POINTS OF SMALL ORDER ON THREE-DIMENSIONAL ABELIAN VARIETIES; WITH AN APPENDIX BY YURI ZARHIN

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Abstract. Let A be a three-dimensional abelian variety defined over a number field K, and let $\ell \in \{3, 5\}$. We classify the images of the mod ℓ representations of those three-dimensional abelian varieties which possess an ℓ-torsion point modulo $\mathfrak{p}$ for almost all primes $\mathfrak{p}$ of K, but for which there does not exist a K-isogenous A' with a rational point of order ℓ.

1. Introduction

In this paper we complete a classification begun in [3] and [4] on the local-to-global properties of torsion points on three dimensional abelian varieties. In particular, let A be an abelian variety defined over a number field K. Fix an integer $m \geq 2$ and suppose that for a set of good primes $\mathfrak{p}$ of density 1 the number of $\mathbf{F}_\mathfrak{p}$-rational points on $A_\mathfrak{p}$ is divisible by m. Does there exist an abelian variety A'/K which is K-isogenous to A with $\#A'_{\text{tors}}(K)$ divisible by m?

This problem was first investigated by Katz in [8] where the answer was shown to be "Yes" when A is an elliptic curve and, in the case where m is a prime number ℓ, for two-dimensional abelian varieties. Furthermore, it was shown that the answer is "No" when $\ell > 2$ and A has dimension three or greater. In [3] we classified the abelian threefolds for which this local-to-global divisibility fails for all primes $\ell > 5$. In [4] we showed that when $\ell = 2$, the answer is "Yes" for three-dimensional abelian varieties and "No" for all dimensions ≥ 4.

In this paper we complete the classification of the images of the mod ℓ representations of three-dimensional abelian varieties for which this divisibility fails when $\ell \in \{3, 5\}$ and investigate their endomorphism rings. Our main result is the following.

Theorem 1. *Let A be a three-dimensional abelian variety defined over a number field K and let $\ell \in \{3, 5\}$. Suppose that for a set of primes $\mathfrak{p}$ of density 1 the divisibility $\#A_\mathfrak{p}(\mathbf{F}_\mathfrak{p}) \equiv 0(\ell)$ holds and that there does not exist a K-isogenous A' which possesses a K-rational ℓ-torsion point. Then the Levi component of the image of the mod ℓ representation of A is one of the following:*

$\ell = 3$	$\ell = 5$
$\mathbf{Z}/2 \times \mathbf{Z}/2$	$\mathbf{Z}/2 \times \mathbf{Z}/2$
D_4	D_4
S_4	$S_4,\ S_5$

The proof relies on a group-theoretic reformulation of the question due to Katz in [8]. The classification is then arrived at using a combination of group theory and explicit computations using the software package Magma. In Section 2 we review the group-theoretic reformulation as well as the relevant background on abelian varieties. We then divide the proof of the theorem over the next several sections based on the structure of $\operatorname{im} \overline{\rho_\ell}$. We point out that there always exists an abelian variety over some number field with the prescribed mod ℓ representation. This follows easily from Galois theory. However, it is much more difficult to determine whether such abelian varieties exist over $\mathbf{Q}$. We do not address this question in this paper.

We end this section by noting that the answer to the original question remains unknown in the case of two-dimensional abelian varieties and composite m (it suffices to take m to be a prime power by factorization of isogenies). In fact, the answer is known to be "No" when ℓ is a power of 2 due to an example of Serre [8]. In a forthcoming paper [5] we revisit this problem in the case of prime powers ℓ^n, for $\ell > 3$.

The paper is organized as follows: the next section is devoted to the group-theoretic interpretation of the problem; we then find all counterexamples in terms of the image of the mod ℓ representation. In the final section we make some remarks about the endomorphism rings of these abelian varieties following an argument of Zarhin [14, 15], and the paper ends with an appendix by Zarhin. We point out that our classification theorem is more general than the one given in [3]. In particular, the assumption $\ell > 5$ in [3] was made so that $\ell \nmid \# \operatorname{im} \overline{\rho_\ell}$ in the majority of cases. Moreover, we do not restrict to the case $\operatorname{im} \overline{\rho_\ell} \subset \mathrm{Sp}_6(\mathbf{F}_\ell)$, *i.e.* we do not assume $\det \overline{\rho_\ell} = 1$.

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2. Group-theoretic Reformulation of the Problem

Let ℓ be a prime number. If n is a positive integer then we write μ_n for the multiplicative group of nth roots of unity in $\overline{K}$. We write $\mathbf{Z}_\ell(1)$ for the projective limit of groups μ_{ℓ^i}, which is a free $\mathbf{Z}_\ell$-module of rank 1 provided with the natural structure of a Galois module. Given an abelian variety A of positive dimension d defined over a number field K (with algebraic closure $\overline{K}$), the ℓ-adic representation $\rho_\ell : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Aut}(T_\ell(A))$ of the absolute Galois group on the Tate module $T_\ell(A)$ is the representation-theoretic formulation of the natural action of $\mathrm{Gal}(\overline{K}/K)$ on the points of ℓ-power order of A. The Tate module is a rank $2d$ free $\mathbf{Z}_\ell$-module, hence upon choosing a basis for $T_\ell(A)$ and reducing modulo ℓ, we have the mod ℓ representation encoding the action of $\mathrm{Gal}(\overline{K}/K)$ on $A[\ell]$:

$$\overline{\rho_\ell} : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Aut}(T_\ell(A) \otimes \mathbf{F}_\ell) \simeq \mathrm{GL}_{2d}(\mathbf{F}_\ell).$$

The Weil pairing has as arguments elements of Tate modules of a given abelian variety and its dual. Choosing a K-polarization on A, one gets a Galois-equivariant alternating form on $T_\ell(A)$ with values in $\mathbf{Z}_\ell(1)$ (or on $A[\ell]$ with values in μ_ℓ). If the degree of the polarization is not divisible by ℓ then the corresponding alternating bilinear form on $A[\ell]$ is nondegenerate and the Galois image lies in the group of symplectic similitudes of $A[\ell]$. However, such a polarization may not exist even replacing A by any abelian variety B that is K-isogenous to A [6, 13]. Still, dividing (if necessary) a K-polarization by a suitable power of ℓ, we get a K-polarization on A that is not divisible by ℓ and therefore the corresponding Galois-equivariant alternating bilinear form on $A[\ell]$ is not identically zero. However, this form may be degenerate, which means that its kernel W is a proper Galois-invariant subspace in $A[\ell]$ of even dimension, while the induced Galois-invariant alternating form on the quotient $A[\ell]/W$ is nondegenerate. From now on we assume that $d = 3$. Then the image of the semisimplification of $\overline{\rho_\ell}$ is contained in $\mathrm{GSp}_4(\mathbf{F}_\ell) \times \mathrm{GL}_2(\mathbf{F}_\ell)$ (if the kernel is 2-dimensional) or $\mathrm{GL}_4(\mathbf{F}_\ell) \times \mathrm{GL}_2(\mathbf{F}_\ell)$ (if the kernel is 4-dimensional). In the latter case we can say more.

