ON THE PLESKEN LIE ALGEBRA DEFINED OVER A FINITE FIELD

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Abstract. Let G be a finite group. The Plesken Lie algebra is a subalgebra of the complex group algebra $\mathbf{C}[G]$ and admits a direct-sum decomposition into simple Lie algebras. We describe finite-field versions of the Plesken Lie algebra via traditional and computational methods. The computations motivate our conjectures on the general structure of the modular Plesken Lie algebra.

1. Introduction

Let G be a finite group, k a field, and $k[G]$ the group algebra of G. Then $k[G]$ assumes the structure of a Lie algebra via the bracket (commutator) operation. This leads to the natural question: what is the Lie algebra structure of $k[G]$? If k is a characteristic-0 splitting field of G (for example, k algebraically closed), then ordinary representation theory answers this question – $k[G]$ is a direct sum of matrix algebras, the summands endowed with the natural Lie algebra structure of $\mathfrak{gl}(V)$, where V is an irreducible representation of G. On the other hand, if the characteristic of k divides the order of G then the question is much more subtle. For instance, in [14], the authors give necessary and sufficient conditions for $k[G]$ to be nilpotent and for $k[G]$ to be solvable. Their results are for arbitrary characteristic, but the arguments where $\operatorname{char} k \mid \#G$ are more intricate. In a different (but related) direction, the "Lie-representations" of finite groups in positive characteristic are an active area of research (see, for example, [3]).

In this paper we study the structure of a certain Lie subalgebra of $k[G]$ in positive characteristic. This Lie algebra was introduced in [4], and, in a more general setting, in [12]. We follow the notation and conventions of [4] and set

$$\mathscr{L}(G) = \operatorname{span}_k\{g - g^{-1} \mid g \in G\}$$

to be the linear subspace of $k[G]$ spanned by the $g - g^{-1}$; one easily checks that $\mathscr{L}(G)$ is a Lie subalgebra of $k[G]$. Following [4], we call this the *Plesken* Lie algebra of G.

Given that the structure of finite-dimensional Lie algebras over $\mathbf{C}$ is well known, it is natural to try to determine the structure of $\mathscr{L}(G)$ when $k = \mathbf{C}$. The main result of [4] is a structure theorem relating the constituents of the direct-sum decomposition of $\mathscr{L}(G)$ to the complex-irreducible representations of the finite group G. However, in positive characteristic one would expect the Lie algebra to have a different flavor, especially when the characteristic divides the order of G. This is the starting point for our paper, and for the purposes of computation we focus on finite fields of prime order.

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Let p be a prime number and set $k = \mathbf{F}_p$. There are two reasonable ways to define the Plesken Lie algebra over $\mathbf{F}_p$. One can start with the complex Lie algebra $\mathscr{L}(G)$ equipped with a choice of Chevalley basis and construct the lattice $\mathscr{L}_{\mathbf{Z}}(G) \subset \mathscr{L}(G)$ relative to this basis. Since $\mathscr{L}(G)$ is a direct sum of complex Lie algebras, this direct sum will be preserved under the reduction of $\mathscr{L}_{\mathbf{Z}}(G)$ modulo p via

$$\mathscr{L}_p(G) := \mathscr{L}_{\mathbf{Z}}(G) \otimes \mathbf{F}_p \simeq \mathscr{L}_{\mathbf{Z}}(G)/p\mathscr{L}_{\mathbf{Z}}(G).$$

In this way the main result of [4] is preserved via "reduction modulo p" (see below for details).

Alternatively, one can start with the group algebra $\mathbf{F}_p[G]$ and construct the linear span of the $\widehat{g} \in \mathbf{F}_p[G]$; we will denote this Lie-subalgebra of $\mathbf{F}_p[G]$ by $\Lambda_p(G)$. One can then ask how $\mathscr{L}_p(G)$ and $\Lambda_p(G)$ are related. As one would expect, the answer depends on whether p divides the order of G. In the ordinary case, any difference between $\mathscr{L}_p(G)$ and $\Lambda_p(G)$ is due to the fact that $\mathbf{F}_p$ may not be a splitting field for the representations of G. However, since G is finite, there will be a positive (computable) density of primes p for which the direct-sum constituents of $\Lambda_p(G)$ mirror those of $\mathscr{L}_p(G)$. Thus, [4, thm. 5.1] holds *mutatis mutandis* for the $\mathbf{F}_p$-Lie algebra $\Lambda_p(G)$ when $\mathbf{F}_p$ is a splitting field for G; see Section 2 for details.

If $p \mid \#G$, then the situation is far more complicated. The group algebra $\mathbf{F}_p[G]$ is not semisimple, hence its simple composition factors as a $\mathbf{F}_p[G]$-module do not coincide with its direct summands, which is the basis for the structure theorem in the semisimple case. Thus, the Lie algebra structure of $\Lambda_p(G)$ should be different in modular characteristic. To start, $\mathbf{F}_p[G]$ decomposes as a sum of blocks, each of which is an $\mathbf{F}_p$ Lie algebra. Thus, one could (in theory) get information on the dimensions of the summands of $\Lambda_p(G)$ by intersecting with the blocks of $\mathbf{F}_p[G]$. However, the block theory of finite groups can be extremely complicated and it is unlikely that one could deduce a general structure theorem for $\Lambda_p(G)$ in this way.

Another potential source of difficulty in determining the Lie algebra structure of $\Lambda_p(G)$ is the current status of the classification of finite dimensional simple Lie algebras in positive characteristic. In this case, the classification is complete in characteristic ≥ 5 over algebraically closed fields [15], but for characteristics 2 and 3, even in the restricted case, we do not have a complete classification.

