FIXED-POINT SUBGROUPS OF $\mathrm{GL}_3(q)$

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ABSTRACT. Let V be a vector space over a field k. We call a subgroup $G \subset \mathrm{GL}(V)$ a *fixed-point subgroup* if $\det(1-g) = 0$ for all $g \in G$. Let q be a power of a prime. In this paper we classify the fixed-point subgroups of $\mathrm{GL}_3(q)$.

1. INTRODUCTION

1.1. **Motivation.** Let $X/\mathbf{Q}$ be a smooth, projective algebraic variety and ℓ a rational prime. Then there are ℓ-adic representations

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Aut}\left(\mathrm{H}^i_{\text{ét}}\left(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell\right)\right)$$

on the étale cohomology groups of X, and the image of such a representation can have interesting consequences for the arithmetic of the variety X. For example, when X is an elliptic curve and ρ_ℓ is the ℓ-adic representation on the Tate module, then the image of ρ_ℓ gives information on the ℓ-power torsion structure of $X(\mathbf{Q})$. A concrete instance of this stems from a question of Lang, answered by Katz in [10]. Namely, if X is an elliptic curve over $\mathbf{Q}$ such that for all but finitely many primes p the numbers $\#X_p(\mathbf{F}_p)$ are divisible by ℓ^n (where X_p denotes the reduction of X modulo a good primes p), then it is true that at least one of the curves X' in the isogeny class of X has $\#X'(\mathbf{Q})$ divisible by ℓ^n. By translating to Galois representations, this result amounts to a classification of subgroups G of $\mathrm{GL}_2(\mathbf{Z}_\ell)$ such that $\det(1-g) \equiv 0 \pmod{\ell^n}$ for all $g \in G$. One can ask for a similar classification of subgroups of symplectic similitude groups with a view towards higher-dimensional abelian varieties with divisibilities on their number of points mod p; we provided such classifications in dimensions 4 and 6 in [3, 4, 5, 6] for the groups $\mathrm{GSp}_4(\mathbf{F}_\ell)$ and $\mathrm{GSp}_6(\mathbf{F}_\ell)$.

This raises a natural question: If k is a finite field, can one classify the *irreducible* subgroups of $\mathrm{GL}_n(k)$ such that every element has a fixed point? (By "irreducible subgroup" we mean a subgroup $G \subset \mathrm{GL}_n(k)$ that acts irreducibly on the underlying vector space k^n.) Let us call a subgroup G of $\mathrm{GL}_n(k)$ a *fixed-point subgroup* if every element fixes a point in its natural representation.

By an exercise of Serre [14, Ex. 1] there are no irreducible fixed-point subgroups of $\mathrm{GL}_2(k)$. One of the main results of [10] is that there are no irreducible fixed-point subgroups of $\mathrm{GSp}_4(\mathbf{F}_\ell)$, where $\mathbf{F}_\ell$ is the field of ℓ elements. In [3, 4, 5] we classified the fixed-point subgroups of $\mathrm{GSp}_6(\mathbf{F}_\ell)$ and showed that none are irreducible. However, we recall an example of [5], originally communicated to us by Serre in [15]. If $\mathrm{L}_3(2)$ is the simple group of order 168, then the Steinberg representation

$$\mathsf{St} : \mathrm{L}_3(2) \to \mathrm{Sp}_8(\mathbf{F}_2)$$

is absolutely irreducible and $\mathsf{St}(\mathrm{L}_3(2))$ is a fixed-point subgroup of $\mathrm{Sp}_8(\mathbf{F}_2)$. As an application of this observation, if A is an abelian fourfold defined over a number field K such that the image of the mod 2 representation

$$\overline{\rho_2} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(A[2])$$

coincides with $\mathsf{St}(\mathrm{L}_3(2))$, then A has the property that for all but finitely many primes $\mathfrak{p}$, the number of points $\#\overline{A}_{\mathfrak{p}}(\mathbf{F}_{\mathfrak{p}})$ on the reduction modulo $\mathfrak{p}$ of A is even, while no member of the isogeny class of A has an even number of K-rational torsion points. An interesting related question is whether such a fourfold can be realized over $\mathbf{Q}$.

Leaving the case of abelian varieties and symplectic groups, we focus on three-dimensional representations, which arise naturally in an arithmetic context as well. Using [7] as motivation, one can consider modular forms for congruence subgroups $\Gamma_0(N)$ of $\mathrm{SL}_3(\mathbf{Z})$. Given a cuspidal eigenform $f \in \mathrm{H}^3(\Gamma_0(N), \mathbf{C})$, let $\mathbf{Q}_f$ denote the number field generated by the Hecke eigenvalues of f. Let $\lambda \in \mathbf{Q}_f$ be a prime dividing ℓ. Then we have attached to f the λ-adic Galois representation

$$\rho_\lambda : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_3(\mathbf{Q}_{f,\lambda}).$$

(See [7, §3] for an explicit example of how such compatible families of representations arise.) The residual representation $\overline{\rho_\lambda}$ then provides a natural setting for studying subgroups of $\mathrm{GL}_3(k)$, where k is a finite field. In the aforementioned example, if $\operatorname{im}\overline{\rho_\lambda}$ is a fixed-point subgroup of $\mathrm{GL}_3(k)$ then we get additional information on congruence properties of the number of points on the variety modulo p, for all but finitely many p, by the Chebatorev Density Theorem. For this reason, and the ones mentioned above with respect to abelian varieties, the fixed-point subgroups of linear groups have special arithmetic interest.

In this paper we continue our classification of fixed-point linear groups and determine all fixed-point subgroups of $\mathrm{GL}_3(k)$, where k is a finite field. Unlike the classifications in [3, 4, 5] for the groups $\mathrm{GSp}_4(\mathbf{F}_\ell)$ and $\mathrm{GSp}_6(\mathbf{F}_\ell)$, the main theorem of this paper allows for k to be an arbitrary finite field of any characteristic.

1.2. **The Main Theorem.**

We postpone a review of notation until the next section, except to remark that the maximal subgroups of a finite linear group fall into 8 geometric classes $\mathcal{C}_1, \ldots, \mathcal{C}_8$, together with a class $\mathcal{S}$ of exceptional subgroups; we refer the reader to [1] for the details of the classification.

There are certain subgroups of $\mathrm{GL}_3(q)$ that are easily identifiable as fixed-point subgroups, for example those conjugate to

$$\begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \mathrm{GL}_3(q),$$

and we do not wish to include them in our classification. We therefore declare a subgroup G of $\mathrm{GL}_3(q)$ to be a *trivial fixed-point group* if the semisimplification of the underlying 3-dimensional representation contains the trivial representation. Henceforth we work exclusively with *semisimple* groups in this paper as these will have the same fixed-point properties as the parabolic groups they lie in. Our main theorem is as follows; see the following sections for all notational definitions.

Theorem 1.2.1. *The maximal, nontrivial, semisimple fixed-point subgroups of* $\mathrm{GL}_3(q)$ *are as follows:*

Dimensions of Simple Factors	*Isomorphism Type*	*Conditions*
$(1,1,1)$	$C_2 \times C_2$	q *odd*
$(2,1)$	D_{q-1}	q *odd*
	D_{q+1}	q *odd*
Irreducible	Sym_4	q *odd*
	$\mathrm{SO}_3(q)$	q *odd*
	Alt_5	$5 \in \mathbf{F}_q^{\times 2}$, $p \neq 2,3$

Remark 1.2.2. As a corollary of our theorem we obtain that there are no irreducible fixed-point subgroups of $\mathrm{GL}_3(q)$ in characteristic 2. In particular, the groups $\mathrm{SO}_3(q)$ and Sym_4, each of which is naturally a subgroup of $\mathrm{GL}_3(q)$, fix a line in characteristic 2 (for the former, see [1, Thm. 1.5.41] while the latter is deduced from the Brauer Table of Sym_4 in characteristic 2).