Let $\lambda : A \to A^t$ be a K-polarization that is not divisible by ℓ and let M be the ℓ-component of its kernel. Let M_ℓ be the kernel of multiplication by ℓ in M. In our case M_ℓ is four-dimensional and M is isomorphic (as a commutative group) to a direct sum $(\mathbf{Z}/\ell^i)^2 \oplus (\mathbf{Z}/\ell^j)^2$ with $1 \le i \le j$. The polarization λ gives rise to the corresponding Riemann form e_λ - the alternating "biadditive" nondegenerate Galois-equivariant form on M with values in μ_{ℓ^j} [12, Sect. 23], [11, Sect. 1]. The Galois image in $\mathrm{Aut}(M)$ lies in $\mathrm{Aut}(M, e_\lambda)$. If $i = j$ then $\mathrm{Aut}(M, e_\lambda)$ is isomorphic to $\mathrm{GSp}_4(\mathbf{Z}/\ell^j)$ and its image in $\mathrm{Aut}(M_\ell)$ is isomorphic to $\mathrm{GSp}_4(\mathbf{Z}/\ell)$. If $i < j$ then M_ℓ contains a Galois-invariant two-dimensional subspace $\ell^{j-1}M$, which implies that the Galois image in the automorphism group of the semisimplification of M_ℓ lies in $\mathrm{GL}_2(\mathbf{F}_\ell) \times \mathrm{GL}_2(\mathbf{F}_\ell)$. Taking into account that $A[\ell]/M_\ell$ is two-dimensional, we obtain that the semisimplification of the Galois image in $\mathrm{Aut}(A[\ell])$ lies either in $\mathrm{GSp}_4(\mathbf{F}_\ell) \times \mathrm{GL}_2(\mathbf{F}_\ell)$ or $\mathrm{GL}_2(\mathbf{F}_\ell) \times \mathrm{GL}_2(\mathbf{F}_\ell) \times \mathrm{GL}_2(\mathbf{F}_\ell)$. In the rest of the paper, we may omit the latter case since it is addressed by [3, Lemma 22].

Let $\mathfrak{p}$ be a prime of good reduction for A and denote by $\mathbf{F}_\mathfrak{p}$ the residue field at $\mathfrak{p}$. In [8], Katz shows that the condition $\# A_\mathfrak{p}(\mathbf{F}_\mathfrak{p}) \equiv 0(\ell)$ for a set of $\mathfrak{p}$ of density 1 is equivalent to $\det(1 - \sigma) = 0$ for all $\sigma \in \operatorname{im} \overline{\rho_\ell}$. Furthermore, the condition that there exists a K-isogenous abelian variety A' such that A' has a K-rational ℓ-torsion point is equivalent to the Jordan-Hölder series of $T_\ell(A) \otimes \mathbf{F}_\ell$ containing the trivial representation. Hence, A fails the local-to-global divisibility condition when $\operatorname{im} \overline{\rho_\ell}$ is a subgroup G of $\mathrm{GSp}_{2d}(\mathbf{F}_\ell)$ for which every element has 1 as an eigenvalue and such that the Jordan-Hölder series of $\mathbf{F}_\ell[G]$ does not contain the trivial representation.

Not every element of $\mathrm{GSp}_{2d}(\mathbf{F}_\ell)$ has 1 as an eigenvalue, hence any A which fails this local-to-global divisibility condition must have $\operatorname{im} \overline{\rho_\ell}$ a proper subgroup of $\mathrm{GSp}_{2d}(\mathbf{F}_\ell)$. We now take $d = 3$ and $\ell = 3$ or 5.

The subgroup structure of the classical groups is described in [10], and for convenience we record here the maximal subgroups of $\mathrm{Sp}_6(\mathbf{F}_3)$ and $\mathrm{Sp}_6(\mathbf{F}_5)$ according to the classification scheme in [10].

Type	$\mathrm{Sp}_6(\mathbf{F}_3)$	$\mathrm{Sp}_6(\mathbf{F}_5)$
$\mathcal{C}_1$	Parabolics, $\mathrm{Sp}_4(\mathbf{F}_3) \times \mathrm{SL}_2(\mathbf{F}_3)$	Parabolics, $\mathrm{Sp}_4(\mathbf{F}_5) \times \mathrm{SL}_2(\mathbf{F}_5)$
$\mathcal{C}_2$	$\mathrm{SL}_2(\mathbf{F}_3) \wr S_3$, $\mathrm{GL}_3(\mathbf{F}_3).2$	$\mathrm{SL}_2(\mathbf{F}_5) \wr S_3$, $\mathrm{GL}_3(\mathbf{F}_5).2$
$\mathcal{C}_3$	$\mathrm{SL}_2(\mathbf{F}_{27}).3$, $\mathrm{GU}_3(\mathbf{F}_3)$	$\mathrm{SL}_2(\mathbf{F}_{125}).3$, $\mathrm{GU}_3(\mathbf{F}_5)$
$\mathcal{C}_4$	None	$\mathrm{O}_3(\mathbf{F}_5) \otimes \mathrm{SL}_2(\mathbf{F}_5)$
$\mathcal{S}$	$2.A_5$, $\mathrm{SL}_2(\mathbf{F}_{13})$, $\mathrm{SL}_2(\mathbf{F}_{13})$	$2.A_5$, $2.J_2$

Thus, it suffices to search inside these maximal subgroups for counterexamples. Typically these groups are still too large to satisfy the eigenvalue condition and so we will iteratively search the lattice of maximal subgroups for maximal counterexamples.