In this paper we begin to outline the relationship between the simple factors of the composition series of the Lie algebra $\Lambda_p(G)$ with those of the composition series of $\mathbf{F}_p[G]$ as an $\mathbf{F}_p[G]$-module and lay the groundwork for future study. Because of the ambiguity in using the term 'composition series' for both objects, we have taken care to state which algebraic object we mean whenever we use the term. Our study of the modular structure of $\Lambda_p(G)$ is primarily computational and our conjectures on the modular structure are based on the data gathered on many classes of groups, which will be presented below. In the next section we recall the main result of [4] and work out the ordinary structure of $\Lambda_p(G)$. We then move to the modular case where we present our computational data and conjectures. All computations were performed with the computer-algebra package Magma and sample code appears in Appendix A.

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the functionality of the Magma packages for group algebras and Lie algebras in order for us to tackle high-dimensional examples. We are very grateful for all of his work.

2. The Ordinary Structure of $\Lambda_p(G)$

The 'ordinary' of the title of this section refers to the ordinary representation theory of a finite group G; that is, the characteristic of the field where the representations are defined does not divide the order of the group. We define an *ordinary prime* to be a prime number p that does not divide the order of G, and a *modular prime* as one that does divide $\#G$.

We begin by recalling the construction of the Plesken algebra as in [4]. If G is a finite group then the group algebra $\mathbf{C}[G]$ inherits the structure of a Lie algebra via the bracket operation $[x, y] = xy - yx$. The Plesken algebra is then defined as $\mathscr{L}(G) = \mathrm{span}_{\mathbf{C}}\{\widehat{g} \mid g \in G\} \subset \mathbf{C}[G]$. Given this finite dimensional complex Lie algebra, one can study its direct-sum decomposition, which is the main result of [4]:

$$\mathscr{L}(G) = \bigoplus_{\chi\in\mathfrak{R}} \mathfrak{o}(\chi(1)) \oplus \bigoplus_{\chi\in\mathfrak{Sp}} \mathfrak{sp}(\chi(1)) \oplus \bigoplus_{\chi\in\mathfrak{C}}{}' \mathfrak{gl}(\chi(1)),$$

where $\mathfrak{R}, \mathfrak{Sp}$, and $\mathfrak{C}$ are the sets of irreducible characters of real, symplectic, and complex types, respectively, and where the prime signifies that there is just one summand $\mathfrak{gl}(\chi(1))$ for each pair $\{\chi, \overline{\chi}\}$ from $\mathfrak{C}$. For example, the algebra $\mathscr{L}(A_4)$ has dimension 4 and admits the decomposition $\mathscr{L}(A_4) = \mathfrak{o}(1) \oplus \mathfrak{o}(3) \oplus \mathfrak{gl}(1) = \mathfrak{o}(3) \oplus \mathfrak{gl}(1)$, corresponding to the three 1-dimensional representations (one real, two complex) and the 3-dimensional real representation (note $\mathfrak{o}(1) = 0$).

Given a finite dimensional, semisimple, complex Lie algebra L defined over an algebraically closed field of characteristic 0, one can choose an integral basis (*Chevalley basis*) with respect to which the structure constants of L are integral. This allows for the "reduction modulo p" of L by choosing a Chevalley basis for L, taking the integral span $L_{\mathbf{Z}}$ of this basis, then tensoring with $\mathbf{F}_p$.

In general, the Plesken algebras are not semisimple because of the factors of complex type. These reductive algebras admit the decomposition $\mathfrak{gl}(\chi(1)) = \mathfrak{sl}(\chi(1)) \oplus \mathfrak{s}(\chi(1))$, where $\mathfrak{s}(\chi(1))$ is a one-dimensional abelian Lie algebra. It is not hard to show, following the proofs in [10, §25], that integral bases for $\mathfrak{gl}$-type Lie algebras exist and can be expressed as the union of a Chevalley basis for $\mathfrak{sl}$ and the singleton $\{\mathrm{diag}(1, \ldots, 1)\}$. This observation shows that the complex Lie algebra $\mathscr{L}(G)$ admits an integral basis. Moreover, each of the direct summands of $\mathscr{L}(G)$ do as well. The following Lemma is simply the "reduction modulo p" of [4, thm. 5.1].

Lemma 2.1. *Let G be a finite group, $\mathscr{L}(G)$ the complex Plesken algebra and $\mathscr{L}_p(G)$ its reduction modulo p via an integral basis. Let $\mathfrak{X} = \mathfrak{X}_1 \coprod \mathfrak{X}_{-1} \coprod \mathfrak{X}_0$ be the set of complex irreducible characters of G partitioned by Schur indicator. Then*

$$\mathscr{L}_p(G) = \bigoplus_{\chi\in\mathfrak{X}_1} \mathfrak{o}(\chi(1), p) \oplus \bigoplus_{\chi\in\mathfrak{X}_{-1}} \mathfrak{sp}(\chi(1), p) \oplus \bigoplus_{\chi\in\mathfrak{X}_0}{}' \mathfrak{gl}(\chi(1), p),$$

where the prime signifies that there is just one summand $\mathfrak{gl}(\chi(1), p)$ for each pair $\{\chi, \overline{\chi}\}$ with $\mathrm{ind}\,\chi = 0$.