1.3. **Notation and Setup.** Let k be a finite field of characteristic p (we choose p instead of ℓ for consistency in the group theory literature) and write $k = \mathbf{F}_q$, where $q = p^n$. We follow the classification and notational scheme of [1] which is based on Aschbacher's original classification of the subgroups of the finite classical groups.

Since $\mathrm{GL}_3(q)$ is not a fixed-point group itself, any fixed-point subgroup must lie in a maximal subgroup of $\mathrm{GL}_3(q)$ and hence in one of the 8 geometric classes $\mathcal{C}_1, \dots, \mathcal{C}_8$, or the exceptional class $\mathcal{S}$. We use the standard notation from finite group theory, basing much of our notational scheme on that of [1]. In particular, we set

- Alt_n: The alternating group on n letters.
- Sym_n: The symmetric group on n letters.
- C_n: The cyclic group of order n.
- E_q: Elementary abelian group of exponent p and rank n.
- A^{m+n}: If A is elementary abelian, then A^{m+n} has elementary abelian normal subgroup A^m and quotient A^n.
- p_+^{1+2n}: Extra-special p group of order p^{1+2n} and exponent p.
- d: the center of $\mathrm{SL}_3(q)$.
- $Z(q)$: the center (scalar matrices) of $\mathrm{GL}_3(q)$.
- $\mathrm{L}_n(q)$: The projective special linear group $\mathrm{PSL}_n(q)$.
- $A \wr B$: The wreath product of A and B, where $B \hookrightarrow \mathrm{Perm}(A \times \cdots \times A)$.
- $N{\cdot}Q$ denotes a non-split extension of Q by N.
- $N:Q$ denotes a split extension of Q by N.
- $N.Q$ denotes an arbitrary extension of Q by N.

Our strategy for proving Theorem 1.2.1 is roughly as follows. Given a subgroup G of $\mathrm{GL}_3(q)$, we intersect with $\mathrm{SL}_3(q)$ and use the classification of maximal subgroups of $\mathrm{SL}_3(q)$ outlined in [1, Chapter 2] to determine the fixed-point subgroups of $\mathrm{SL}_3(q)$. We then lift back to $\mathrm{GL}_3(q)$ to find the maximal fixed-point subgroups. The issue is that we may encounter *novel* subgroups – maximal subgroups M of $\mathrm{GL}_3(q)$ such that $M \cap \mathrm{SL}_3(q)$ is not maximal in $\mathrm{SL}_3(q)$. We will address any novelties as they arise. Toward that end, we record the maximal subgroups of $\mathrm{SL}_3(q)$ in Table 1.3 below; see [1, Table 8.3] for complete details on the subgroup structure of $\mathrm{SL}_3(q)$.

Remark. There is a typographical error in [1, Table 8.3]: in class $\mathcal{C}_1$, the group labelled $E_q^3 : \mathrm{GL}_2(q)$ should be $E_q^2 : \mathrm{GL}_2(q)$. We have corrected this in Table 1.3.

We also note that the papers [9, 13] provide a classification of the ternary linear groups over finite fields, from which one could recover [1, Table 8.3]; however, we prefer to begin with the classification scheme of [1] due to the modern notation and language.

Class	Isomorphism Type
$\mathcal{C}_1$	$E_q^2 : \mathrm{GL}_2(q)$, $E_q^{1+2} : (q-1)^2$, $\mathrm{GL}_2(q)$
$\mathcal{C}_2$	$(q-1)^2 : \mathrm{Sym}_3$, $q \geq 5$
$\mathcal{C}_3$	$(q^2+q+1).3$, $q \neq 4$
$\mathcal{C}_5$	$\mathrm{SL}_3(q_0).\left(\frac{q-1}{q_0-1}, 3\right)$ if $q = q_0^r$, r prime.
$\mathcal{C}_6$	$3_+^{1+2}.Q_8.\frac{(q-1,9)}{3}$ when $p = q \equiv 1 \pmod 3$
$\mathcal{C}_8$	$d \times \mathrm{SO}_3(q)$ when q is odd $(q_0-1, 3) \times \mathrm{SU}_3(q)$ when $q = q_0^2$
$\mathcal{S}$	$d \times \mathrm{L}_2(7)$ when $q = p \equiv 1, 2, 4 \pmod 7$, $q \neq 2$. $3^{\cdot}A_6$ when $q = p \equiv 1, 4 \pmod{15}$ or $q = p^2$, $p = 2, 3 \pmod 5$, $p \neq 3$

FIGURE 1. Maximal Subgroups of $\mathrm{SL}_3(q)$

The groups in class $\mathcal{C}_1$ are the parabolic subgroups of $\mathrm{GL}_3(q)$ and we treat them separately in the next section. We then focus the rest of the paper on the irreducible fixed-point subgroups of $\mathrm{GL}_3(q)$.

2. Parabolic Fixed-Point Subgroups of $\mathrm{GL}_3(q)$

Let G be a semisimple subgroup of $\mathrm{GL}_3(q)$. We break the proof of Theorem 1.2.1 into two cases, depending on whether the action of G on k^3 is reducible or irreducible. In case of a reducible representation, G lies in a parabolic subgroup (Type $\mathcal{C}_1$) of $\mathrm{GL}_3(q)$, and the irreducible factors are either all one-dimensional, or consist of a 2-dimensional and 1-dimensional factor. (In both cases, we replace the representations with their semisimplifications.) Moreover, we require the classification of subgroups of a direct product, given by Goursat's Lemma [2, p. 864].

Theorem (Goursat's Lemma)**.** *Let A and B be finite groups. The subgroups G of $A \times B$ are in one-to-one correspondence with the tuples (G_1, G_2, G_3, ψ) where $G_1 \subset A$, $G_2 \subset B$, $G_3 \lhd G_2$, and $\psi : G_1 \to G_2/G_3$ is a surjective homomorphism.*

Beginning with the case where G is a subgroup of the diagonal subgroup C_{q-1}^3 of $\mathrm{GL}_3(q)$, we write $G \subset (C_{q-1} \times C_{q-1}) \times C_{q-1}$. We can describe G via two "Goursat-tuples":

$$(H_1, H_2, H_3, \psi), \text{ where } H_1 \subset C_{q-1} \times C_{q-1},\ H_2 \subset C_{q-1}, \text{ and}$$
$$(D_1, D_2, D_3, \phi), \text{ where } D_1 \subset C_{q-1},\ D_2 \subset C_{q-1},$$

and (D_1, D_2, D_3, ϕ) is the Goursat-tuple corresponding to $H_1 \subset C_{q-1} \times C_{q-1}$.

Lemma 2.0.1. *Suppose G, acting diagonally on $\mathbf{F}_q^3$, is a fixed-point subgroup that does not fix a line. Then q is odd and $G \simeq C_2 \times C_2$.*

Proof. With all notation as above, we assume $G \subset (C_{q-1} \times C_{q-1}) \times C_{q-1}$ is given by the Goursat-tuple (H_1, H_2, H_3, ψ) where $H_1 \subset (C_{q-1} \times C_{q-1})$, and H_1 is given by the Goursat-tuple (D_1, D_2, D_3, ϕ). Let S be the subset of H_1 consisting of pairs (x, y) such that neither x nor y is 1. We will show that unless G is the group specified in the statement of the Lemma, the size of S forces G to fix a line. We make several elementary observations:

(1) S lies in $\ker\psi$ (G is a fixed-point group).
(2) H_3 is trivial (G contains the elements of the form $(\ker\psi, H_3)$, $S \subset \ker\psi$, and G is a fixed-point group).
(3) Therefore $\psi : H_1 \to H_2$ is a surjective homomorphism.

