) = c(x - r_1)(x - r_2)\cdots(x - r_n),$$

for some complex numbers $c, r_1, r_2, \ldots, r_n$ (these numbers might or might not be real numbers, and there might be repeats, corresponding to multiplicity higher than 1).

2. If $p(x)$ has real coefficients, and if r is a root of $p(x)$, the $\bar{r}$ is also a root of $p(x)$.

10. The Complex Numbers: Polar Form

Let $z = a + bi$ be a complex number. Suppose that (r, θ) are the polar coordinates of the point (a, b) in $\mathbb{R}^2$.

1. $a = r\cos\theta$ and $b = r\sin\theta$.
2. $r = \|z\| = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}$.
3. The **argument** of z is the angle θ.
4. The **polar form** of z is $z = r(\cos\theta + i\sin\theta)$.

11. The Complex Numbers: Multiplication and Division in Polar Form

Let $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$ be complex numbers, and let n be a positive integer.

1. $$zw = rs(\cos(\theta + \phi) + i\sin(\theta + \phi)).$$

2. $$\frac{z}{w} = \frac{r}{s}(\cos(\theta - \phi) + i\sin(\theta - \phi)).$$

3. $$z^n = r^n(\cos n\theta + i\sin n\theta).$$

4. The number z has n distinct n^{th} roots, which are given by

$$w_k = r^{\frac{1}{n}}(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\left(\frac{\theta + 2k\pi}{n}\right)$$

for each positive integer k such that $0 \le k \le n - 1$.

12. Euler's Formula

Let $a + bi$ be a complex number.

$$e^{a+bi} = e^a(\cos b + i \sin b).$$

13. Second Order Linear Differential Equations

1. A **second order linear differential equation** is an ODE of the form

$$a(x)y'' + b(x)y' + c(x)y = g(x),$$

for some functions $a(x)$, $b(x)$, $c(x)$ and $g(x)$.

2. A **homogeneous** second order linear differential equation is one where $g(x) = 0$.

14. Homogeneous Second Order Linear Differential Equations

1. A **homogeneous second order linear differential equation** is an ODE of the form

$$a(x)y'' + b(x)y' + c(x)y = 0,$$

for some functions $a(x)$, $b(x)$ and $c(x)$.

2. If y_1 and y_2 are solutions, and if A and B are real numbers, then $y = Ay_1 + By_2$ is a solution.

15. Homogeneous Second Order Linear Differential Equations with Constant Coefficients

1. A **homogeneous second order linear differential equation with constant coefficients** is an ODE of the form

$$ay'' + by' + cy = 0,$$

for some real numbers a, b and c.

2. The **characteristic equation** of this ODE is

$$ar^2 + br + c = 0.$$

3. To solve the ODE, solve the characteristic equation for all its roots. Some of the roots will be real, and some will be complex (which come in pairs of the form $a \pm bi$). Each root has a multiplicity, which can be found by factoring the characteristic equation. For each root, obtain the following solutions:

root	multiplicity	solution
r	1	e^{rx}
r	2	e^{rx}, xe^{rx}
$a \pm bi$	1	$e^{ax}\cos(bx), e^{ax}\sin(bx)$.

4. If y_1 and y_2 are solutions obtained in this way, then the general solution of the ODE is $y = Ay_1 + By_2$, where A and B are constants.

16. Non-Homogeneous Second Order Linear Differential Equations

We are given a non-homogeneous second order linear ODE of the form

$$a(x)y'' + b(x)y' + c(x)y = g(x),$$

for some functions $a(x)$, $b(x)$, $c(x)$ and $g(x)$.

Let y_p be a particular solution of this ODE; that is, the solution y_p has no constants in it. Let y_c be the general solution of the associated homogeneous linear ODE, which is

$$a(x)y'' + b(x)y' + c(x)y = 0.$$

Then $y = y_c + y_p$ is the general solution of the original non-homogeneous linear ODE.

17. Undetermined Coefficients: Second Order

1. A **second order linear differential equation with constant coefficients** is an ODE of the form
$$ay'' + by' + cy = g(x),$$
for some real numbers a b and c, and some function $g(x)$.

2. To find a particular solution, here are some common types of functions $g(x)$, and what to guess for particular solutions:

function	guess
$p_m(x)$	$A_0 + A_1 x + \cdots A_m x^m$
ae^{rx}	Ae^{rx}
$p_m(x)e^{rx}$	$(A_0 + A_1 x + \cdots A_m x^m)e^{rx}$
$a \sin kx$	$A \sin kx + B \cos kx$
$a \cos kx$	$A \sin kx + B \cos kx$
$ae^{rx} \sin kx$	$e^{rx}(A \sin kx + B \cos kx)$
$ae^{rx} \cos kx$	$e^{rx}(A \sin kx + B \cos kx)$
$p_m(x) \sin kx$	$(A_0 + A_1 x + \cdots A_m x^m) \sin kx + (B_0 + B_1 x + \cdots B_m x^m) \cos kx$
$p_m(x) \cos kx$	$(A_0 + A_1 x + \cdots A_m x^m) \sin kx + (B_0 + B_1 x + \cdots B_m x^m) \cos kx$

3. To solve the ODE, first find the general solution y_c of the associated homogeneous ODE. Then make a guess y_p for a particular solution of the original ODE, but if any part of the guess overlaps with y_c, multiply that part of the guess by whatever power of x is needed to avoid any overlap. Solve for the constants in y_p. Then $y = y_c + y_p$ is the general solution of the original non-homogeneous ODE.