Proof. It is not hard to check that there exists an integral basis $\mathscr{B}$ for $\mathscr{L}(G)$ such that the projection of $\mathscr{B}$ onto any of its direct summands is an integral basis for that summand. Therefore, choose an integral basis for $\mathscr{L}(G)$ that is compatible with the integral bases of the constituents of its direct-sum factors. The tensor product distributes across the direct-sum decomposition of $\mathscr{L}(G)$ and one checks that the Lie algebra structure is preserved also. □

Remark 2.2. Lemma 2.1 holds for all primes p, even for those primes dividing $\#G$. We will recover this decomposition for ordinary primes, but for modular primes the decomposition of $\mathscr{L}_p(G)$ does not mimic that of $\Lambda_p(G)$.

We now turn our attention to $\Lambda_p(G)$ for ordinary primes. If p is ordinary for G, then the group algebra $\mathbf{F}_p[G]$ is semisimple and the representation theory of G over $\mathbf{F}_p$ resembles the representation theory over $\mathbf{C}$. Let m denote the least common multiple of the orders of the elements of G and suppose that $p \equiv 1 \pmod{m}$ so that $\mathbf{F}_p$ contains a primitive mth root of unity and hence is a splitting field for G. Thus, the absolutely irreducible representations of G are realized over $\mathbf{F}_p$ (often, however, this condition is not necessary for $\mathbf{F}_p$ to be a splitting field for G).

The Schur indicator of a complex representation of G can be extended to positive characteristic. In particular, if χ is an irreducible character of G, then $\operatorname{ind}(\chi) = \pm 1$ if χ is orthogonal or symplectic, respectively, and $\operatorname{ind}(\chi) = 0$ otherwise. (In characteristic 0, $\operatorname{ind}(\chi) = 0$ means the representation is unitary, but this is not necessarily the case in positive characteristic; however, they still come in an even number of Galois-conjugate sets.) Moreover, if $p = 2$, then orthogonal and symplectic characters coincide. We write $\mathfrak{X}_0$, $\mathfrak{X}_1$, and $\mathfrak{X}_{-1}$ for the sets of characters of Schur indicator 0,1, and -1, respectively. With this notation in place, we state the following Lemma.

Lemma 2.3. *If $p \nmid \#G$ and $\mathbf{F}_p$ is a splitting field for G, then $\Lambda_p(G) = \mathscr{L}_p(G)$.*

Proof. Since $p \nmid \#G$, the group algebra $\mathbf{F}_p[G]$ is semisimple. Moreover, since $\mathbf{F}_p$ is a splitting field, the simple constituents of $\mathbf{F}_p[G]$ are absolutely simple. Thus, the decomposition of $\mathbf{F}_p[G]$ mirrors that of $\mathbf{C}[G]$ and decomposes as $\mathbf{F}_p[G] = \bigoplus_{j=1}^r \operatorname{End}(V_j)$, for a set of representatives V_j of the irreducible $\mathbf{F}_p$-representations of G. At this point the proof of [4, thm. 5.2] carries over exactly to this case: $\Lambda_p(G)$ is the -1-eigenspace of the anti-involution $g \mapsto g^{-1}$ and the Schur indicator of each V_j dictates the type of bilinear form preserved therein; the Lie algebra associated to $\operatorname{End}(V_j)$ is the full Lie algebra associated to the form. Thus,

$$\Lambda_p(G) = \bigoplus_{\chi \in \mathfrak{X}_1} \mathfrak{o}(\chi(1), p) \oplus \bigoplus_{\chi \in \mathfrak{X}_{-1}} \mathfrak{sp}(\chi(1), p) \oplus \bigoplus_{\chi \in \mathfrak{X}_0} {}'\mathfrak{gl}(\chi(1), p),$$

which is what we wanted to show. □

This lemma says the condition that $\mathbf{F}_p$ be a splitting field is enough to recover the decomposition obtained through a Chevalley basis. Indeed, the splitting of $\mathbf{C}[G]$ into a direct sum of irreducible submodules is defined over a finite extension K of $\mathbf{Q}$, where K is a subfield of the cyclotomic field $\mathbf{Q}(\zeta_m)$ and m is as above. In particular, Chevalley bases for the decomposition of $\mathscr{L}_p(G)$ are defined over K.

If $\mathbf{F}_p$ is not a splitting field for G, then we must pass to a finite extension k of $\mathbf{F}_p$ to realize all absolutely irreducible representations. In that case Galois-conjugate representations may fuse to form larger direct summands that are defined over subfields of k. For example, let $G = \mathrm{L}_2(8)$, the simple group of order 504. Using Atlas notation, the character table of G is [5, p. 6]:

TABLE 1. Character Table of $L_2(8)$

	ind	1A	2A	3A	7A	B*2	C*4	9A	B*2	C*4
χ_1	+	1	1	1	1	1	1	1	1	1
χ_2	+	7	-1	-2	0	0	0	1	1	1
χ_3	+	7	-1	1	0	0	0	-y9	*2	*4
χ_4	+	7	-1	1	0	0	0	*4	-y9	*2
χ_5	+	7	-1	1	0	0	0	*2	*4	-y9
χ_6	+	8	0	-1	1	1	1	-1	-1	-1
χ_7	+	9	1	0	y7	*2	*4	0	0	0
χ_8	+	9	1	0	*4	y7	*2	0	0	0
χ_9	+	9	1	0	*2	*4	y7	0	0	0