We will give several estimates of $\#S$ below, and so we set the following notation:

$$h_i = \#H_i, \qquad d_i = \#D_i, \qquad k = \#\ker\psi, \qquad l = \#\ker\phi.$$

Combining the observations we immediately see that

$$k = h_1/h_2 \geq \#S + 1, \tag{2.0.2}$$

where the '+1' is due to the identity of H_1.

Since H_1 is given by the Goursat-tuple (D_1, D_2, D_3, ϕ), we can write $h_1 = d_1 d_3$. We can estimate the size of S by writing $\#S = h_1 -$ the number of elements (x, y) of H_1 with at least one x or y trivial; that is:

$$\#S = d_1 d_3 - l - d_3 + 1.$$

Comparing this to (2.0.2), we get our first estimate

$$d_1 d_3/h_2 \geq d_1 d_3 - l - d_3 + 2. \tag{2.0.3}$$

But since $l \leq d_1$, we can refine (2.0.3) to get our second estimate

$$d_1 d_3/h_2 \geq d_1 d_3 - d_1 - d_3 + 2 = (d_1 - 1)(d_3 - 1) + 1. \tag{2.0.4}$$

It is easy to check that the only integer triples (d_1, d_3, h_2) with $d_3 \geq 1$ and $d_1, h_2 \geq 2$ (recall G is a non-trivial fixed-point group) satisfying (2.0.4) are of the form $(d_1, 2, 2)$ or $(2, d_3, 2)$.

If q is even then there is no such subgroup of G since $q-1$ is odd, so we suppose q is odd. We will work through the details of the case $(d_1, 2, 2)$ and omit those of the case $(2, d_3, 2)$ since they are nearly identical. Therefore we consider the group

$$H_1 = \{(g, \phi(g)) \mid g \in D_1 \text{ and } \phi : D_1 \to D_2/\{\pm 1\}\}.$$

In general, there are $2d_1/d_2 + 1$ pairs in H_1 with a 1 in one of the components. Therefore, there are

$$2d_1 - (2d_1/d_2 + 1)$$

with both components nontrivial. We impose this condition on the estimate of k:

$$k \geq 2d_1 - (2d_1/d_2 + 1) + 1 = 2d_1 - 2d_1/d_2.$$

Notice that if $d_2 > 2$ then G would be a trivial fixed-point group since we would have $\ker\psi = H_1$ and so H_2 would coincide with H_3, which is trivial. Therefore we may assume $d_2 = 2$.

Since $d_2 = d_3 = 2$, this means $D_2 = D_3 = \{\pm 1\}$ and so H_1 is a direct product: $H_1 = D_1 \times \{\pm 1\}$. Including the identity, there are at least d_1 elements of H_1 that must lie in $\ker\psi$:

$$(1, 1), (g, -1), \ldots, (g^{d_1 - 1}, -1),$$

where g is a generator of D_1. Since $\ker\psi$ is a subgroup of H_1, it follows that $(g^2, 1) \in \ker\psi$ as well. Unless $g^2 = 1$, this forces $\ker\psi = H_1$ and G to be a trivial fixed-point group. We conclude that $H_1 = \{\pm 1\} \times \{\pm 1\}$. Together with $[H_2 : H_3] = 2$ and $H_3 = 1$ we get exactly the group $C_2 \times C_2$ as claimed in the Lemma, which is given explicitly in terms of matrices as

$$\begin{pmatrix} \epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_1\epsilon_2 \end{pmatrix},$$

where $\epsilon_i \in \{\pm 1\}$. □

Next suppose that $G \subset \mathrm{GL}_2(q) \times \mathrm{GL}_1(q)$ is semisimple with irreducible projection onto $\mathrm{GL}_2(q)$. Since G is a subgroup of a direct product, it is given by a Goursat-tuple (H_1, H_2, H_3, ψ), with $H_1 \subset \mathrm{GL}_2(q)$. As above, we only classify those G which are not direct products, that is $H_2 \neq H_3$.

Observation. If $G \subset \mathrm{GL}_2(q) \times \mathrm{GL}_1(q)$ is a fixed-point subgroup that does not fix a line and is given by the Goursat-tuple (H_1, H_2, H_3, ψ), then H_3 is trivial. This follows because any $g \in H_1$ without a fixed point is paired via ψ with H_3.

Lemma 2.0.5. *With all notation as above, if G is a fixed-point subgroup of $\mathrm{GL}_2(q) \times \mathrm{GL}_1(q)$ that does not fix a line, then H_1 is a proper subgroup of $\mathrm{GL}_2(q)$.*

Proof. It is an elementary counting problem to show that more than half the elements of $\mathrm{GL}_2(q)$ do not have a fixed point once $q > 2$ – use the fact that there are

- $q^2(q-1)^2/2$ elements with eigenvalues in a quadratic extension
- $q-2$ non-trivial central elements, and
- $(q-2)(q-1)(q+1)$ non-diagonalizable elements without a fixed point.

Dividing by the size of $\mathrm{GL}(2, q)$, we get

$$\frac{q^2(q-1)^2/2 + (q-2) + (q-2)(q-1)(q+1)}{q^4 - q^3 - q^2 + q} = \frac{q^3 - 3q}{2q^3 - 2q^2 - 2q + 2} > \frac{1}{2}.$$

Thus if $H_1 = \mathrm{GL}_2(q)$, then $\ker\psi = \mathrm{GL}_2(q)$ and so $H_2 = H_3$. But since H_3 is trivial by the observation above, we have that H_2 is trivial. When $q = 2$, H_2 is trivial. □

By Lemma 2.0.5, H_1 must lie in a proper subgroup of $\mathrm{GL}_2(q)$ and hence lies in a maximal subgroup of $\mathrm{GL}_2(q)$. By [12, Thm. 2.3], the subgroups H of $\mathrm{GL}_2(q)$ not containing $\mathrm{SL}_2(q)$ are described as follows (we use PH to denote the image of H in $\mathrm{PGL}_2(q)$):

(1) If H contains an element of order q then either G lies in a Borel subgroup or $\mathrm{SL}_2(q) \subset H$;
(2) PH is cyclic and H is contained in a Cartan group;
(3) PH is dihedral and H is contained in the normalizer of a Cartan group but not in the Cartan subgroup itself;
(4) PH is is isomorphic to Alt_4, Sym_4, or Alt_5.

Returning to our setup, if H_1 lies in a Borel or a Cartan, then H_1 is not irreducible. We therefore focus only on cases (3) and (4) of the subgroup classification of $\mathrm{GL}_2(q)$. We recall from [16, §3] the explicit description of the normalizers of the Cartan subgroups of $\mathrm{GL}_2(q)$ and adopt that notation in what follows.

