MATH 245 Intermediate Calculus Fall 2018

Study Sheet for Exam #3

The Exam is in class, Thursday, 20 December 2018

GUIDELINES THAT WILL BE ON THE EXAM

EXAM GUIDELINES

In order for this exam to be an honest and accurate reflection of your understanding of the material, you are asked to adhere to the following guidelines:

- The exam is closed book.
- The study sheet is not allowed during the exam.
- Books, notes and online resources are not allowed during the exam.
- Electronic devices (calculators, cell phones, tablets, laptops, etc.) are not allowed during the exam.
- For the duration of the exam, you may not discuss the exam, or related material, with anyone other than the course instructor.
- Giving help to others taking this exam is as much a violation of these guidelines as receiving help.
- Late exams will be allowed only if you discuss it with the course instructor before hand, or if an emergency occurs.
- **Violation of these guidelines will result, at minimum, in a score of zero on this exam.**
- **There will be no opportunity to retake this exam.**

Further comments:

- Write your solutions carefully and clearly.
- Show all your work. You will receive partial credit for work you show, but you will not receive credit for what you do not write down. In particular, correct answers with no work will not receive credit.

TOPICS

1. Sequences
2. Series (convergence of series, telescoping series, geometric series, p-series)
3. Convergence tests for series (Divergence Test, Comparison Test, Limit Comparison Test, Integral Test, Alternating Series Test, Ratio Test)
4. Absolute and conditional convergence
5. Power series (interval of convergence and radius of convergence)
6. Differentiation and integration of power series
7. Representing a function as a power series, Taylor series and Maclaurin Series
8. Power series solutions of ordinary differential equations

TIPS FOR STUDYING FOR THE EXAM

×	**Bad**	Forgetting about the homework and the previous quizzes.
✓	**Good**	Making sure you know how to do all the problems on the homework and previous quizzes; seeking help seeking help from the instructor and the tutors about the problems you do not know how to do.
×	**Bad**	Doing all the practice problems from some of the sections, and not having enough time to do practice problems from the rest of the sections.
✓	**Good**	Doing a few practice problems of each type from every sections.
×	**Bad**	Studying only by reading the book.
✓	**Good**	Doing a lot of practice problems, and reading the book as needed.
×	**Bad**	Studying only by yourself.
✓	**Good**	Trying some practice problems by yourself (or with friends), and then seeking help from the instructor and the tutors about the problems you do not know how to do.
×	**Bad**	Doing practice problems while looking everything up in the book.
✓	**Good**	Doing some of the practice problems the way you would do them on the quiz or exam, which is with closed book and no calculator.
×	**Bad**	Staying up late (or all night) the night before the exam.
✓	**Good**	Studying hard up through the day before the exam, but getting a good night's sleep the night before the exam.

Ethan's Office Hours

- **Monday:** 4:30-6:00
- **Tuesday:** 5:00-6:00
- **Wednesday:** 2:00-3:30
- **Or by appointment**

Tutor

- Weronica Nguyen
- **Office hours**: Wednesday: 6:00-7:00, Mathematics Common Room (third floor of Albee)
- **Email to Make an Appointment**: tn3599 "at" bard "dot" edu.

PRACTICE PROBLEMS FROM
STEWART, CALCULUS CONCEPTS AND CONTEXTS, 4TH ED.

Section 8.1: 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 41, 43

Section 8.2: 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 37, 39, 49, 51, 53, 65