Here, y9 is a root of the polynomial x^3-3x-1 and y7 a root of x^3+x^2-2x-1. The algebra $\Lambda_p(\mathrm{L}_2(8))$ has dimension 220. If $p=71$, for example, then both polynomials split modulo p and $\Lambda_{71}(\mathrm{L}_2(8))$ admits the decomposition

$$\Lambda_{71}(\mathrm{L}_2(8)) = \mathfrak{o}(7,71)^{\oplus 4} \oplus \mathfrak{o}(8,71) \oplus \mathfrak{o}(9,71)^{\oplus 3}.$$

However, if $p=5$ then neither polynomial splits and $\Lambda_5(\mathrm{L}_2(8))$ decomposes as

$$\Lambda_5(\mathrm{L}_2(8)) = \mathfrak{o}(7,5) \oplus \mathfrak{l}_1 \oplus \mathfrak{o}(8,5) \oplus \mathfrak{l}_2,$$

where $\mathfrak{l}_1$ is a 63-dimensional Lie algebra decomposing over $\mathbf{F}_5(\mathtt{y9})$ into $\mathfrak{o}(7,5)^{\oplus 3}$ and $\mathfrak{l}_2$ is a 108-dimensional Lie algebra decomposing over $\mathbf{F}_5(\mathtt{y7})$ into $\mathfrak{o}(9,5)^{\oplus 3}$.

3. The modular structure of $\Lambda_p(G)$

For the rest of the paper, p denotes a modular prime. Thus, $\mathbf{F}_p[G]$ is no longer semisimple, but decomposes as a sum of blocks: $\mathbf{F}_p[G] = \bigoplus_j b_j$. The blocks are simultaneously vector subspaces and Lie subalgebras of $\mathbf{F}_p[G]$. Moreover, the Lie subalgebra $\Lambda_p(G)$ of $\mathbf{F}_p[G]$ decomposes as a direct sum by intersecting with the blocks of $\mathbf{F}_p[G]$; we write $\Lambda_p(G) = \bigoplus_j \lambda_j$, where $\lambda_j = b_j \cap \Lambda_p(G)$. However, the dimensions of the λ_j do not necessarily coincide with the dimensions of simple Lie algebras in positive characteristic (see below for explicit examples). Because of the non-semisimplicity of the algebra $\mathbf{F}_p[G]$, in order to speak of simple factors we must pass to a Lie algebra composition series of $\mathbf{F}_p[G]$ and $\Lambda_p[G]$. Just as the main result of [4] relates the simple Lie algebra factors of $\mathscr{L}(G)$ to the ordinary representation theory of G, we put forth the following:

Problem. Determine a dictionary between the Lie algebra composition factors of $\Lambda_p(G)$ and the modular representation theory of G.

Our aim in this section is to provide substantial computational data to support some of our conjectures on the structure $\Lambda_p(G)$. Recall that in addition to the classical algebras, there are exceptional simple Lie algebras in positive characteristic. When $p \geq 5$, the finite-dimensional simple Lie algebras over an algebraically closed field of characteristic p have been classified. Any such algebra is either of classical or Cartan type (Witt; Special; Hamiltonian; Contact) for $p \geq 7$ or, additionally, of Melikian type if $p=5$ [15]. Recall that for classical Lie algebras, those of type A_{mp-1}, where m is a positive integer, fail to be simple in characteristic p (there is a one-dimensional center). However, the quotient by this center is simple. By abuse of terminology we refer to these "projective" Lie algebras as classical.

Table 2 gives the dimensions of the exceptional algebras and gives the conditions under which they are simple and restricted, the latter being an important distinction in the classification of modular Lie algebras (see [15] for more details). With the exception of the Melikian algebras, these Lie algebras are parameterized by $m, \underline{n}$ with $\underline{n} = [n_1, \ldots, n_m] \in \mathbf{Z}^m_{>0}$. Set $N = \sum n_i$.

Table 2. Exceptional Modular Lie Algebras

Lie algebra	Dimension	Simple	Restricted
$W(m, \underline{n})$	p^N	$p \neq 2$ and $m \neq 1$	$W(m, [1, \ldots, 1])$
$S(m, \underline{n})$	$(m-1)p^N + 1$	$m \geq 3$	$S(m, [1, \ldots, 1])^{(1)}$
$H(m, \underline{n})$	$p^N - 1$	$p > 2$, $m \geq 2$	$H(m, [1, \ldots, 1])^{(2)}$
$K(m, \underline{n})$	p^N	$p > 2$, $m \geq 3$	$K(m, [1, \ldots, 1])^{(1)}$
$M(n_1, n_2)$	$5^{n_1+n_2+1}$	$n_i > 0$	$M(n_1, n_2)$

We remark in passing that the group algebra $\mathbf{F}_p[G]$ is restricted since it is an associative algebra endowed with the usual bracket and p-operations. This does not immediately imply that $\Lambda_p(G)$ is restricted, but one checks that $\Lambda_p(G)$ is closed under associative pth powers, which means that it is indeed restricted.