Let $C_s(q)$ and $C_{ns}(q)$ denote the maximal split and nonsplit Cartan subgroups, respectively. Then $C_s(q) \simeq \mathbf{F}_q^\times \times \mathbf{F}_q^\times$ and $C_{ns}(q) \simeq \mathbf{F}_{q^2}^\times$ and each Cartan group has index 2 in its normalizer, which we denote by $C_s^+(q)$ and $C_{ns}^+(q)$, respectively, borrowing the notation of [16]. Each normalizer has a distinguished dihedral subgroup, $\mathsf{D}_s(q)$ and $\mathsf{D}_{ns}(q)$, respectively, where

$$\mathsf{D}_s(q) \cap C_s(q) = C_s(q) \cap \mathrm{SL}_2(q) \text{ and } \mathsf{D}_{ns}(q) \cap C_{ns}(q) = C_{ns}(q) \cap \mathrm{SL}_2(q).$$

That is, the "rotation" group of $\mathsf{D}_s(q)$ (resp. $\mathsf{D}_{ns}(q)$) consists of the elements of $C_s(q)$ (resp. $C_{ns}(q)$) of determinant (norm) 1. It follows that

$$\mathsf{D}_s(q) \simeq D_{q-1}$$
$$\mathsf{D}_{ns}(q) \simeq D_{q+1}.$$

Each dihedral group admits a surjective homomorphism to C_2 and when q is odd we can realize that homomorphism in the Goursat-tuples

$$D_{q-1} \simeq (\mathsf{D}_s(q), \{\pm 1\}, 1, \psi) \text{ and } D_{q+1} \simeq (\mathsf{D}_{ns}(q), \{\pm 1\}, 1, \psi).$$

It is easy to check that both dihedral groups are fixed-point subgroups of $\mathrm{GL}_2(q) \times \mathrm{GL}_1(q)$ with irreducible projection to $\mathrm{GL}_2(q)$ that do not fix a line in $\mathbf{F}_q^3$. We will show in Proposition 2.0.6 below that these are the only such groups. In preparation for the proof we make some observations.

Observations. Let $G \subset \mathrm{GL}_2(q) \times \mathrm{GL}_1(q)$ have Goursat-tuple (H_1, H_2, H_3, ψ) and suppose H_1 is an irreducible subgroup of $\mathrm{GL}_2(q)$ that normalizes a Cartan subgroup. Let G be a fixed-point group.

(1) The normalizer of the split Cartan group has exactly $3q-4$ elements with a fixed point; by fixing a basis, we can write these elements explicitly as
$$\left\{ \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix} \right) \mid x \text{ or } y = 1 \right\} \cup \left\{ \left(\begin{smallmatrix} 0 & z \\ z^{-1} & 0 \end{smallmatrix} \right) \mid z \neq 0 \right\}$$
(2) The normalizer of the non-split Cartan group has exactly q non-trivial elements with a fixed point, all of which belong to the non-trivial coset of the Cartan subgroup.
(3) If G does not fix a line, then H_1 must contain at least
$$\#H_1 \cdot \frac{\#H_2 - 1}{\#H_2}$$
elements with a fixed point.

Proposition 2.0.6. *Let $G \subset \mathrm{GL}_2(q) \times \mathrm{GL}_1(q)$ be a fixed point group with Goursat-tuple (H_1, H_2, H_3, ψ) and suppose H_1 normalizes a Cartan subgroup.*

If q is even, then H_2 is trivial and so G fixes a line in $\mathbf{F}_q^3$. If q is odd then either H_2 is trivial (and so G fixes a line), or G is dihedral with Goursat-data $(\mathsf{D}_s(q), \{\pm 1\}, 1, \psi)$ or $(\mathsf{D}_{ns}(q), \{\pm 1\}, 1, \psi)$.

Proof. We only sketch the proof since it comes down an exercise in matrix manipulation. Suppose H_2 is non-trivial. Because H_1 normalizes a split Cartan subgroup, its maximal order is $2(3q-4)$ in the split case and $2(q+1)$ in the non-split case, by combining Observations (2) and (4) above. When q is odd, in order to create a subgroup H_1 (and not merely a subset) satisfying the hypotheses of the Proposition, matrix manipulation shows that H_1 must be a subgroup of $\mathsf{D}_s(q)$ in the split case and $\mathsf{D}_{ns}(q)$ in the non-split case and $H_2 = \{\pm 1\}$. When q is even, $\#H_2$ is odd and so at least $2/3$ of the elements of H_1 must have a fixed point and H_1 must admit a cyclic odd-order quotient with all non-kernel elements having a fixed point. No such subgroup exists. $\square$

We conclude this section by analyzing the subgroups of $\mathrm{GL}_2(q)$ with projective image Alt_4, Sym_4, and Alt_5. Let $PH \in \{\mathrm{Alt}_4, \mathrm{Sym}_4, \mathrm{Alt}_5\}$. The central extensions of PH are classified by the Schur multiplier. Neither Alt_4 nor Alt_5 has an ordinary 2-dimensional irreducible representation, hence any central extension $H \subset \mathrm{GL}_2(q)$ of PH must be non-trivial for these groups. When $PH = \mathrm{Sym}_4$, the trivial central extensions of Sym_4 do occur as subgroups of $\mathrm{GL}_2(q)$.

In all cases, the Schur multiplier of PH has exponent 2, hence any central extension has the form $2.PH$ times a group of scalar matrices, the order of which can be deduced from [16, Lemma 3.21]. The isomorphism types of $2.PH$ that occur as subgroups of $\mathrm{GL}_2(q)$ are as follows

$$2.\mathrm{Alt}_4 \simeq \mathrm{SL}_2(3)$$
$$2.\mathrm{Alt}_5 \simeq \mathrm{SL}_2(5)$$
$$2.\mathrm{Sym}_4 \simeq \begin{cases} 2_1.\mathrm{Sym}_4 \simeq \mathrm{Alt}_4 \rtimes C_4 \\ 2_2.\mathrm{Sym}_4 \simeq \mathrm{SL}_2(3).C_2 \text{ (nonsplit)} \\ 2_3.\mathrm{Sym}_4 \simeq \mathrm{GL}_2(3) \\ 2_4.\mathrm{Sym}_4 \simeq C_2 \times \mathrm{Sym}_4 \end{cases}$$

The complexity of the groups $2.\mathrm{Sym}_4$ is due to the fact that the Schur multiplier $H^2(\mathrm{Sym}_4, C_2) \simeq C_2 \times C_2$. We now investigate the groups H for their fixed-point properties.

2.1. **Projective Image Alt_4.** Let q be coprime to 6. Let H be a subgroup of $\mathrm{GL}_2(q)$ such that $PH \simeq \mathrm{Alt}_4$. There are three inequivalent absolutely irreducible ordinary representations σ_1, σ_2, and σ_3 of $\mathrm{SL}_2(3)$, with character values as follows (ω denotes a fixed primitive 3rd root of unity):

Class	1	2	$3A$	$3B$	4	$6A$	$6B$
χ_1	2	-2	-1	-1	0	1	1
χ_2	2	-2	$1+\omega$	$-\omega$	0	ω	$-1-\omega$
χ_3	2	-2	$-\omega$	$1+\omega$	0	$-1-\omega$	ω

The representation σ_1 is defined over $\mathbf{Z}$ and $\sigma_1(\mathrm{SL}_2(3)) \subset \mathrm{SL}_2(q)$, while $\sigma_2(\mathrm{SL}_2(3))$ and $\sigma_3(\mathrm{SL}_2(3))$ define subgroups of $\mathrm{GL}_2(q)$ when $q \equiv 1 \pmod 3$. In any of the three representations, the only class with a fixed point is the identity.

Lemma 2.1.1. *Let H be a maximal preimage of Alt_4 in $\mathrm{GL}_2(q)$. Let $G \subset \mathrm{GL}_2(q) \times \mathrm{GL}_1(q)$ be a fixed point subgroup of $\mathrm{GL}_3(q)$ with Goursat-tuple (H_1, H_2, H_3, ψ). Suppose H_1 is an irreducible subgroup of H. Then H_2 is trivial.*

Proof. If H is a maximal preimage of Alt_4, then it is a product of scalar matrices and the non-trivial extension $2.\mathrm{Alt}_4$ of Alt_4. Since all elements of H without a fixed point must belong to $\ker\psi$, it follows that $2.\mathrm{Alt}_4$ is a subgroup of $\ker\psi$ as well as the group of scalar matrices. Thus ψ is the trivial homomorphism, whence H_2 is trivial. □

Now we consider the special cases of modular characteristic. If q is even then any group H such that $PH = \mathrm{Alt}_4$ is not irreducible in $\mathrm{GL}_2(q)$ [11, Lemma 6.1]. If q is a power of 3 then the isomorphism $2.\mathrm{Alt}_4 \simeq \mathrm{SL}_2(3)$ shows that $2.\mathrm{Alt}_4$ occurs naturally as a subfield subgroup of $\mathrm{GL}_2(q)$. The same counting argument of Lemma 2.0.5 shows that more than half the elements of H do not have a fixed point, and hence H cannot give rise to a non-trivial fixed-point subgroup of $\mathrm{GL}_3(q)$.