We remind the reader of several conventions from finite group theory. We write $A.B$ for the middle term of a short exact sequence of groups with kernel A and quotient B; notation $A \otimes B$ is used for the image of the tensor product representation of groups A and B. Finally, if P is a parabolic subgroup of a classical group G then the *Levi factor* of P is the complement to the unipotent radical in P. For more details see [1, p. 257].

3. Determination of Counterexamples

One of the results from elementary group theory that we use extensively is *Goursat's Lemma*, which states that the subgroups G of a direct product $A \times B$ are in one-to-one correspondence with quadruples (G_1, G_2, G_3, ψ), where $G_1 \subset A$, $G_3 \lhd G_2 \subset B$, and $\psi : G_1 \longrightarrow G_2/G_3$ a surjective homomorphism [2]. For the rest of the paper, a matrix group for which all $g \in G$ satisfy $\det(1-g) = 0$ will be called a *fixed-point group.* A *Goursat-subgroup* will be a subgroup of a direct product which itself is not direct product (so ψ is non-trivial).

Observe that if $G \subset \mathrm{GSp}_4(\mathbf{F}_\ell) \times \mathrm{GL}_2(\mathbf{F}_\ell)$ is a counterexample, then G must be a Goursat-subgroup since otherwise one of the projections would have to be a fixed-point group, giving rise to a fixed-point subgroup of $\mathrm{GSp}_4(\mathbf{F}_\ell)$ or $\mathrm{GL}_2(\mathbf{F}_\ell)$. Katz proved [8] that in these cases the Jordan-Hölder series contains the trivial representation. Moreover, we have the following estimate on the size of such a counterexample. Let $P \subset G_1$ be the set of elements that do not have 1 as an eigenvalue. Then every $\psi(p)$ must be a coset in G_2/G_3 consisting entirely of elements having 1 as an eigenvalue. A little algebra reveals that $p \in \ker \psi$ and that G_3 must consist entirely of elements having 1 as an eigenvalue, whence the estimate

$$\frac{\#G_1}{[G_2 : G_3]} \geq \#P + 1. \tag{1}$$

For a proof of this fact, we refer the reader to [3, p. 743]. We now continue by determining the counterexamples in $\mathrm{GL}_3(\mathbf{F}_\ell)$ since they will arise in many different contexts. In fact, this Proposition classifies all the counterexamples that occur, and the remainder of the paper will be to show that there are no additional counterexamples.

Proposition 1. *Let $\ell \in \{3, 5\}$, V be a three-dimensional $\mathbf{F}_\ell$-vector space, and G a maximal fixed-point subgroup of $\mathrm{Aut}(V)$. If the Jordan-Hölder series of the $\mathbf{F}_\ell[G]$-module V does not contain the trivial representation, then G is one of the following:*

Dimension of Jordan-Hölder Factors	*Levi(G)*	*G*
(1,1,1)	$\mathbf{Z}/2 \times \mathbf{Z}/2$	$\ell^3{:}(\mathbf{Z}/2 \times \mathbf{Z}/2)$
(2,1)	D_4	$\ell^2{:}D_4$
(3)	S_4	S_4

In addition, when $\ell = 5$, the group $G = S_5$ is a maximal, irreducible, fixed-point subgroup of $\mathrm{Aut}(V)$.

Proof. Since the underlying question is one of eigenvalues and components of the Jordan-Hölder series, it suffices to work with the Levi component of G. We divide the proof into three cases according to the dimensions of the simple factors of V. We start with the case of three one-dimensional factors, so that the Levi component is a subgroup of $(\mathbf{F}_\ell^\times)^3$. In this case we refer to [3, Lemma 5], which shows that the only counterexample in this case is when $G \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$ via

$$\mathbf{Z}/2 \times \mathbf{Z}/2 \longrightarrow \begin{pmatrix} \epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_1\epsilon_2 \end{pmatrix},$$

where $\epsilon_i \in \{\pm 1\}$. Of course, once can enlarge G by adding the full unipotent radical (so that $G \simeq \ell^3{:}(2\times 2)$), which preserves the counterexample.

Next, we assume that the dimensions of the simple factors are 2 and 1, so that the Levi component is a subgroup of $\mathrm{GL}_2(\mathbf{F}_\ell) \times \mathbf{F}_\ell^\times$. Note that G must be a Goursat-subgroup since otherwise it would give rise to a fixed-point subgroup of $\mathrm{GL}_2(\mathbf{F}_\ell)$, and by [8, Lemma 1] therefore has a trivial Jordan-Hölder factor. At this point we make a distinction between $\ell = 3$ and $\ell = 5$; we give details when $\ell = 3$ and omit the nearly-identical argument for $\ell = 5$.

When $\ell = 3$, we look for subgroups satisfying the estimate in (1) with $G_1 \subset \mathrm{GL}_2(\mathbf{F}_3)$, and $G_3 \lhd G_2 \subset \{\pm 1\}$ (here $G_3 = 1$ since G is a Goursat-subgroup). When $G_1 = \mathrm{GL}_2(\mathbf{F}_3)$, we have $\#P = 27$, giving:

$$\frac{48}{2} \geq \frac{\#\,\mathrm{GL}_2(\mathbf{F}_3)}{[G_2 : G_3]} \not\geq 28 = \#P + 1.$$

It follows that G_1 must be contained in some maximal subgroup of $\mathrm{GL}_2(\mathbf{F}_3)$, hence is a subgroup of D_6, $Q_8.2$, or $\mathrm{SL}_2(\mathbf{F}_3)$. The Jordan-Hölder factors of D_6 are 1-dimensional (and in fact give rise to a counterexample $3{:}(2 \times 2)$ as in the argument above), so this group has already been described in the first part of the proof. The groups $Q_8.2$ and $\mathrm{SL}_2(\mathbf{F}_3)$ fail the estimate (1). It is easy to check that $Q_8.2$ contains a maximal counterexample isomorphic to D_4 via

$$\text{“}r\text{”} \leftrightarrow \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} \qquad \text{“}s\text{”} \leftrightarrow \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & -1 \end{pmatrix}.$$

The maximal subgroups of $\mathrm{SL}_2(\mathbf{F}_3)$ are $\mathbf{Z}/6$ and Q_8, which cannot give rise to counterexamples since $\mathbf{Z}/6$ is cyclic and the only fixed-point element of Q_8 is the identity.