We now begin with an illustrative example. Let $G = \mathrm{SL}_2(\mathbf{F}_5)$ and take $p = 5$. The ordinary character table of G is given in Table 3 [5, p. 2]. The group algebra

Table 3. Character Table of $\mathrm{SL}_2(\mathbf{F}_5)$

	ind	$1A_0$	$1A_1$	$2A_0$	$3A_0$	$3A_1$	$5A_0$	$5A_1$	$5B_0$	$5B_1$
χ_1	+	1	1	1	1	1	1	1	1	1
χ_2	+	3	3	-1	0	0	$-b_5$	$-b_5$	$-b_5^*$	$-b_5^*$
χ_3	+	3	3	-1	0	0	$-b_5^*$	$-b_5^*$	$-b_5$	$-b_5$
χ_4	+	4	4	0	1	1	-1	-1	-1	-1
χ_5	+	5	5	1	-1	-1	0	0	0	0
χ_6	$-$	2	-2	0	-1	1	b_5	$-b_5$	b_5^*	$-b_5^*$
χ_7	$-$	2	-2	0	-1	1	b_5^*	$-b_5^*$	b_5	$-b_5$
χ_8	$-$	4	-4	0	1	-1	-1	1	-1	1
χ_9	$-$	6	-6	0	0	0	1	-1	1	-1

admits the decomposition $\mathbf{F}_5[G] = b_1 \oplus b_2 \oplus b_3$ with $\dim b_1 = 35$, $\dim b_2 = 60$ and $\dim b_3 = 25$. The block b_1 contains the characters $\chi_1, \ldots, \chi_4$; b_2 contains $\chi_6, \ldots, \chi_9$; b_3 has defect 0 and contains only the Steinberg χ_5. Turning to the Plesken Lie algebra, we get the decomposition

$$\Lambda_5(G) = \lambda_1 \oplus \lambda_2 \oplus \lambda_3 = \lambda_1 \oplus \lambda_2 \oplus \mathfrak{o}(5,5),$$

where λ_1 has dimension 12 and λ_2 has dimension 37. Note that this is consistent with the classical dimensions associated to the characters belonging to the blocks b_1 and b_2:

$$\dim \mathfrak{o}(1) + 2\dim \mathfrak{o}(3) + \dim \mathfrak{o}(4) = 0 + 2 \times 3 + 6 = 12$$
$$2\dim \mathfrak{sp}(2) + \dim \mathfrak{sp}(4) + \dim \mathfrak{sp}(6) = 2 \times 3 + 10 + 21 = 37.$$

Furthermore, if we compute the Lie algebra composition factors of the direct summands of $\Lambda_5(G)$ we get the following:

$$\Lambda_5(G) = \lambda_1 \oplus \lambda_2 \oplus \mathfrak{o}(5,5) \sim \underbrace{(\mathrm{Ab}(3), \mathrm{Ab}(3), \mathrm{Ab}(3), \mathfrak{o}(3,5))}_{\dim=12} \oplus \underbrace{(\mathrm{Ab}(3), \mathrm{Ab}(3), \mathrm{Ab}(8), \mathrm{Ab}(10), \mathfrak{sp}(2,5), \mathfrak{sp}(4,5))}_{\dim=37} \oplus \mathfrak{o}(5,5),$$

where $\mathrm{Ab}(n)$ denotes an abelian Lie algebra of dimension n. Thus, as we stated above, the simple Lie algebra composition factors of the modular Plesken Lie algebra need not be the same as those of the ordinary Plesken Lie algebra. Note the dimensions of the of the abelian terms are the same as the those of classical algebras.

This example illustrates an important aspect of the modular structure of $\Lambda_p(G)$. Suppose that ρ is a complex irreducible representation of a finite group G that affords the character χ and let $p \mid \#G$. If the reduction of ρ modulo p is irreducible (so that $\#G/\deg\rho$ is prime to p), then ρ gives rise to a block of defect 0. This block in turn is a minimal ideal of $\mathbf{F}_p[G]$ and is isomorphic to a matrix algebra over $\mathbf{F}_p$. Suppose further that $\mathbf{F}_p$ is a splitting field for G. Then, $\Lambda_p(G)$ will admit a direct summand of orthogonal, symplectic, or general linear type according to the value of $\operatorname{ind}\chi$. The finite groups of Lie type are examples of groups that have an irreducible representation with this property.

Let q be a power of a prime p and $\mathbf{G}(\mathbf{F}_q)$ a finite group of Lie type over $\mathbf{F}_q$. The representation theory of these groups is an immense subject with many open questions; see [11] for an overview. Associated to each $\mathbf{G}(\mathbf{F}_q)$ is the *Steinberg representation* which has degree equal to the largest power of p dividing $\#\mathbf{G}(\mathbf{F}_q)$. While the modular group algebras in defining characteristic are quite mysterious in general, we can predict one of the simple Lie algebras in the composition series of the Plesken Lie algebra in this case.

Theorem 3.1. *Let q be a power of a prime p, $\mathbf{G}(\mathbf{F}_q)$ a finite group of Lie type, and suppose p^r is the largest power of p dividing $\#\mathbf{G}(\mathbf{F}_q)$. Then $\Lambda_q(\mathbf{G}(\mathbf{F}_q))$ admits $\mathfrak{o}(p^r, p)$ as a direct summand.*

Proof. Since the Steinberg representation V has degree p^r, it remains irreducible upon reduction modulo p and the corresponding ideal in the group algebra $\mathbf{F}_q[\mathbf{G}(\mathbf{F}_q)]$ is a matrix algebra. Moreover, V is self-dual, which ensures that a non-degenerate bilinear form f is preserved. When q is odd, the form is symmetric since there is no symplectic structure on odd-dimensional vector spaces. If q is even, then f is symmetric, since otherwise $f(x,y) - f(y,x) = f(x,y) + f(y,x)$ would be a non-zero, symmetric, non-degenerate bilinear form preserved by $\mathbf{G}(\mathbf{F}_q)$, hence an $\mathbf{F}_q$-multiple of f, contradicting the assumption that f is not symmetric. Thus, the Schur indicator of V is equal to 1 in all cases, which proves the Proposition. □

While this result holds for groups of Lie type, the essence of the theorem holds for any group algebra which has a block of defect 0 – $\Lambda_p(G)$ admits a direct summand isomorphic to a classical Lie algebra which can be predicted from the ordinary character table of G. For blocks of higher defect, however, the situation is much more complicated. We now present some conjectures on specific families of groups with the aim that they will lead to a general structure theorem for $\Lambda_p(G)$. In particular, we expect that the block theory of $\mathbf{F}_p[G]$ will play a fundamental role in the Lie algebra structure of $\Lambda_p(G)$.