2.2. **Projective Image** Alt_5**.** Let q be coprime to 30. Then there are two inequivalent ordinary absolutely irreducible representations σ_1 and σ_2 of $\mathrm{SL}_2(5)$, with the following character data.

class	1	2	3	4	$5A$	$5B$	6	$10A$	$10B$
χ_1	2	-2	-1	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_2	2	-2	-1	0	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

In both representations the only element with a fixed point is the identity.

Lemma 2.2.1. *Let H be a maximal preimage of* Alt_5 *in* $\mathrm{GL}_2(q)$*. Let $G \subset \mathrm{GL}_2(q) \times \mathrm{GL}_1(q)$ be a fixed point subgroup of* $\mathrm{GL}_3(q)$ *with Goursat-tuple (H_1, H_2, H_3, ψ). Suppose H_1 is an irreducible subgroup of H. Then H_2 is trivial.*

Proof. The proof is identical to that of Lemma 2.1.1. □

In modular characteristic, if q is even then the isomorphism $\mathrm{Alt}_5 \simeq \mathrm{SL}_2(4) = \mathrm{PSL}_2(4)$ shows that Alt_5 occurs as a subfield subgroup of $\mathrm{SL}_2(q)$ (once $q > 4$). The same counting argument of Lemma 2.0.5 shows that more than half the elements of H do not have a fixed point, and hence H cannot give rise to a non-trivial fixed-point subgroup of $\mathrm{GL}_3(q)$. The same argument applies when q is a power of 5 via the isomorphism $2.\mathrm{Alt}_5 \simeq \mathrm{SL}_2(5)$.

If q is a power of 3 then $2.\mathrm{Alt}_5$ only occurs as a subgroup of $\mathrm{GL}_2(q)$ when q is an *even* power of 3, since it is required that $5 \in (\mathbf{F}_q^\times)^2$. And if q is an even power of 3, then $\mathbf{F}_q$ contains $\mathbf{F}_9$, so it suffices to work in $\mathrm{GL}_2(9)$. In $\mathrm{GL}_2(9)$, the group $2.\mathrm{Alt}_5$ has 15 elements without a fixed point, hence $\ker\psi = 2.\mathrm{Alt}_5$ and so H_2 is trivial.

2.3. **Projective Image** Sym_4**.** Let q be coprime to 6. We consider the four groups $2_i.\mathrm{Sym}_4$ separately for $i = 1, 2, 3, 4$. The group $2_1.\mathrm{Sym}_4$ has no faithful irreducible degree 2 ordinary representations and we do not consider unfaithful representations in this analysis for fixed-point subgroups.

The group $2_2.\mathrm{Sym}_4$ has two faithful irreducible ordinary degree-2 representations σ_1, σ_2 with character data:

Class	1	2	3	$4A$	$4B$	6	$8A$	$8B$
χ_1	2	-2	-1	0	0	1	$\sqrt{2}$	$-\sqrt{2}$
χ_2	2	-2	-1	0	0	1	$-\sqrt{2}$	$\sqrt{2}$

In the representations σ_1 and σ_2, the group $2_2.\mathrm{Sym}_4$ has no non-trivial elements with a fixed point.

Lemma 2.3.1. *Let H be a maximal preimage of* Sym_4 *in* $\mathrm{GL}_2(q)$ *that contains* $2_2.\mathrm{Sym}_4$*. Let $G \subset \mathrm{GL}_2(q) \times \mathrm{GL}_1(q)$ be a fixed point subgroup of* $\mathrm{GL}_3(q)$ *with Goursat-tuple (H_1, H_2, H_3, ψ). Suppose H_1 is an irreducible subgroup of H. Then H_2 is trivial.*

Proof. The proof is identical to that of Lemma 2.1.1. □

The group $2_3.\mathrm{Sym}_4$ has two faithful irreducible ordinary degree-2 representations σ_1, σ_2 with character data:

Class	1	$2A$	$2B$	3	4	6	$8A$	$8B$
χ_1	2	-2	0	-1	0	1	$-\sqrt{-2}$	$\sqrt{-2}$
χ_2	2	-2	0	-1	0	1	$\sqrt{-2}$	$-\sqrt{-2}$

In both representations there are exactly 35 elements without a fixed point.

Lemma 2.3.2. *Let H be a maximal preimage of Sym_4 in $\mathrm{GL}_2(q)$ that contains $2_3.\mathrm{Sym}_4$. Let $G \subset \mathrm{GL}_2(q) \times \mathrm{GL}_1(q)$ be a fixed point subgroup of $\mathrm{GL}_3(q)$ with Goursat-tuple (H_1, H_2, H_3, ψ). Suppose H_1 is an irreducible subgroup of H. Then H_2 is trivial.*

Proof. Any element of $2_3.\mathrm{Sym}_4$ without a fixed point belongs to $\ker\psi$, whence $\ker\psi$ contains $2_3.\mathrm{Sym}_4$ and the scalar matrices. Thus $\ker\psi = H_1$ and so H_2 is trivial. □

The group $2_4.\mathrm{Sym}_4$ has two unfaithful irreducible degree-2 representations and we do not consider unfaithful representations in this analysis.

We finish this section by considering the groups $2_2.\mathrm{Sym}_4$ and $2_3.\mathrm{Sym}_4$ in modular characteristic. If q is even then neither $2_2.\mathrm{Sym}_4$ nor $2_3.\mathrm{Sym}_4$ is irreducible [11, Lemma 6.1]. If q is a power of 3 then the isomorphism $2_3.\mathrm{Sym}_4 \simeq \mathrm{GL}_2(3)$ shows that H_1 occurs as a subfield subgroup of $\mathrm{GL}_2(q)$. The same counting argument of Lemma 2.0.5 shows that more than half the elements of H do not have a fixed point, and hence H cannot give rise to a non-trivial fixed-point subgroup of $\mathrm{GL}_3(q)$. Finally, $2_2.\mathrm{Sym}_4$ contains $\mathrm{SL}_2(3)$ as index-2 subgroup and the full group $2_2.\mathrm{Sym}_4$ is contained in $\mathrm{GL}_2(9)$. Again, the same counting argument of Lemma 2.0.5 shows that there are no non-trivial fixed-point subgroups in this case.

3. The Irreducible Fixed-Point Subgroups of $\mathrm{GL}_3(q)$

In this section we complete the proof of Theorem 1.2.1 in a case-by-case analysis based on the maximal subgroup classes.

3.1. **Subgroups of Type $\mathcal{C}_2$.** The maximal subgroup of $\mathrm{GL}_3(q)$ of type $\mathcal{C}_2$ is isomorphic to $\mathrm{GL}_1(q) \wr \mathrm{Sym}_3$ as long as $q \geq 5$. If G is a subgroup of $\mathrm{GL}_1(q) \wr \mathrm{Sym}_3$, then G fits into a split short exact sequence

$$1 \to G_0 \to G \to P \to 1,$$

where G_0 is a subgroup of $\mathrm{GL}_1(q)^3$ and P is subgroup of Sym_3. If G is a fixed-point subgroup of $\mathrm{GL}_1(q) \wr \mathrm{Sym}_3$, then so is G_0. By Lemma 2.0.1, either G_0 fixes a line or $G_0 \simeq C_2 \times C_2$.