The final case is when G acts irreducibly on V. The maximal irreducible subgroups of $\mathrm{GL}_3(\mathbf{F}_3)$ are 13:6, $S_4 \times 2$, and $\mathrm{SL}_3(\mathbf{F}_3)$. The only fixed-point subgroup of 13:6 is cyclic of order 3. Moreover, $S_4 \times 2$ is not a fixed-point subgroup since it contains a non-trivial central element, but its maximal subgroup isomorphic to S_4 is an irreducible fixed-point subgroup, hence is a counterexample. Finally, $\mathrm{SL}_3(\mathbf{F}_3)$ is not a fixed-point group and its maximal irreducible subgroups are 13:3 and S_4, both of which were analyzed previously.

When $\ell = 5$, we first restrict to $\mathrm{SL}_3(\mathbf{F}_5)$. By Clifford's Theorem, the restriction of an irreducible subgroup of $\mathrm{GL}_3(\mathbf{F}_5)$ to $\mathrm{SL}_3(\mathbf{F}_5)$ remains irreducible. The maximal irreducible subgroups of $\mathrm{SL}_3(\mathbf{F}_5)$ are 31:3, $4^2{:}S_3$, and S_5 ($\simeq \mathrm{SO}_3(\mathbf{F}_5)$). The group 31:3 is not a fixed point group and any subgroup is cyclic, hence it contains no counterexamples. The maximal fixed-point, irreducible subgroup of $4^2{:}S_3$ is S_4 and is therefore a counterexample. Finally, the group S_5 is an irreducible fixed-point group. It is now easy to check (using the isomorphism $\mathrm{GL}_3(\mathbf{F}_5) \simeq \mathrm{SL}_3(\mathbf{F}_5) \times 4$) that these SL_3-counterexamples are in fact the maximal GL_3-counterexamples. This completes the proof of the Proposition. □

We now proceed with an analysis of the subgroups of $\mathrm{GSp}_6(\mathbf{F}_3)$ and $\mathrm{GSp}_6(\mathbf{F}_5)$ based on the geometric type of the subgroup as described in the table at the end of Section 2. In particular, we will show that there are no further counterexamples.

3.1. **Type $\mathcal{C}_1$.** Let $\ell = 3$ or 5. Observe that if G is a parabolic subgroup of $\mathrm{GSp}_6(\mathbf{F}_\ell)$ then it suffices to restrict to its Levi component for questions of eigenvalues and Jordan-Hölder series. Observe further that the "shape" of a parabolic in Sp_6 is the same as that in GSp_6 and, upon choosing a basis, differs only by a diagonal element of GSp_6 [7]. We divide this subsection further according to the type of subgroup in $\mathcal{C}_1$.

3.1.1. $G \subset \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$. The groups $\mathcal{P}_i$, $i = 1, 2, 3$ are the stabilizers of certain decompositions of six-dimensional symplectic space: a product of a hyperbolic plane and symplectic 4-space; three hyperbolic planes; and a maximal isotropic space, respectively. Let $G \subset \mathcal{P}_i$ be a counterexample, *i.e.* G is a fixed-point subgroup of $\mathcal{P}_i \subset \mathrm{GSp}_6(\mathbf{F}_\ell)$ whose Jordan-Hölder series does not contain a trivial factor. Then the Levi component of $G \cap \mathrm{Sp}_6(\mathbf{F}_\ell)$ is a subgroup of $\mathrm{Sp}_4(\mathbf{F}_\ell) \times \{\pm 1\}$, $\mathrm{GL}_2(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$, or $\mathrm{GL}_3(\mathbf{F}_\ell)$.

In $\mathrm{GSp}_6(\mathbf{F}_\ell)$, the Levi component of $\mathcal{P}_3$ consists of elements of type $(g, \lambda g^*)$, where $\lambda \in \mathbf{F}_\ell^\times$ and $g \in \mathrm{GL}_3(\mathbf{F}_\ell)$. Notice that $\det(1-g) = 0$ if and only if $\det(1-g^*) = 0$, so any fixed-point subgroup of $\mathcal{P}_3$ is automatically a fixed-point subgroup of $\mathcal{P}_3 \cap \mathrm{Sp}_6(\mathbf{F}_\ell)$. Moreover, a fixed-point subgroup G of $\mathcal{P}_3$ is a counterexample if and only if $G \cap \mathrm{Sp}_6(\mathbf{F}_\ell)$ is. It therefore suffices to work inside $\mathrm{Sp}_6(\mathbf{F}_\ell)$, but here we refer to Proposition 1 for a list of the counterexamples in this case.

Similarly, any fixed-point subgroup of $\mathcal{P}_2$ is a fixed-point subgroup of $\mathcal{P}_2 \cap \mathrm{Sp}_6(\mathbf{F}_\ell)$ and any counterexample in $\mathcal{P}_2$ is a counterexample in $\mathcal{P}_2 \cap \mathrm{Sp}_6(\mathbf{F}_\ell)$. It therefore suffices to search for counterexamples inside $\mathrm{GL}_2(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$. However, the part of the proof of Proposition 1 that determined the counterexamples with simple factors of dimensions 2 and 1 is easily tailored to this case. We obtain identical counterexamples, where the "G_2" of Proposition 1 (which was isomorphic to $\{\pm 1\} \subset \mathbf{F}_\ell^\times$) is replaced by the center of $\mathrm{SL}_2(\mathbf{F}_\ell)$.

Finally, the case of $\mathcal{P}_1$ is subsumed by that of $\mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$, which we treat in the next subsection.

3.1.2. *$G \subset \mathrm{GSp}_4(\mathbf{F}_\ell) \times \mathrm{GL}_2(\mathbf{F}_\ell)$.* Let $\ell \in \{3, 5\}$. In this case, the setup is as follows: $G_1 \subset \mathrm{Sp}_4(\mathbf{F}_\ell)$, $G_3 \lhd G_2 \subset \mathrm{SL}_2(\mathbf{F}_\ell)$, and $\psi : G_1 \longrightarrow G_2/G_3$ a surjective homomorphism; we will later lift to $\mathrm{GSp}_4(\mathbf{F}_\ell) \times \mathrm{GL}_2(\mathbf{F}_\ell)$. We may restrict to the case where G_1 acts irreducibly since any reducible action would give rise to a subgroup of the parabolic subgroup $\mathcal{P}_2$ of $\mathrm{GSp}_6(\mathbf{F}_\ell)$. The maximal irreducible subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ are as follows [9]:

Maximal Irreducible Subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell)$

$\ell = 3$	$\ell = 5$
$\mathrm{SL}_2(\mathbf{F}_3) \wr S_2$	$\mathrm{SL}_2(\mathbf{F}_5) \wr S_2$
$\mathrm{SL}_2(\mathbf{F}_9).2$	$\mathrm{SL}_2(\mathbf{F}_{25}).2$
$2^{1+4}.\Omega_4^-(\mathbf{F}_2)$	$2^{1+4}.\Omega_4^-(\mathbf{F}_2)$
	$\mathrm{GL}_2(\mathbf{F}_5).2$
	$\mathrm{GU}_2(\mathbf{F}_{25}).2$
	$2.A_6$

We omit $\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_2$ from our analysis since a subgroup of $(\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_2) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ is also a subgroup of $\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_3$ and will be treated fully in Section 3.2. The following Proposition and its Corollary show that we have already determined all counterexamples of type $\mathcal{C}_1$.