Section 8.3: 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31

Section 8.4: 3, 5, 7, 9, 13, 15, 21, 23, 25, 27, 29, 31, 33, 37

Section 8.5: 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25

Section 8.6: 3, 5, 7, 9, 27,

Section 8.7: 5, 7, 9, 11, 13, 15, 17, 21, 23, 25, 27, 29, 39, 41, 43, 45, 47, 49

Using Series to Solve Differential Equations Handout: 1, 3, 5, 7, 9, 11

SOME IMPORTANT CONCEPTS AND FORMULAS

1. Sequences

1. A **sequence** of real numbers is a collection of real numbers of which there is a first, a second, a third and so on, with one real number for each element of the natural numbers. A sequence is written $a_1, a_2, a_3, \ldots$, and also $\{a_n\}_{n=1}^{\infty}$.
2. The index n of a sequence could start at any number, not just 1.
3. In mathematical usage, the terms "sequence" and "series" mean different things, and should be used according to their precise meanings.
4. As sequence can be defined **explicitly**, which means that the sequence is given by a formula for a_n in terms of n, or **recursively**, which means that the sequence is given by specifying a_1 together with a formula for a_{n+1} in terms of a_n.

2. Sequences: Limits

1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, and let L be a real number. The number L is the **limit** of $\{a_n\}_{n=1}^{\infty}$, written
$$\lim_{n\to\infty} a_n = L,$$
if the value of a_n gets closer and closer to a number L as the value of n gets larger and larger. If $\lim_{n\to\infty} a_n = L$, the sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to L. If $\{a_n\}_{n=1}^{\infty}$ converges to some real number, the sequence $\{a_n\}_{n=1}^{\infty}$ is **convergent**; otherwise $\{a_n\}_{n=1}^{\infty}$ is **divergent**.
2. The above definition, and in particular the use of the phrase "gets closer and closer," is informal. A rigorous definition of limits will be seen in a Real Analysis course.
3. If a sequence has a limit, the limit is unique.
4. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Let $f\colon [1,\infty) \to \mathbb{R}$ be a function such that $f(n) = a_n$ for all natural numbers n. If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.

3. Sequences: Basic Limits

1. $\lim_{n\to\infty} \frac{1}{n} = 0$.
2. $$\lim_{n\to\infty} r^n = \begin{cases} 0, & \text{if } |r| < 1 \\ 1, & \text{if } r = 1 \\ \text{does not exist,} & \text{otherwise.} \end{cases}$$

4. Sequences: Properties of Limits

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be sequences, and let k be a real number. Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent.

1. $\{a_n + b_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$.

2. $\{a_n - b_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} (a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$.

3. $\{ka_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} ka_n = k \lim\limits_{n\to\infty} a_n$.

4. $\{a_n b_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} a_n b_n = [\lim\limits_{n\to\infty} a_n] \cdot [\lim\limits_{n\to\infty} b_n]$.

5. If $\lim\limits_{n\to\infty} b_n \neq 0$, then $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$.

6. If $f(x)$ is a continuous function, then $\lim\limits_{n\to\infty} f(a_n) = f(\lim\limits_{n\to\infty} a_n)$.

7. If $a_n \leq b_n$ for all natural numbers n, then $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$.

8. (Squeeze Theorem) If $a_n \leq c_n \leq b_n$ for all natural numbers n, and if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$, then $\{c_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$.

5. Series

1. A **series** of real numbers is a formal sum of a sequence of real numbers, written

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots .$$

2. The index n of a series could start at any number, not just 1.

6. Series: Convergence

Let $\sum_{n=1}^{\infty} a_n$ be a series.

1. For each natural number k, the k^{th} **partial sum** of $\sum_{n=1}^{\infty} a_n$, denoted s_k, is defined by

$$s_k = \sum_{i=1}^{k} a_i = a_1 + a_2 + \cdots + a_k.$$

2. The **sequence of partial sums** of $\sum_{n=1}^{\infty} a_n$ is the sequence $\{s_n\}_{n=1}^{\infty}$.

3. Let L be a real number. The number L is the **sum** of $\sum_{n=1}^{\infty} a_n$, written

$$\sum_{n=1}^{\infty} a_n = L,$$

if $\lim_{n\to\infty} s_n = L$. If $\sum_{n=1}^{\infty} a_n = L$, the series $\sum_{n=1}^{\infty} a_n$ **converges** to L. If $\sum_{n=1}^{\infty} a_n$ converges to some real number, the series $\sum_{n=1}^{\infty} a_n$ is **convergent**; otherwise $\sum_{n=1}^{\infty} a_n$ is **divergent**.