Lemma 3.1.1. *Suppose G_0 fixes a line. Then any lift G of G_0 to $\mathrm{GL}_1(q) \wr \mathrm{Sym}_3$ fixes a line as well. Therefore there are no irreducible fixed-point subgroups of Type $\mathcal{C}_2$ when q is even, or when G_0 fixes a line.*

Proof. If G_0 fixes a line then consider the permutation group P. If P is trivial or has order 2, then G is reducible and fixes a line. So we assume P contains a 3-cycle. Choosing a basis with respect to which G_0 fixes the first coordinate, we see that G contains matrices of the form

$$M(\alpha,\beta) \stackrel{\text{def}}{=} \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{smallmatrix}\right) \qquad \text{and} \qquad s \stackrel{\text{def}}{=} \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right).$$

In order for a matrix of the form $M(\alpha,\beta)s$ to have a fixed point, we must take $\alpha\beta = 1$. Continuing, the product $M(\alpha,\alpha^{-1})s^2M(\alpha,\alpha^{-1})^2s$ has a fixed point if and only if $\alpha \in \{\pm 1\}$. Finally,

$$sM(-1,-1)s^2 = \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right),$$

which shows G_0 can only contain the identity matrix $M(1,1)$. But the full permutation group Sym_3 fixes a line in this representation. Thus, there are no irreducible fixed-point subgroups G such that G_0 fixes a line. □

Lemma 3.1.2. *Let q be odd. Suppose G is an irreducible fixed-point subgroup of $\mathrm{GL}_1(q) \wr \mathrm{Sym}_3$. Then G is isomorphic to Alt_4 or Sym_4.*

Proof. By Lemmas 2.0.1 and 3.1.1, we can assume $G_0 \simeq C_2 \times C_2$, given explicitly by

$$\left\{ \begin{pmatrix} \epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_1\epsilon_2 \end{pmatrix} : \epsilon_i \in \{\pm 1\} \right\}.$$

An easy calculation shows that the full wreath product $(C_2 \times C_2) \rtimes \mathrm{Sym}_3 \simeq \mathrm{Sym}_4$ is an irreducible fixed-point subgroup of $\mathrm{GL}_1(q) \wr S_3$, as well as its subgroup Alt_4. □

Remarks. It remains to discuss what happens for $q < 5$. When $q = 2$, the group of type $\mathcal{C}_2$ is not maximal in $\mathrm{SL}_3(q)$, but belongs to the reducible maximal subgroup class of type $\mathcal{C}_1$. When $q = 3$ the classes $\mathcal{C}_2$ and $\mathcal{C}_8$ coincide, in light of the isomorphism $\mathrm{SO}_3(3) \simeq \mathrm{Sym}_4$, so this group can be considered as an irreducible fixed-point subgroup of Type $\mathcal{C}_8$ as well. When $q = 4$, the group $\mathrm{GL}_1(4) \wr \mathrm{Sym}_3$ is not a maximal subgroup of $\mathrm{GL}_3(4)$ [1, Prop. 2.3.6].

3.2. **Subgroups of Type $\mathcal{C}_3$.** There are no irreducible fixed-point subgroups of $\mathrm{GL}_3(q)$ in this class, as we now show. The maximal subgroup of $\mathrm{GL}_3(q)$ in this class is isomorphic to $\mathrm{GL}_1(q^3).3$, with outer automorphisms given by the Galois group $\mathrm{Gal}(\mathbf{F}_{q^3}/\mathbf{F}_q)$.

Remark. When $q = 4$ the restriction of $\mathrm{GL}_1(4).3$ to $\mathrm{SL}_3(4)$ is not maximal (see Table 1.3) in $\mathrm{SL}_3(4)$, but $\mathrm{GL}_1(4).3$ *is* maximal in $\mathrm{GL}_3(4)$.

Let $G \subset \mathrm{GL}_1(q^3).3$ be a fixed-point subgroup. Then G fits into the short exact sequence

$$1 \to N \to G \to Q \to 1,$$

where N is a cyclic group of order dividing q^3-1 and Q is either trivial or isomorphic to C_3.

Let g be a generator for the group $\mathrm{GL}_1(q^3)$ and σ a generator of $\mathrm{Gal}(\mathbf{F}_{q^3}/\mathbf{F}_q)$; in this representation the eigenvalues of g have the form $\gamma, \gamma^\sigma, \gamma^{\sigma^2}$. Because $\mathrm{GL}_1(q^3)$ is cyclic, and because the eigenvalues of any power of g are powers of γ, γ^σ, and γ^{σ^2}, it follows that the only element of $\mathrm{GL}_1(q^3)$ with a fixed point is the identity. The trivial group lifts to a cyclic group of order 3 inside $\mathrm{GL}_1(q^3).3$, and every element of such a C_3 has a fixed point, but the group is not irreducible.

3.3. **Subgroups of Type $\mathcal{C}_5$.** These are the field-restriction subgroups of $\mathrm{GL}_3(q)$. That is, if we can write $q = q_0^r$, then $\mathrm{GL}_3(q_0)$ is naturally a subgroup of $\mathrm{GL}_3(q)$. When r is prime the group generated by $\mathrm{GL}_3(q_0)$ and the center $Z(q)$ of $\mathrm{GL}_3(q)$ is the maximal subgroup of $\mathrm{GL}_3(q)$ of type $\mathcal{C}_5$.

Suppose r is prime and let $\mathcal{G} = \langle \mathrm{GL}_3(q_0), Z(q) \rangle$. Let G be an irreducible fixed point subgroup of $\mathcal{G}$. Because no nontrivial element of $Z(q)$ has a fixed point, it follows that G is an irreducible fixed-point subgroup of $\mathrm{GL}_3(q_0)$. Since we seek to classify the subgroups of Type $\mathcal{C}_5$, we may assume (by descent) that G is an irreducible fixed-point subgroup of $\mathrm{GL}_3(p)$, hence lies in a subgroup class other than $\mathcal{C}_5$. Therefore, the class $\mathcal{C}_5$ contains no irreducible fixed-point subgroups of $\mathrm{GL}_3(q)$ that are not already contained in another class.

3.4. **Subgroups of Type $\mathcal{C}_6$.** There are no irreducible, fixed-point subgroups of $\mathrm{GL}_3(q)$ in this class, as we now show. We first classify the fixed point subgroups of $\mathrm{SL}_3(q)$ in this class and then lift them to $\mathrm{GL}_3(q)$. Recall from Table 1.3 that $q = p \equiv 1 \pmod 3$.