Proposition 2. *Let $G \subset \mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ be a fixed-point subgroup. Then the projection G_1 of G to $\mathrm{Sp}_4(\mathbf{F}_\ell)$ is reducible.*

Proof. The proof is broken into several cases according to the maximal subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell)$. Let $\ell \in \{3, 5\}$ and consider the group $\mathrm{SL}_2(\mathbf{F}_{\ell^2}).2$ and the associated exact sequence:

$$1 \longrightarrow \mathrm{SL}_2(\mathbf{F}_{\ell^2}) \longrightarrow \mathrm{SL}_2(\mathbf{F}_{\ell^2}).2 \xrightarrow{\pi} S_2 \longrightarrow 1.$$

Let H be the subgroup of $G \subset \mathrm{SL}_2(\mathbf{F}_{\ell^2}).2 \times \mathrm{SL}_2(\mathbf{F}_\ell)$ consisting of pairs (g_1, g_2) with $g_1 \in \ker \pi$ and $g_2 \in G_2 \subset \mathrm{SL}_2(\mathbf{F}_\ell)$. Suppose further that G (and therefore H) is a fixed-point group. We claim that H cannot be a counterexample.

To see this, note that any counterexample H of this type must be a Goursat-subgroup of $\mathrm{SL}_2(\mathbf{F}_{\ell^2}) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ and therefore corresponds to a quadruple (H_1, H_2, H_3, ψ), where $\ker \psi$ is a nontrivial subgroup of H_1 consisting of all the elements without a fixed-point. The fixed-point elements of $\mathrm{SL}_2(\mathbf{F}_{\ell^2})$ form a cyclic subgroup of order 1, ℓ or ℓ^2. By the subgroup structure of the special linear groups, the only possibility is for H_1 to have only one fixed-point element. However, the estimate (1) says that strictly more than half of the elements of H_1 must have a fixed-point. Thus H_1 is trivial which proves our claim.

Therefore H must be a fixed-point subgroup of $\mathrm{SL}_2(\mathbf{F}_{\ell^2}) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ that is not a counterexample. If the trivial representation occurs in the first component, then H_1 is not irreducible, and neither is G_1 by Clifford's theorem. If the trivial representation occurs in the second component, then all elements of the non-trivial coset in the first component must have eigenvalue 1. But all of these elements square to elements of eigenvalue 1. Therefore all of H_1 has eigenvalue 1 and by [8] cannot give rise to a counterexample. Moreover any lift to $\mathrm{SL}_2(\mathbf{F}_{\ell^2}).2$ will not be a counterexample.

Next suppose $G_1 \subset 2^{1+4}.\Omega_4^-(\mathbf{F}_2)$; recall that $\Omega_4^-(2) \simeq A_5$. Due to the complicated nature of this group we appeal to MAGMA for eigenvalue information. In characteristic 3 there are 471 elements having 1 as an eigenvalue, none of which come from the conjugacy classes of elements of order 5. Since at least half the elements of G_1 must have 1 as an eigenvalue, and those without 1 as an eigenvalue must lie in a normal subgroup, it suffices to search among the irreducible subgroups of index divisible by 5 with trivial center and having more than half of the elements with eigenvalue 1 as candidates for G_1. There are none.

In characteristic 5, there are 455 elements of eigenvalue 1, none of which come from the conjugacy classes of order 3. A similar analysis with Magma shows there are no candidates for G_1.

The remainder of the proof is devoted to the subgroups in characteristic 5. We start with the case $G_1 \subset \mathrm{GU}_2(\mathbf{F}_{25}).2$. Suppose first that $G \subset \mathrm{GU}_2(\mathbf{F}_{25}).2 \times \mathrm{SL}_2(\mathbf{F}_5)$ is a counterexample corresponding to the Goursat-tuple (G_1, G_2, G_3, ψ). Let H be subgroup of G with Goursat-tuple (H_1, H_2, H_3, ψ) where $H_1 \subset \mathrm{GU}_2(\mathbf{F}_{25})$ (so that H is an index-2 subgroup of G).

If H is a counterexample, then at least half (but not all) of the elements of H_1 must have 1 as an eigenvalue. An argument nearly identical to the "(2,1)" case of Proposition 1 applies here and shows that the only counterexamples are isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}/2$ or D_4. Neither of these groups have an irreducible H_1. By Clifford's theorem, we need only examine D_4, but any lift of D_4 to $\mathrm{GU}_2(\mathbf{F}_{25}).2$ is not a fixed-point group and so no irreducible counterexamples arise in this way.

If H is not a counterexample, then it has a trivial Jordan-Hölder factor. If it is in the first component, then by Clifford's theorem any lift to $\mathrm{GU}_2(\mathbf{F}_{25}).2$ is not irreducible. If it is in the second component, then H lifts to a Goursat-subgroup $G \subset \mathrm{GU}_2(\mathbf{F}_{25}) \times \{\pm 1\}$ such that any element of G_1 pairing with -1 has 1 as an eigenvalue. Any such G_1 consists entirely of elements having 1 as an eigenvalue. By [8] such a subgroup of $\mathrm{Sp}_4(\mathbf{F}_5)$ necessarily has a trivial Jordan-Hölder factor.