4. If a series has a sum, the sum is unique.

5. Changing or deleting a finite numbers of terms in a series will not affect whether the series is convergent or divergent (though it might change the sum of the series if the series is convergent).

7. Harmonic Series

1. The **harmonic series** is the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots .$$

2. The harmonic series is divergent.

8. Geometric Series

1. A **geometric series** is any series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots ,$$

where a and r are real numbers.

2. A geometric series converges to $\frac{a}{1-r}$ if $|r| < 1$, and is divergent if $|r| \geq 1$.

9. Series: Properties

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series, and let k be a real number. Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent.

1. $\sum_{n=1}^{\infty} (a_n + b_n)$ is convergent and $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.
2. $\sum_{n=1}^{\infty} (a_n - b_n)$ is convergent and $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.
3. $\sum_{n=1}^{\infty} ka_n$ is convergent and $\sum_{n=1}^{\infty} ka_n = k \sum_{n=1}^{\infty} a_n$.

10. Divergence Test

Let $\sum_{n=1}^{\infty} a_n$ be a series.

1. If $\lim\limits_{n \to \infty} a_n \neq 0$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.
2. **Caution**: If $\lim\limits_{n \to \infty} a_n = 0$, you CANNOT conclude that the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

11. Integral Test

Let $\sum_{n=1}^{\infty} a_n$ be a series, and let $f : [1, \infty) \to \mathbb{R}$ be function that satisfies the following four properties:

(1) $f(n) = a_n$ for all natural numbers n.

(2) $f(x)$ is continuous on $[1, \infty)$.

(3) $f(x) > 0$ on $[1, \infty)$.

(4) $f(x)$ is decreasing on $[1, \infty)$.

Then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\displaystyle\int_1^{\infty} f(x)\, dx$ is convergent.

12. p-Series

1. A p**-series** is any series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots,$$

   where p is a real number.

2. A p-series is convergent if $p > 1$, and is divergent if $p \leq 1$.

13. Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series. Suppose that $a_n \geq 0$ and $b_n \geq 0$ for all natural numbers n. Suppose that $a_n \leq b_n$ for all natural numbers n.

1. If $\sum\limits_{n=1}^{\infty} b_n$ is convergent, then $\sum\limits_{n=1}^{\infty} a_n$ is convergent.
2. If $\sum\limits_{n=1}^{\infty} a_n$ is divergent, then $\sum\limits_{n=1}^{\infty} b_n$ is divergent.
3. **Caution**: If $\sum\limits_{n=1}^{\infty} a_n$ is convergent or if $\sum\limits_{n=1}^{\infty} b_n$ is divergent, you CANNOT conclude anything about the other series by the Comparison Test.

14. Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series. Suppose that $a_n \geq 0$ and $b_n \geq 0$ for all natural numbers n. Suppose that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = L,$$

for some $L \in \mathbb{R}$ or $L = \infty$.

1. Suppose that $0 < L < \infty$. Then either both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, or both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent.
2. Suppose that $L = 0$. If $\sum\limits_{n=1}^{\infty} a_n$ is convergent, then $\sum\limits_{n=1}^{\infty} b_n$ is convergent.
3. Suppose that $L = \infty$. If $\sum\limits_{n=1}^{\infty} a_n$ is divergent, then $\sum\limits_{n=1}^{\infty} b_n$ is divergent.

15. Alternating Series

An **alternating series** is any series of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \qquad \text{or} \qquad \sum_{n=1}^{\infty} (-1)^n a_n,$$

where $a_n > 0$ for all natural numbers n.

16. Alternating Series Test

Let $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ be an alternating series, where $a_n > 0$ for all natural numbers n.

1. Suppose that the alternating series satisfies the following two properties:

(a) the sequence $\{a_n\}_{n=1}^{\infty}$ is decreasing.

(b) $\lim_{n\to\infty} a_n = 0$.