Lemma 3.4.1. *Let G be a nontrivial fixed-point subgroup of $3_+^{1+2}.Q_8.\frac{(q-1,9)}{3} \subset \mathrm{SL}_3(q)$. Then $G \simeq Q_8$ or $G \simeq C_3$.*

Proof. This is a finite computation, easily performed in Magma, and we omit the details. The result is that there are, up to isomorphism, two fixed-point subgroups of $3_+^{1+2}.Q_8.\frac{(q-1,9)}{3}$: a cyclic group of order 3, and Q_8. □

Lemma 3.4.2. *There is no irreducible fixed-point subgroup of $\mathrm{GL}_3(q)$ of Type $\mathcal{C}_6$ that restricts to Q_8.*

Proof. The group Q_8 is normal in any subgroup of $\mathrm{GL}_3(q)$ that restricts to $Q_8 \subset \mathrm{SL}_3(q)$. The three-dimensional representation of Q_8 decomposes into a 2-dimensional factor and a 1-dimensional factor. By Clifford's theorem, any lift of Q_8 to $\mathrm{GL}_3(q)$ retains this decomposition, whence there are no irreducible subgroups of $\mathrm{GL}_3(q)$ restricting to Q_8. □

Lemma 3.4.3. *There is no irreducible fixed-point subgroup of $\mathrm{GL}_3(q)$ of Type $\mathcal{C}_6$ that restricts to the fixed-point $C_3 \subset 3_+^{1+2}.Q_8.\frac{(q-1,9)}{3} \subset \mathrm{SL}_3(q)$.*

Proof. Because $q \equiv 1 \pmod 3$, the representation of the fixed-point C_3 is completely reducible and decomposes into three 1-dimensional representations, one of which is trivial. By Clifford's theorem, the representation of any subgroup of $\mathrm{GL}_3(q)$ restricting to C_3 is either a sum of three one-dimensional representations, or is irreducible. If it were irreducible, the three one-dimensional representations of C_3 (upon restriction) would be conjugate. Since only one of the three is trivial, and a non-trivial representation cannot be conjugate to a trivial, it follows that the representation of any overgroup $C_3.m$ of C_3 is not irreducible. This proves the lemma. □

3.5. **Subgroups of Type $\mathcal{C}_8$.** There are two isomorphism types of maximal subgroups of $\mathrm{SL}_3(q)$ of Type $\mathcal{C}_8$, namely $d \times \mathrm{SO}_3(q)$ and $(q_0-1,3) \times \mathrm{SU}_3(q_0)$ if $q = q_0^2$. Moreover, this class contains no novel subgroups. We first consider the case of $d \times \mathrm{SO}_3(q)$.

The group $\mathrm{SO}_3(q)$ is a fixed-point group [8, Prop. 6.10] and is irreducible in odd characteristic. If $q \not\equiv 1 \pmod 3$ then $\mathrm{SO}_3(q)$ is maximal in $\mathrm{SL}_3(q)$, while if $q \equiv 1 \pmod 3$ then $d \times \mathrm{SO}_3(q)$ is maximal, with d a scalar group of order 3. Because d is scalar, the maximal fixed-point subgroup of $d \times \mathrm{SO}_3(q)$ is $\mathrm{SO}_3(q)$. Thus, for all q, the maximal fixed-point subgroup of $\mathrm{SL}_3(q)$ of Type $\mathcal{C}_8$ is $\mathrm{SO}_3(q)$.

It remains to determine whether there exist fixed-point groups H that fit into the sequence

$$\mathrm{SO}_3(q) \subset H \subset \mathrm{GL}_3(q)$$

of proper containments. The groups of Type $\mathcal{C}_8$ are *scalar-normalizing* [1, Def. 4.4.4] in the sense that any such group H has the presentation $\mathrm{SO}_3(q)Z$, where Z is a subgroup of the scalars of $\mathrm{GL}_3(q)$. Thus, any overgroup H properly containing $\mathrm{SO}_3(q)$

is necessarily not a fixed-point group (some non-trivial element of Z multiplies the identity of $\mathrm{SO}_3(q)$). We therefore have the following result.

Lemma 3.5.1. *Let q be a power of an odd prime. The maximal irreducible fixed point subgroup of $\mathrm{GL}_3(q)$ containing $\mathrm{SO}_3(q)$ is $\mathrm{SO}_3(q)$.*

Next we consider the case of the subgroup $(q_0-1,3)\times\mathrm{SU}_3(q_0)$ of $\mathrm{SL}_3(q)$ and more generally the subgroup $\mathrm{GU}_3(q_0)$ of $\mathrm{GL}_3(q)$. We will show that there are no additional irreducible fixed-point subgroups arising in this class that have not already been classified. First, the group $\mathrm{GU}_3(q_0)$ is not itself a fixed-point group hence any irreducible fixed-point subgroup must lie in one of its maximal subgroups. First we list the non-parabolic maximal subgroups of $\mathrm{GU}_3(q_0)$ and $\mathrm{SU}_3(q_0)$.

- Type $\mathcal{C}_2$: $\mathrm{GU}_1(q_0)\wr\mathrm{Sym}_3$ is the maximal subgroup of $\mathrm{GU}_3(q_0)$ of Type $\mathcal{C}_2$. The same argument as in Lemma 3.1.2 applies and shows the maximal irreducible fixed-point subgroup is isomorphic to Sym_4.
- Type $\mathcal{C}_3$: $\mathrm{GU}_1(q_0^3).3$ is the maximal subgroup of $\mathrm{GU}_3(q_0)$ of Type $\mathcal{C}_3$. The same argument as in Section 3.2 applies and shows there are no irreducible fixed point subgroups in this class.
- Type $\mathcal{C}_5$: There are two subgroups of $\mathrm{SU}_3(q_0)$ in this class: $\mathrm{SU}_3(q_1).\left(\frac{q+1}{q_1+1},3\right)$ (if $q_0=q_1^r$ for prime r) and $\mathrm{SO}_3(q)$.
- Type $\mathcal{C}_6$: There is one maximal subgroup of $\mathrm{SU}_3(q_0)$ in this class: $3_+^{1+2}.Q_8.\frac{(q+1,9)}{3}$.
- Type $\mathcal{S}$: There are four isomorphism classes of subgroups of Type $\mathcal{S}$ of $\mathrm{SU}_3(q)$:
  - $d\times\mathrm{L}_2(7)$ (d conjugates; $q=p\equiv 3,5,6 \pmod 7$, $q\neq 5$), where $d=\gcd(q+1,3)$
  - $3^{\cdot}\,\mathrm{Alt}_6$ (3 conjugates; $q=p\equiv 11,14 \pmod{15}$)
  - $3^{\cdot}\,\mathrm{Alt}_6^{\cdot}\,2_3$ (3 conjugates; $q=5$)
  - $3^{\cdot}\,\mathrm{Alt}_7$ (3 conjugates; $q=5$)

Many of the same arguments as in the previous sections apply here as well. In particular:

- In Class $\mathcal{C}_5$ the same descent argument as in Section 3.3 shows that it is enough to classify the maximal subgroups of $\mathrm{SU}_3(p)$.
- In Class $\mathcal{C}_6$ the same argument as in Section 3.4 applies as well: the only fixed-point subgroups of $3_+^{1+2}.Q_8.\frac{(q+1,9)}{3}$ are C_3 and Q_8 and, by the same Clifford's theorem argument, any lift to $\mathrm{GU}_3(q)$ is reducible.

It remains to analyze the subgroups of Type $\mathcal{S}$. We delay our treatment of $d\times\mathrm{L}_2(7)$ and $3^{\cdot}\,\mathrm{Alt}_6$ until the next section so that we can give a unified treatment of these two groups; they occur as maximal subgroups of $\mathrm{SL}_3(q)$ for certain q and $\mathrm{SU}_3(q)$ for others. We now consider the two subgroups $3^{\cdot}\,\mathrm{Alt}_6\,.2_3$ and $3^{\cdot}\,\mathrm{Alt}_7$ of $\mathrm{SU}_3(5)$.