Next let $G_1 \subset \mathrm{GL}_2(\mathbf{F}_5).2$ corresponding to the counterexample $G \subset \mathrm{GL}_2(\mathbf{F}_5).2 \times \mathrm{SL}_2(\mathbf{F}_5)$. Consider the subgroup H of G consisting of elements of the form

$$\begin{pmatrix} A & & \\ & A^* & \\ & & B \end{pmatrix},$$

where $A \in \mathrm{GL}_2(\mathbf{F}_5)$ and $B \in \mathrm{SL}_2(\mathbf{F}_5)$. Then $[G : H] = 2$ and by assumption H is a fixed-point subgroup of $\mathrm{Sp}_4(\mathbf{F}_5) \times \mathrm{SL}_2(\mathbf{F}_5)$. If H is itself a counterexample, then H is one of the groups outlined in Section 3.1.1 (which come from Proposition 1). It is easy to check that any overgroup G of H does not have an irreducible G_1.

Alternatively, the Jordan-Hölder series of H could contain the trivial representation. If it is contained in the "Sp_4" part of H, then by Clifford's theorem G_1 is not irreducible. Thus, it suffices to check if the trivial representation is contained in $\mathrm{SL}_2(\mathbf{F}_\ell)$. If so then in order for G to be a counterexample, it must be the case that G is partitioned into two cosets: H, and a collection of matrices of the form

$$\begin{pmatrix} & a & \\ b & & \\ & & -I \end{pmatrix},$$

where each of the the 4×4 matrices $\left(\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right)$ have 1 as an eigenvalue. It is easy to check that this forces G_1 to consist entirely of elements having 1 as an eigenvalue. By [8], a subgroup of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ with this property necessarily has trivial Jordan-Hölder factor. Therefore, there do not exist counterexamples of this type with an irreducible G_1.

Finally, suppose $G_1 \subset 2.A_6$. The maximal subgroups of $2.A_6$ are $2.S_4$, $2.S_4$, $2.F_{36}$, $2.A_5$ and $2.A_5$. One first checks that the only element of $2.S_4$ (either copy) having 1 as an eigenvalue is the identity. The estimate

$$\frac{\#G_1}{2} \geq \frac{\#G_1}{[G_2 : G_3]} \geq \#P + 1$$

implies that G_1 has order at most 2 and hence is not irreducible.

The group $2.F_{36}$ contains 5 elements having 1 as an eigenvalue – the identity plus four elements coming from a conjugacy class of elements having order 3. Applying the estimate again implies that $\#G_1 \leq 9$ and is divisible by 3. However, no group of order 9, 6, or 3 can give rise to a counterexample (in the non-cyclic cases, no groups of this order have irreducible 4-dimensional representations over $\mathbf{F}_5$).

A similar analysis shows that no subgroup of $2.A_5$ gives rise to a counterexample. This finishes the proof of the Proposition. $\square$

Corollary 1. *Let $G \subset \mathrm{GSp}_4(\mathbf{F}_\ell) \times \mathrm{GL}_2(\mathbf{F}_\ell)$ be a fixed-point subgroup. Then the projection G_1 of G to $\mathrm{GSp}_4(\mathbf{F}_\ell)$ is reducible.*

Proof. The lift of any fixed-point subgroup H of $\mathrm{Sp}_4(\mathbf{F}_\ell)\times\mathrm{SL}_2(\mathbf{F}_\ell)$ to $\mathrm{GSp}_6(\mathbf{F}_\ell)$ differs from H by a diagonal element, which preserves the dimensions of Jordan-Hölder factors. $\square$

3.2. **Type $\mathcal{C}_2$.** It suffices to classify the irreducible counterexamples of this type since otherwise G is contained in some parabolic subgroup and these have already been enumerated. We show that there are no such counterexamples.

Proposition 3. *Let $\ell \in \{3,5\}$ and suppose G is a subgroup of $\mathrm{GSp}_6(\mathbf{F}_\ell)$ of type $\mathcal{C}_2$. If G acts irreducibly, then G cannot be a counterexample.*

Proof. First suppose that $G \subset \mathrm{GL}_3(\mathbf{F}_\ell).2 \subset \mathrm{Sp}_6(\mathbf{F}_\ell)$ is an irreducible fixed-point group and consider the exact sequence

$$1 \longrightarrow \mathrm{GL}_3(\mathbf{F}_\ell) \longrightarrow \mathrm{GL}_3(\mathbf{F}_\ell).2 \xrightarrow{\pi} \mathrm{GL}_3(\mathbf{F}_\ell).2/\,\mathrm{GL}_3(\mathbf{F}_\ell) \longrightarrow 1.$$

The intersection $G\cap\ker\pi$ is a fixed-point subgroup of $\mathrm{GL}_3(\mathbf{F}_\ell)$ and by Clifford's theorem the Jordan-Hölder factors of the module $\mathbf{F}_\ell[G\cap\ker\psi]$ must have dimensions 3 and 3. However, a representative of the non-trivial coset of $\mathrm{GL}_3(\mathbf{F}_\ell).2/\,\mathrm{GL}_3(\mathbf{F}_\ell)$ is $\left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$, which squares to the non-trivial central element of $\mathrm{Sp}_6(\mathbf{F}_\ell)$, contradicting the fixed-point assumption on G. Since $\mathrm{Sp}_6(\mathbf{F}_\ell)$ contains no such irreducible counterexamples, neither does $\mathrm{GSp}_6(\mathbf{F}_\ell)$.

Next suppose $G \subset \mathrm{SL}_2(\mathbf{F}_\ell) \wr S_3 \subset \mathrm{Sp}_6(\mathbf{F}_\ell)$ is an irreducible counterexample. Any such G would restrict to a fixed-point subgroup $G\cap\mathrm{SL}_2(\mathbf{F}_\ell)^3$. If $G\cap\mathrm{SL}_2(\mathbf{F}_\ell)^3$ is itself a counterexample, then it follows that $G\cap\mathrm{SL}_2(\mathbf{F}_\ell) \simeq \mathbf{Z}/2\times\mathbf{Z}/2$ (this is an easy consequence of the completely reducible case of Proposition 1; alternatively it is proved in [3, Lemma 22]). It is easy to check that in this case $G \simeq S_4$ with Jordan-Hölder factors of dimensions 3 and 3. On the other hand, if $G\cap\mathrm{SL}_2(\mathbf{F}_\ell)^3$ is not a counterexample, then its Jordan-Hölder series contains a trivial factor (in fact, it contains at least 2 trivial factors since one occurs inside SL_2). By Clifford's theorem, all factors are 1-dimensional. But the natural permutation representation on these three hyperbolic planes decomposes into two irreducible 3-dimensional representations, hence G could not have been irreducible. $\square$

3.3. **Type $\mathcal{C}_3$.** In this section we analyze the field-extension subgroups of $\mathrm{GSp}_6(\mathbf{F}_\ell)$ and show that they do not contain any irreducible fixed-point subgroups.