3.1. **Group rings with the dimension property.** For this subsection, k denotes an arbitrary field of positive characteristic and G a finite group. In [9], the author defines an important class of groups, whose group rings are called *group rings with the dimension property*. One of several characterizations of these group rings is that $k[G]$ has the dimension property precisely when all irreducible representations of G over k that belong to the principal block have k-dimension 1 (see [9, cor. 2.4] for this, and equivalent statements). For example, *supersolvable* groups are an important class of groups with this property, and their group rings have been studied extensively from a computational point of view (see *e.g.* [1], [13]).

Using MAGMA, we ran through all groups G and all primes p for which $\mathbf{F}_p[G]$ has the dimension property, up to the computational limit of the program. In every case, the Lie algebra composition factors of λ_1 had dimension 1. This suggests a further connection between the Lie algebra composition factors of $\Lambda_p(G)$ and the representations of G. Based on this, we pose the following.

Conjecture 3.2. *Let $\mathbf{F}_p[G]$ have the dimension property with principal block b_1. Then λ_1 is a solvable Lie algebra, with all Lie algebra composition factors having dimension 1.*

For group rings without the dimension property, then the Lie algebra composition factors of λ_1 need not have dimension 1. For example, it can be shown using the code in Appendix A that $\Lambda_5(A_5) = \lambda_1 \oplus \mathfrak{o}(5,5)$ and that the composition factors of the 12-dimensional Lie algebra λ_1 consist of nine dimension-1 factors and one dimension-3 factor isomorphic to $\mathfrak{sl}(2,5)$.

3.2. **Group algebras consisting of a single block.** We start with examples of groups whose $\mathbf{F}_p$-group algebras consist of a single block and give a general conjecture based on these and numerous other examples. We first quote a result from [2, 6.2.2].

Theorem 3.3 (6.2.2. of [2])**.** *Suppose D is a normal subgroup of G. Then every idempotent in $Z(kG)$ lies in $kC_G(D)$. In particular, if $C_G(D) \leq D$ then kG has only one block.*

Thus, to find examples of groups satisfying the hypotheses of Theorem 3.3, it is sufficient to exhibit G with $C_G(O_p(G)) \subseteq O_p(G)$, where $O_p(G)$ is the largest normal p-subgroup of G.

Example 3.4. Let $p > 2$ be a prime. Then the group algebras

$$\mathbf{F}_p[(D_p)^n] \ (p \geq 3, n \geq 1), \mathbf{F}_3[S_3 \wr S_3], \mathbf{F}_3[S_3 \wr S_2], \mathbf{F}_p[p\text{-group}], \mathrm{AGL}_1(\mathbf{F}_p)$$

(where D_p denotes the dihedral group of order $2p$ and $\mathrm{AGL}_1(\mathbf{F}_p)$ the affine general linear group on the 1-dimensional vector space $\mathbf{F}_p$) are examples of solvable G for which $\mathbf{F}_p[G]$ has only one block. To see this, one can apply Theorem 3.3 above, or use a direct argument using the criteria of [6, (85.11), (85.12)] (the ordinary character tables of these groups are well-known). Using these and other examples as test cases, we computed the composition factors of $\Lambda_p(G)$, and showed they were all abelian. For non-solvable examples, we present the following.

Example 3.5. Let $n \geq 5$ and $G = \mathbf{Z}/p \wr A_n$, so that $\#G = p^n n!/2$ and $O_p(G) \simeq (\mathbf{Z}/p)^n$. Then G is a non-solvable group whose $\mathbf{F}_p$-group algebra consists of a single block. We look at two special cases:

$$\Lambda_2(\mathbf{Z}/2 \wr A_5) \qquad \text{and} \qquad \Lambda_3(\mathbf{Z}/3 \wr A_5).$$

These Plesken Lie algebras have dimensions 884 and 7222, respectively, and the former is solvable while the latter is not. The dimensions of the composition factors are given by the following tables:

Dimension	Multiplicity	Type
1	356	Abelian
2	104	Abelian
4	24	Abelian
8	28	Abelian
Total	884	

Dimension	Multiplicity	Type
1	314	Abelian
3	4	A_1(over $\mathbf{F}_9$)
4	269	Abelian
6	262	Abelian
9	200	Abelian
24	102	Abelian
Total	7222	

Remark 3.6. The groups in the preceding example were initially too large (even when $p = 2$) for Magma to create and store $\Lambda_p(G)$, let alone determine the isomorphism type. As a result of our collaboration with the developers, Magma is now able create $\Lambda_p(G)$ for large groups and to compute the isomorphism types of the Lie algebra composition factors in a reasonably short computing time (54 seconds and 16 hours, respectively).

Based on these examples (and the preceding ones for non-solvable groups), we present the following conjecture.