In both cases, we search in the subgroup lattices of $3^{\cdot}\,\mathrm{Alt}_6\,.2_3$ and $3^{\cdot}\,\mathrm{Alt}_7$ for fixed-point subgroups. One can check that the conjugacy classes of elements of order 1, 2, 5 have fixed points, while some of the classes of order 3, 4, and 6 do as well. The result of the search is that the following are the isomorphism types of fixed-point subgroups of $3^{\cdot}\,\mathrm{Alt}_6\,.2_3$ and $3^{\cdot}\,\mathrm{Alt}_7$:

$$\{C_j\}_{j=1,\dots,5},\ F_{20}.$$

Setting aside the group C_5, each of the fixed-point groups listed above is reducible and the semisimplification of each representation consists of three 1-dimensional representations, one of which is trivial. The identical Clifford's theorem argument of Section 3.4 shows that none of these groups lifts to an irreducible fixed-point subgroup of $\mathrm{GL}_3(25)$. For the group C_5, the semisimplification consists of three trivial representations, so the Clifford's theorem argument does not immediately rule out an irreducible fixed-point subgroup of $\mathrm{GL}_3(25)$. However, a search for all subgroups of $\mathrm{GL}_3(25)$ of the form $C_5.m$ that are irreducible fixed-point subgroups reveals none. All computations for this section were performed in Magma.

3.6. **Subgroups of Type $\mathcal{S}$.** We complete the classification of irreducible fixed-point subgroups of $\mathrm{GL}_3(q)$ with the groups of Type $\mathcal{S}$, and we incorporate two of the type $\mathcal{S}$ subgroups of $\mathrm{GU}_3(q)$ into this section as well. We recall the conditions under which each of these groups occur.

Subgroup of $\mathrm{SL}_3(q)$	Conditions
$(q-1,3)\times \mathrm{L}_2(7)$	$q=p\equiv 1,2,4 \pmod 7$, $q\neq 2$
$3^{\cdot}\,\mathrm{Alt}_6$	$q=p\equiv 1,4 \pmod{15}$ $q=p^2$, $p\equiv 2,3 \pmod 5$, $p\neq 3$

Subgroup of $\mathrm{SU}_3(q)$	Conditions
$(q+1,3)\times \mathrm{L}_2(7)$	$q=p\equiv 3,5,6 \pmod 7$, $q\neq 5$
$3^{\cdot}\,\mathrm{Alt}_6$	$q=p\equiv 11,14 \pmod{15}$

The simple group $\mathrm{L}_2(7)$ of order 168 has an absolutely irreducible 3-dimensional representation over $\mathbf{F}_q$ when $-7\in(\mathbf{F}_q^\times)^2$ and the group $3^{\cdot}\,\mathrm{Alt}_6$ has an absolutely irreducible representation over $\mathbf{F}_q$ when $-3,5\in(\mathbf{F}_q^\times)^2$. The conditions on q reflect these requirements. We start with $d\times \mathrm{L}_2(7)$.

Lemma 3.6.1. *Let G be a maximal, irreducible, fixed-point subgroup of $(q-1,3)\times \mathrm{L}_2(7)\subset \mathrm{SL}_3(q)$ or $(q+1,3)\times \mathrm{L}_2(7)\subset \mathrm{SU}_3(q)$, subject to the conditions on q in the tables above. Then $G\simeq \mathrm{Sym}_4$.*

Proof. In either case, a fixed-point subgroup intersects the center trivially, hence $G\subset \mathrm{L}_2(7)$. The maximal subgroup Sym_4 of $\mathrm{L}_2(7)$ is an absolutely irreducible fixed-point subgroup, so it remains to show that no element of order 7 has a fixed point for any allowable q.

Fix a primitive 7th root of unity $\omega\in\overline{\mathbf{F}}_q$. Then the characteristic polynomial on either class of order 7, evaluated at 1 is given by

$$\frac{4}{3}\omega(\omega-1)(\omega^4+2\omega^3+\omega^2+2\omega+1)\neq 0.$$

The inequality follows from the observation that if $\omega^4+2\omega^3+\omega^2+2\omega+1=0$, then the resultant

$$\mathrm{Res}(\omega^4+2\omega^3+\omega^2+2\omega+1,\omega^6+\omega^5+\omega^4+\omega^3+\omega^2+\omega+1)=7^2=0,$$

which is impossible since q is coprime to 7. $\square$

Lemma 3.6.2. *Let G be a maximal, irreducible, fixed-point subgroup of $3^{\cdot}\,\mathrm{Alt}_6\subset \mathrm{SL}_3(q)$. Then $G\simeq \mathrm{Alt}_4$ or Alt_5.*

Proof. Since $3^{\cdot}\,\mathrm{Alt}_6$ contains the center of $\mathrm{SL}_3(q)$, any fixed-point subgroup must be proper. There are five maximal subgroups of $3^{\cdot}\,\mathrm{Alt}_6$:

$$3_+^{1+2}.4,\qquad d\times \mathrm{Alt}_4\ \text{(two copies)},\qquad d\times \mathrm{Alt}_5\ \text{(two copies)}.$$

The group $3_+^{1+2}.4$ was analyzed previously in Section 3.4 and does not possess any irreducible fixed-point subgroups. On the other hand, one can check that the (centerless) groups Alt_4 and Alt_5 of the remaining cases are each irreducible, fixed-point subgroups. □

The groups of Type $\mathcal{S}$ are scalar normalizing [1, 4.5.2] and so there are no irreducible, fixed-point overgroups $H \subset \mathrm{GL}_3(q)$ that properly contain the Alt_4, Sym_4, or Alt_5 of the Lemmas above. This completes the classification of nontrivial fixed-point subgroups of $\mathrm{GL}_3(q)$ stated in Theorem 1.2.1.

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References

[1] J.N. Bray, D.F. Holt, C.M. Roney-Dougal. The maximal subgroups of the low-dimensional finite classical groups. With a foreword by Martin Liebeck. London Mathematical Society Lecture Note Series, 407. Cambridge University Press, Cambridge, 2013.
[2] G. Butler and J. McKay, The transitive groups of degree up to eleven. *Comm. Algebra.* **11** (1983), 863-911
[3] J. Cullinan. Local-global properties of torsion points on three-dimensional abelian varieties. *J. Algebra.* **311** (2007), 736-774
[4] J. Cullinan, Points of small order on three-dimensional abelian varieties; with an appendix by Yuri Zarhin. J. Algebra **324** (2010), no. 3, 565-577.
[5] J. Cullinan, A computational approach to the 2-torsion structure of abelian threefolds. *Math. of Comp.* **78** (2009), 1825-1836.
[6] J. Cullinan, Symplectic stabilizers with applications to abelian varieties. Int. J. Number Theory 8 (2012), no. 2, 321-334.
[7] L. Dieulefait, N. Vila, On the images of modular and geometric three-dimensional Galois representations. Amer. J. Math. **126** (2004), no. 2, 335-361.
[8] L.C. Grove, *Classical Groups and Geometric Algebra*, American Mathematical Society, 2002
[9] R.W. Hartley, Determination of the ternary collineation groups whose coefficients lie in the $GF(2^n)$. Ann. of Math. **27** (1925/6), 140-158.
[10] N.M. Katz, Galois properties of torsion points on abelian varieties. *Invent. math.* **62** (1981), 481-502
[11] C. Khare, J. Wintenberger, Jean-Pierre Serre's modularity conjecture. I. Invent. Math. 178 (2009), no. 3, 485-504.
[12] S. Lang. Introduction to modular forms. Grundlehren der mathematischen Wissenschaften, No. 222. Springer-Verlag, Berlin-New York, 1976.
[13] H.H. Mitchell, Determination of the ordinary and modular ternary linear groups. Trans. Amer. Math. Soc. **12** (1911), 207-242.
[14] J-P. Serre, *Abelian ℓ-Adic Representations and Elliptic Curves*, A.K. Peters, Ltd., 1998
[15] J-P. Serre. Letter to J. Cullinan, 2006
[16] A. Sutherland, Computing images of Galois representations attached to elliptic curves. Forum Math. Sigma **4** (2016), e4, 79 pp.