Lemma 1. *Let $\ell \in \{3,5\}$. The group $\mathrm{GSp}_6(\mathbf{F}_\ell)$ contains no irreducible fixed-point subgroups of SL_2-type.*

Proof. Consider the field-extension embedding $\mathscr{F} : \mathrm{SL}_2(\mathbf{F}_{\ell^3}).3 \hookrightarrow \mathrm{Sp}_6(\mathbf{F}_\ell)$. If $\alpha \in \mathbf{F}_{\ell^3}$, let $L_\alpha \in M_3(\mathbf{F}_\ell)$ be the linear transformation "multiplication by α". If L_α has 1 as an eigenvalue, then the characteristic polynomial $c_{L_\alpha}(x)$ of L_α vanishes at 1 and hence has $(x-1)$ as a factor. If $\alpha \neq 1$, then α must be the root of an irreducible quadratic polynomial over $\mathbf{F}_\ell$, which is impossible as $\alpha \in \mathbf{F}_{\ell^3}$. Thus, the only $\alpha \in \mathbf{F}_{\ell^3}$ for which L_α has 1 as an eigenvalue is $\alpha = 1$. This means the fixed-point subgroups of $\mathrm{SL}_2(\mathbf{F}_{\ell^3})$ are in one-to-one correspondence with the fixed-point subgroups of $\mathscr{F}(\mathrm{SL}_2(\mathbf{F}_{\ell^3}))$. Therefore, the maximal fixed-point subgroups of $\mathscr{F}(\mathrm{SL}_2(\mathbf{F}_{\ell^3}))$ are the Sylow-ℓ subgroups, whose Jordan Hölder factors are all trivial.

When $\ell = 3$, any lift to $\mathscr{F}(\mathrm{SL}_2(\mathbf{F}_{27}).3)$ is a 3-group, hence has trivial Jordan-Hölder factors. A further lift to $\mathrm{GSp}_6(\mathbf{F}_3)$ is not irreducible. Similarly, when $\ell = 5$ a lift to $\mathscr{F}(\mathrm{SL}_2(\mathbf{F}_{125}).3)$ either has 1-dimensional or 3-dimensional Jordan-Hölder factors and a subsequent lift to $\mathrm{GSp}_6(\mathbf{F}_5)$ preserves these dimensions. $\square$

For the unitary groups we proceed similarly and give details when $\ell = 3$ and sketch the idea when $\ell = 5$.

Lemma 2. *The group $\mathrm{GSp}_6(\mathbf{F}_3)$ contains no irreducible, fixed-point subgroups of unitary $\mathcal{C}_3$-type.*

Proof. The maximal subgroups of $\mathrm{GU}_3(\mathbf{F}_9).2$ are of the form $M.2$, where M is a maximal subgroup (not necessarily proper) of $\mathrm{GU}_3(\mathbf{F}_9)$, plus an additional (irreducible) index-2 subgroup. If G is an irreducible (six-dimensional) subgroup of $\mathrm{GU}_3(\mathbf{F}_9).2$ that is not contained in $\mathrm{GU}_3(\mathbf{F}_9)$, then $[G : G\cap\mathrm{GU}_3(\mathbf{F}_9)] = 2$. Therefore, by Clifford's theorem, any irreducible subgroup of $\mathrm{GU}_3(\mathbf{F}_9).2$ either restricts to an irreducible subgroup of $\mathrm{GU}_3(\mathbf{F}_9)$ or to one with two 3-dimensional stable subspaces.

It is known that $\mathrm{GU}_3(\mathbf{F}_9) \simeq 4\times\mathrm{SU}_3(\mathbf{F}_9)$ and the maximal subgroups of $\mathrm{SU}_3(\mathbf{F}_9)$, together with dimensions of the Jordan-Hölder factors (of it's natural module), are as follows. Note that the dimensions are the same for the central lifts to $\mathrm{GU}_3(\mathbf{F}_9)$:

Subgroup	$3^3{:}8$	$\mathrm{GU}_2(\mathbf{F}_9)$	$4^2.S_3$	$\mathrm{PSL}_2(\mathbf{F}_7)$
Dimensions	(1,1,1)	(2,1)	(3)	(3)

The field-extension embeddings of these groups produce Jordan-Hölder factors of dimensions (2,2,2), (4,2), (6), and (6), respectively. Altogether, this shows that a maximal irreducible (six-dimensional) subgroup of $\mathrm{GU}_3(\mathbf{F}_9).2$ restricts to an irreducible subgroup of $\mathrm{GU}_3(\mathbf{F}_9)$.

The maximal irreducible subgroups of $\mathrm{GU}_3(\mathbf{F}_9)$ are the central lifts of $\mathrm{PSL}_2(\mathbf{F}_7)$ and $4^2.S_3$, however it suffices to analyze $\mathrm{PSL}_2(\mathbf{F}_7)$ and $4^2.S_3$ since the central lifts will not be fixed-point groups. The group $\mathrm{PSL}_2(\mathbf{F}_7)$ is not a fixed-point group because of the elements of order 7, and any subgroup of index divisible by 7 is not irreducible. Similarly, $4^2.S_3$ is not a fixed-point subgroup and a search of its maximal subgroups reveals that it possesses no irreducible fixed-point subgroups.

Thus, $\mathrm{GU}_3(\mathbf{F}_9)$ has no irreducible fixed-point subgroups and so neither does $\mathrm{GU}_3(\mathbf{F}_9).2$. Finally, because the dimensions of the irreducible factors are preserved in $\mathrm{GSp}_6(\mathbf{F}_3)$, there are no irreducible fixed-point subgroups of $\mathrm{GSp}_6(\mathbf{F}_3)$ of unitary $\mathcal{C}_3$-type. □

The structure of $\mathrm{GU}_3(\mathbf{F}_{25})$ is more complicated than that of $\mathrm{GU}_3(\mathbf{F}_9)$; in particular, $\mathrm{SU}_3(\mathbf{F}_{25})$ is not simple whereas $\mathrm{SU}_3(\mathbf{F}_9)$ is, and the exceptional subgroups of $\mathrm{SU}_3(\mathbf{F}_{25})$ isomorphic to $3.A_7$ and $3.M_{10}$ (3 copies each) fuse in $\mathrm{GU}_3(\mathbf{F}_{25})$. Nonetheless, the analysis of this group is similar to Lemma 2 and the conclusion is the same, so we omit the details.