Conjecture 3.7. *Suppose that G is a solvable group and that $\mathbf{F}_p[G]$ is a group algebra that consists of a single block. Then the Lie algebra $\Lambda_p(G)$ is solvable.*

Remark 3.8. The Lie algebras $\Lambda_p(G)$ for such G as in the conjecture are, in general, non-abelian (*e.g.* $\Lambda_5(D_5 \times D_5)$).

Remark 3.9. If the group algebra consists of more than a single block, then λ_1 may not have trivial composition factors. See the example following Lemma 3.2 above.

For groups with a more complicated block structure, it is considerably more difficult to detect patterns in the simple composition factors of the $\Lambda_p(G)$. Our aim is to provide enough data to support conjectures that could be proved in a subsequent work. In Table 4 we give some data on simple (and related) groups; see Appendix A for details on the construction of the table. In particular, we compute the *absolutely simple* composition factors of $\Lambda_p(G)$ (*i.e.* we work over a splitting field k of G) for odd primes p. For ease of comparison of the modular and ordinary cases, we use A, B, C, D notation and use "PA" for the simple algebras $\mathfrak{psl}(mp)$ in characteristic p.

Conjecture 3.10. *If $\mathbf{F}_p$ is a splitting field for G, then the composition factors of $\Lambda_p(G)$ are isomorphic to classical or projective classical Lie algebras, or are abelian of dimension equal to a classical or projective classical Lie algebra.*

Appendix A. Computations and Code

Our main computational tool was the computer-algebra package Magma. When we began this project, some of the examples were too large for Magma to create. After an initial draft of this paper was completed, we contacted the algebra group at Magma to inquire about improving the functionality of the Lie algebra package; in particular to speed up the creation of $\Lambda_p(G)$ and its composition series and finally to

Table 4. Plesken algebras of some simple (and related) groups

G	p	Composition Factors of $\Lambda_p(G)$	$\mathscr{L}_p(G)$
A_5	3	$\mathrm{Ab}(4); \mathrm{Ab}(3) \times 2; D_2 \times 2$	$B_1 \times 4; B_2$
A_5	5	$\mathrm{Ab}(3) \times 3; B_1; B_2$	$B_1 \times 4; B_2$
S_5	3	$\mathrm{Ab}(4) \times 2; \mathrm{Ab}(3) \times 4; D_2 \times 2; D_3$	$B_1 \times 4; B_2 \times 2; D_3$
S_5	5	$\mathrm{Ab}(3) \times 4; \mathrm{Ab}(9); B_1 \times 2; B_2 \times 2$	$B_1 \times 4; B_2 \times 2; D_3$
$\mathrm{L}_2(7)$	3	$\mathrm{Ab}(1) \times 2; \mathrm{Ab}(7); \mathrm{Ab}(21); PA_2; D_3; B_3$	$\mathrm{Ab}(1) \times 2; PA_2; D_3; B_3; D_4$
$\mathrm{L}_2(7)$	7	$\mathrm{Ab}(1); \mathrm{Ab}(3); \mathrm{Ab}(5) \times 2; \mathrm{Ab}(10); \mathrm{Ab}(15); B_1; B_2; B_3$	$\mathrm{Ab}(1); A_2; D_3; B_3; D_4$
$\mathrm{SL}_2(\mathbf{F}_7)$	3	$\mathrm{Ab}(1) \times 4; \mathrm{Ab}(7); \mathrm{Ab}(15); \mathrm{Ab}(20); \mathrm{Ab}(21)$ $PA_2; A_3; D_3; C_3 \times 2; B_3$	$\mathrm{Ab}(1) \times 2; PA_2; A_3; D_3; C_3 \times 2; B_3; C_4; D_4$
$\mathrm{SL}_2(\mathbf{F}_7)$	7	$\mathrm{Ab}(1) \times 2; \mathrm{Ab}(3) \times 2; \mathrm{Ab}(5) \times 3; \mathrm{Ab}(8);$ $\mathrm{Ab}(10) \times 2; \mathrm{Ab}(12); \mathrm{Ab}(15); \mathrm{Ab}(21)$ $B_1 \times 2; B_2 \times 2; B3_{\times} 2$	$\mathrm{Ab}(1); A_2; A_3; D_3; C_3 \times 2; B_3; C_4; D_4$
A_6	3	$\mathrm{Ab}(4) \times 4; \mathrm{Ab}(3) \times 12; \mathrm{Ab}(9) \times 1; \mathrm{Ab}(24) \times 2$ $B_1 \times 4; B_4$	$B_2 \times 2; D_4 \times 2; B_4; D_5$
A_6	5	$\mathrm{Ab}(8); \mathrm{Ab}(28) \times 2; C_2; B_2; D_4; D_5$	$B_2 \times 2; D_4 \times 2; B_4; D_5$
S_6	3	$\mathrm{Ab}(1); \mathrm{Ab}(4) \times 8; \mathrm{Ab}(6) \times 2; \mathrm{Ab}(3) \times 12$ $\mathrm{Ab}(15) \times 2; \mathrm{Ab}(16); \mathrm{Ab}(24) \times 4$ $B_1 \times 4; D_3; B_4 \times 2$	$B_2 \times 4; B_4 \times 2; D_5 \times 2; D_8 \times 2$
S_6	5	$\mathrm{Ab}(8) \times 2; \mathrm{Ab}(28) \times 2; \mathrm{Ab}(64)$ $B_2 \times 4; D_4 \times 2; D_5 \times 2$	$B_2 \times 4; B_4 \times 2; D_5 \times 2; D_8 \times 2$
$\mathrm{L}_2(8)$	3	$\mathrm{Ab}(7); \mathrm{Ab}(21) \times 4; B_3; B_4 \times 3$	$B_3 \times 4; D_4; B_4 \times 3$
$\mathrm{L}_2(8)$	7	$\mathrm{Ab}(8) \times 3; \mathrm{Ab}(28) \times 3; B_3 \times 4; D_4$	$B_3 \times 4; D_4; B_4 \times 3$
$\mathrm{L}_2(11)$	3	$\mathrm{Ab}(1) \times 2; \mathrm{Ab}(10) \times 3; \mathrm{Ab}(24); \mathrm{Ab}(45)$ $A_4; D_5; D_6 \times 2$	$\mathrm{Ab}(1); A_4; D_5 \times 2; B_5; D_6 \times 2$
$\mathrm{L}_2(11)$	5	$\mathrm{Ab}(1) \times 2; \mathrm{Ab}(11) \times 2; \mathrm{Ab}(55) \times 2; PA_4; D_5 \times 2; B_5$	$\mathrm{Ab}(1) \times 2; PA_4; D_5 \times 2; B_5; D_6 \times 2$
$\mathrm{L}_2(16)$	5	$\mathrm{Ab}(16) \times 2; \mathrm{Ab}(120) \times 2; \mathrm{Ab}(136) \times 4; B_7 \times 8; D_8; B_8$	$B_7 \times 8; D_8; B_8 \times 8$
$\mathrm{L}_2(19)$	5	$\mathrm{Ab}(1) \times 3; \mathrm{Ab}(18); \mathrm{Ab}(36) \times 4; \mathrm{Ab}(80) \times 2;$ $\mathrm{Ab}(153) \times 2; A_8; D_9; D_{10} \times 4$	$\mathrm{Ab}(1); A_2; D_9 \times 4; B_9; D_{10} \times 4$
$\mathrm{L}_2(25)$	5	$\mathrm{Ab}(6) \times 7; \mathrm{Ab}(9) \times 2; \mathrm{Ab}(16) \times 2; \mathrm{Ab}(20);$ $\mathrm{Ab}(24) \times 2; \mathrm{Ab}(25); \mathrm{Ab}(30) \times 2; \mathrm{Ab}(36) \times 8;$ $\mathrm{Ab}(48) \times 2; \mathrm{Ab}(54); \mathrm{Ab}(56) \times 3; \mathrm{Ab}(64) \times 2;$ $\mathrm{Ab}(90); \mathrm{Ab}(96) \times 2; \mathrm{Ab}(120) \times 3; \mathrm{Ab}(144) \times 2;$ $\mathrm{Ab}(160) \times 2; \mathrm{Ab}(210); \mathrm{Ab}(240) \times 2;$ $A_1 \times 4; B_2 \times 2; B_4; D_4 \times 2; D_8; B_7 \times 2; D_8 \times 2$	$B_6 \times 2; D_{12} \times 6; B_{12}; D_{13} \times 5$