18. Springs

The ordinary differential equation for the motion of a spring is
$$mx'' + cx' + kx = F(t),$$
where m is the mass, and c is the resistance, and k is the spring constant, and $F(t)$ is a forcing function.

19. Springs: Free Undamped Motion

The spring has **free undamped motion** when $c = 0$ and $F(t) = 0$. The ordinary differential equation is then

$$mx'' + kx = 0.$$

Let $\omega_0 = \sqrt{\frac{k}{m}}$.

1. The general solution is $x(t) = A\cos\omega_0 t + B\sin\omega_0 t$.
2. Let $C = \sqrt{A^2 + B^2}$, and let α be such that $\cos\alpha = \frac{A}{C}$ and $\sin\alpha = \frac{B}{C}$.
3. The general solution is $x(t) = C\cos(\omega_0 t - \alpha)$.
4. The **amplitude** is C.
5. The **circular frequency** is ω_0.
6. The **phase angle** is α.
7. The **period** is $T = \frac{2\pi}{\omega_0}$.
8. The **frequency** is $\nu = \frac{1}{T} = \frac{\omega_0}{2\pi}$.
9. The **time lag** is $\delta = \frac{\alpha}{\omega_0}$.

20. Springs: Free Damped Motion

The spring has **free undamped motion** when $F(t) = 0$. The ordinary differential equation is then

$$mx'' + cx' + kx = 0.$$

Let $\omega_0 = \sqrt{\frac{k}{m}}$ and $p = \frac{c}{2m}$.

Underdamped Case Suppose that $c^2 < 4km$.

1. Let $\omega_1 = \sqrt{\omega_0^2 - p^2} = \frac{\sqrt{4km-c^2}}{2m}$.
2. The general solution is $x(t) = e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t)$.
3. Let $C = \sqrt{A^2 + B^2}$, and let α be such that $\cos\alpha = \frac{A}{C}$ and $\sin\alpha = \frac{B}{C}$.
4. The general solution is $x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha)$.
5. The **time-varying amplitude** is Ce^{-pt}.
6. The **circular frequency** is ω_1.
7. The **pseudoperiod** is $T = \frac{2\pi}{\omega_1}$.

Overdamped Case Suppose that $c^2 > 4km$.

1. Let $r_1 = -p + \sqrt{p^2 - \omega_0^2}$ and $r_2 = -p - \sqrt{p^2 - \omega_0^2}$.
2. The general solution is $x(t) = Ae^{r_1 t} + Be^{r_2 t}$.

Critically Damped Case Suppose that $c^2 = 4km$.

1. The general solution is $x(t) = (A + Bx)e^{-pt}$.

21. Springs: Undamped Forced Oscillation

The spring has **undamped forced oscillation** when $c = 0$ and $F(t) \neq 0$. Consider the case when $F(t) = F_0 \cos \omega t$. The ordinary differential equation is then

$$mx'' + kx = F_0 \cos \omega t.$$

Let $\omega_0 = \sqrt{\frac{k}{m}}$.

Beats Suppose that $\omega \neq \omega_0$.

1. Let $\omega_1 = \sqrt{\omega_0^2 - p^2} = \frac{\sqrt{4km - c^2}}{2m}$.
2. The general solution is $x(t) = A \cos \omega_0 t + B \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$.
3. Let $C = \sqrt{A^2 + B^2}$, and let α be such that $\cos \alpha = \frac{A}{C}$ and $\sin \alpha = \frac{B}{C}$.
4. The general solution is $x(t) = C \cos(\omega_0 t - \alpha) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$.

Resonance Suppose that $\omega = \omega_0$.

1. Let $\omega_1 = \sqrt{\omega_0^2 - p^2} = \frac{\sqrt{4km - c^2}}{2m}$.
2. The general solution is $x(t) = A \cos \omega_0 t + B \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega t$.
3. Let $C = \sqrt{A^2 + B^2}$, and let α be such that $\cos \alpha = \frac{A}{C}$ and $\sin \alpha = \frac{B}{C}$.
4. The general solution is $x(t) = C \cos(\omega_0 t - \alpha) + \frac{F_0}{2m\omega_0} t \sin \omega t$.