Then the alternating series is convergent.

2. The same result holds for alternating series of the form $\sum_{n=1}^{\infty}(-1)^n a_n$.

17. Remainder Estimate for the Alternating Series Test

Let $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ be an alternating series, where $a_n > 0$ for all natural numbers n. Let m be a natural number.

1. The m^{th} **remainder** of the alternating series, denoted R_m, is defined by

$$R_m = \sum_{n=1}^{\infty}(-1)^{n-1}a_n - s_m = \sum_{n=m+1}^{\infty}(-1)^n a_n.$$

2. Suppose that the alternating series satisfies the hypotheses of the Alternating Series Test, and hence is convergent. Then $|R_m| \le a_{m+1}$.

3. The same result holds for alternating series of the form $\sum_{n=1}^{\infty}(-1)^n a_n$.

18. Absolute Convergence and Conditional Convergence

Let $\sum_{n=1}^{\infty} a_n$ be a series.

1. The series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

2. The series $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ is convergent but not absolutely convergent.

3. If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

4. Any series is either absolutely convergent, conditionally convergent or divergent.

19. Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series. Suppose that $a_n \neq 0$ for all natural numbers n. Suppose that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L,$$

for some real number L or $L = \infty$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
2. If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
3. **Caution**: If $L = 1$, you CANNOT conclude conclude that $\sum_{n=1}^{\infty} a_n$ is either convergent or divergent by the Ratio Test.

20. Power Series

1. A **power series** is any series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots ,$$

where $a, c_0, c_1, c_2, \cdots$ are real numbers.

2. If $a = 0$, a power series has the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots .$$

3. The numbers $c_0, c_1, c_2, \cdots$ are the **coefficients** of the power series.

21. Interval of Convergence and Radius of Convergence of Power Series

1. Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series. Then precisely one of the following happens:

(1) The series is absolutely convergent for all real numbers x, in which case $R = \infty$.

(2) The series is convergent only for $x = a$, in which case $R = 0$.

(3) There is some positive number R such that the series is absolutely convergent for all $|x-a| < R$, and the series is divergent for all $|x-a| > R$.

2. The **radius of convergence** of the power series is R, which is either a real number or ∞.

3. The **interval of convergence** of the power series is set of all numbers x at which the power series is convergent.

4. (1) If $R = \infty$, the interval of convergence is $(-\infty, \infty)$.

(2) If $R = 0$, the interval of convergence is $[a, a]$.

(3) If $0 < R < \infty$, the the interval of convergence is one of $(a-R, a+R)$, or $(a-R, a+R]$, or $[a-R, a+R)$ or $[a-R, a+R]$.

5. To find the interval of convergence and radius of convergence, a method that often works is to use the Ratio Test, which leads to finding the radius convergence, and then, if $0 < R < \infty$, to use other convergence tests to find out convergence or divergence at the endpoints of the interval of convergence.

22. Basic Operations on Power Series

Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ and $g(x) = \sum_{n=0}^{\infty} d_n(x-a)^n$ be power series.

1. $f(x)+g(x) = \sum_{n=0}^{\infty}(c_n+d_n)(x-a)^n$, with interval of convergence the intersection of the intervals of convergence of $f(x)$ and $g(x)$. Similarly for $f(x)-g(x)$.

2. $(x-a)^r f(x) = \sum_{n=0}^{\infty} c_n(x-a)^{n+r}$, with the same interval of convergence as $f(x)$.

3. $f(x)g(x) = \sum_{n=0}^{\infty} e_n(x-a)^n$, where $e_n = \sum_{k=0}^{n} c_k d_{n-k}$, with radius of convergence the smaller of the radii of convergence of $f(x)$ and $g(x)$.

23. Differentiation and Integration of Power Series

Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series. Let R be the radius of convergence of $f(x)$.

1. The power series $\displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ has radius of convergence R, and $f'(x) = \displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ for all $x \in (a-R, a+R)$.