compute the isomorphism types of the composition factors. Before we collaborated with the developers, the highest-dimensional Plesken algebra for which we could compute its composition series was when $G = \mathrm{L}_2(11)$ and $\dim \Lambda_p(\mathrm{L}_2(11)) = 302$. The largest group appearing presently in this paper has order 14 580 and the dimension of its Plesken algebra is 7 222. We now give a brief description of the code used for our computations.

There are different types for this object in Magma (GenAlgebra, GrpAlgebra, AssAlgebra, LieAlgebra) and care needs to be taken when combining functions that are defined for specific types because they are not inherited. For the code below, "`G`" is a permutation group and "`F`" is a finite field. This code will build the Plesken Lie Algebra "`P`".

```
FG := GroupAlgebra(F, G);
L1,h:=Algebra(FG);
FFG,f:=LieAlgebra(L1);
L:=[];
for g in G do
if Order(g) ne 2 and Order(g) ne 1 and Inverse(g) notin L then
Append(~L,g);
end if;
end for;
LL:=[];
for g in L do
Append((~LL,FG!g-FG!Inverse(g) ));
end for;
LLL:=[];
for g in LL do
Append((~LLL, (g@h)@f ));
end for;
P:=sub<FFG|LLL>;
```

In order to clearly show how Table 4 was created, we work through a low-dimensional example. Take `G:=Sym(5);`, `F:=GF(5);`, and build `P` as in the code above. The command

```
C:=CompositionSeries(P);
```

builds and stores the composition series of `P` as a Lie algebra; note that `P` has dimension 47. Then `C` has length 9, and the filtration consists of a chain of Lie algebras of dimensions [10, 13, 16, 25, 28, 31, 34, 44, 47]. Thus, the quotients have dimensions [10, 3, 3, 9, 3, 3, 3, 10, 3], respectively. Note that Magma may build `C` with a different filtration each time; the dimension and structure of the quotients is (obviously) always the same. In order to further study these quotients we build them inside Magma using, for example:

```
qi:=quo<C[i] | C[i-1]>;
```

as `i` ranges over $2\ldots9$ (and `q1:=C[1]`). Now that the quotients are isolated from each other, one can determine their Lie algebra structure using such commands as:

`IsAbelian`, `CompositionSeries`, `IsSimple`,

among others. When applied to `P` above, we obtain the following results:

`q1` – Type B_2;
`q2` – Ab(9);
`q3` – Type B_1;
`q4` – Ab(3);
`q5` – Ab(3);
`q6` – Type B_1;

q7 – Ab(3);
q8 – Type B_2;
q9 – Ab(3).

Altogether, this gives the decomposition as indicated in the table.

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