2. The power series $= \displaystyle\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ has radius of convergence R, and $\displaystyle\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ for all $x \in (a-R, a+R)$.

3. **Caution**: For any particular function $f(x)$, it might be that the above power series are convergent on the endpoints of the interval $(a-R, a+R)$, and it might be that $f'(x)$ or $\int f(x)\,dx$ equals the power series at the endpoints, but that needs to be verified in each case.

24. Representing a Function as a Power Series

1. Let E be a subset of the real numbers, let $f\colon E \to \mathbb{R}$ be a function, and let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series. The function f is **represented** by $\sum_{n=0}^{\infty} c_n(x-a)^n$ if the following three properties hold:

   (1) The radius of convergence of $\sum_{n=0}^{\infty} c_n(x-a)^n$ is positive.

   (2) The interval of convergence of $\sum_{n=0}^{\infty} c_n(x-a)^n$ is a subset of E.

   (3) $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for all x in the interval of convergence.

2. **Caution**: If f is represented by $\sum_{n=0}^{\infty} c_n(x-a)^n$, it is not necessarily the case that the interval of convergence of $\sum_{n=0}^{\infty} c_n(x-a)^n$ is all of E.

3. Not every function is represented by a power series.

4. If a function is represented by a power series, the power series is unique.

25. Taylor Series and Maclaurin Series

Let I be an open interval, let $f\colon I \to \mathbb{R}$ be a function, and let a be in I. Suppose that f is infinitely differentiable.

1. The **Taylor series** of f centered at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots .$$

2. Suppose that 0 is in I. The **Maclaurin series** of f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots .$$

3. **Caution**: The Taylor series and Maclaurin series of a function do not always equal the function.

26. Taylor Series of Some Standard Functions

The following equalities hold for all real numbers x.

1.
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

2.
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

3.
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

27. Binomial Series

Let r be a real number.

1. Let k be an integer. The **binomial coefficient** $\binom{r}{k}$ is defined by

$$\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}.$$

Also, let $\binom{r}{0} = 1$.

2. **Caution**: The number r can be any real number, not necessarily an integer; the number k is always a non-negative integer.

3. (Binomial series)

$$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k = 1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \cdots,$$

for all real numbers x such that $|x| < 1$.

28. Taylor Polynomials and Maclaurin Polynomials

Let I be an open interval, let $f : I \longrightarrow \mathbb{R}$ be a function, and let a be in I. Suppose that f is infinitely differentiable. Let n be a natural number.

1. The n^{th} degree **Taylor polynomial** for $f(x)$ centered at a, denoted $T_n(x)$, is

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \\ &\qquad \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \end{aligned}$$

2. Suppose that 0 is in I. The n^{th} degree **Maclaurin polynomial** for $f(x)$ is

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n. \end{aligned}$$

29. Taylor's Theorem

Let I be an open interval, let $f : I \to \mathbb{R}$ be a function, and let a be in I. Suppose that f is infinitely differentiable. Let n be a natural number.

1. The n^{th} **remainder** of the Taylor series for $f(x)$ centered at a, denoted $R_n(x)$, is defined by

$$R_n(x) = f(x) - T_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

2. (Taylor's Theorem) Suppose that there is some real number M such that $|f^{(n+1)}(x)| \leq M$ for all real numbers x such that $|x - a| \leq d$. Then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

for all real numbers x such that $|x - a| \leq d$.

30. Power Series Solutions of Ordinary Differential Equations

1. A **homogeneous linear differential equation** of order n is an ODE of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0.$$

for some functions $a_0(x), a_1(x), \ldots, a_n(x)$. (These functions are not assumed to be constants.)

2. To use series to solve a homogeneous linear differential equation, let $y = \sum_{n=0}^{\infty} c_n x^n$, find the derivatives of y, substitute into the ODE, and derive relations among the coefficients $c_0, c_1, c_2, \cdots$. It is common to find a recurrence relation among these coefficients, in which c_n is related to some of $c_0, c_1, c_2, \cdots, c_{n-1}$.