3.4. **Type $\mathcal{C}_4$.** The group $\mathrm{O}_3(\mathbf{F}_5)\otimes\mathrm{SL}_2(\mathbf{F}_5)$ is the image of the tensor-product representation of $\mathrm{O}_3(\mathbf{F}_5)\times\mathrm{SL}_2(\mathbf{F}_5)$. Recall the isomorphisms $\mathrm{O}_3(\mathbf{F}_5)\simeq\mathrm{SO}_3(\mathbf{F}_5)\times 2\simeq S_5\times 2$. Let the image of $G\subset\mathrm{O}_3(\mathbf{F}_5)\times\mathrm{SL}_2(\mathbf{F}_5)$ in $\mathrm{O}_3(\mathbf{F}_5)\otimes\mathrm{SL}_2(\mathbf{F}_5)$ be denoted by $\mathcal{G}$. If $(g_1,g_2)\in G$ has eigenvalues $\{\lambda_1,\lambda_2,\lambda_3\},\{\mu_1,\mu_2\}$, then the eigenvalues of the image of (g_1,g_2) in $\mathcal{G}$ are the $\lambda_i\mu_j$. As in the previous sections, it suffices to classify the irreducible counterexamples. The tensor product of two irreducible representations is not necessarily irreducible, but it suffices to analyze only the irreducible subgroups of $\mathrm{O}_3(\mathbf{F}_5)$ and $\mathrm{SL}_2(\mathbf{F}_5)$.

Lemma 3. *Let G be a fixed-point Goursat-subgroup of $\mathrm{O}_3(\mathbf{F}_5)\times\mathrm{SL}_2(\mathbf{F}_5)$ with quadruple (G_1,G_2,G_3,ψ). Then G_3 consists entirely of elements with eigenvalue 1.*

Proof. The identity element of G_1 pairs with all of G_3, hence the eigenvalues of the elements of $\{1\}\otimes G_3$ are simply those of the elements of G_3. This proves the lemma. □

Proposition 4. *There do not exist fixed-point irreducible subgroups of $\mathrm{O}_3(\mathbf{F}_5)\otimes\mathrm{SL}_2(\mathbf{F}_5)$.*

Proof. The maximal irreducible subgroups of $\mathrm{O}_3(\mathbf{F}_5)$ are S_5, $A_5\times 2$, and $S_4\times 2$, while the maximal irreducible subgroups of $\mathrm{SL}_2(\mathbf{F}_5)$ are $\mathbf{Z}/3\rtimes\mathbf{Z}/4$ and $2.A_4$. In light of Lemma 3 and the fact that G is a Goursat-subgroup, it is necessary that $G_3\subset\mathrm{SL}_2(\mathbf{F}_5)$ be trivial. Moreover, this also shows that $\ker\psi$ must be a fixed-point subgroup of $\mathrm{O}_3(\mathbf{F}_5)$.

First consider the possibilities for G_2: no subgroup of S_5, $A_5\times 2$ or $S_4\times 2$ has $\mathbf{Z}/3\rtimes\mathbf{Z}/4$ or $2.A_4$ as a quotient. Thus, G_2 must be a proper irreducible subgroup of either $\mathbf{Z}/3\rtimes\mathbf{Z}/4$ or $2.A_4$. The only possibilities for G_2 are therefore Q_8, $\mathbf{Z}/6$, or $\mathbf{Z}/3$. It is therefore necessary that G_1 be an irreducible subgroup of $S_4\times 2$, $A_5\times 2$, or S_5 with quotient Q_8, $\mathbf{Z}/6$, or $\mathbf{Z}/3$ and with $\ker\psi$ a fixed-point subgroup.

It is easy to rule out all but $A_4\times 2$ and A_4 as possibilities for G_1 with $\mathbf{Z}/6$ or $\mathbf{Z}/3$ for G_2. The tensor-product representations $(A_4\times 2)\otimes\mathbf{Z}/6$ and $A_4\otimes\mathbf{Z}/3$ give rise to fixed-point subgroups of $\mathrm{O}_3(\mathbf{F}_5)\otimes\mathrm{SL}_2(\mathbf{F}_5)$, but are reducible with Jordan-Hölder dimensions 3 and 3. Thus, there are no irreducible fixed-point subgroups of $\mathrm{O}_3(\mathbf{F}_5)\otimes\mathrm{SL}_2(\mathbf{F}_5)$. □

3.5. **Type $\mathcal{S}$.** The two subgroups of $\mathrm{Sp}_6(\mathbf{F}_3)$ of type $\mathcal{S}$ are $2.A_5$ and $\mathrm{SL}_2(\mathbf{F}_{13})$. The orders of the elements of $2.A_5$ in fixed-point conjugacy classes are 1, 3, and 5. The only subgroups consisting entirely of elements of orders 1, 3, or 5 are cyclic and therefore cannot be counterexamples. There are two non-conjugate subgroups of $\mathrm{Sp}_6(\mathbf{F}_3)$ isomorphic to $\mathrm{SL}_2(\mathbf{F}_{13})$. Each has the property that its fixed-point conjugacy classes contain only elements of order 3. This means any fixed-point subgroup of $\mathrm{SL}_2(\mathbf{F}_{13})$ is a 3-group. Since we are working in characteristic 3, the Jordan-Hölder series consists of trivial modules. In each of these cases, the degree-2 lifts of the fixed-point subgroups are not irreducible (Clifford's theorem) and hence do not give rise to any new counterexamples.

The subgroups of $\mathrm{Sp}_6(\mathbf{F}_5)$ of type $\mathcal{S}$ are $2.A_5$ and $2.J_2$. The group $2.A_5$ is isomorphic to $\mathrm{SL}_2(\mathbf{F}_5)$ and the embedding in $\mathrm{Sp}_6(\mathbf{F}_5)$ is via Sym^5, the 5th symmetric power representation. This representation is reducible and decomposes into $\mathrm{Sym}^1 + \mathrm{Sym}^3$. By Clifford's theorem, any lift to $\mathrm{GSp}_6(\mathbf{F}_5)$ preserves the dimensions of the simple modules so we do not get any new counterexamples.

The group $2.J_2$ is not a fixed-point group and so we check its maximal subgroups. The maximal subgroups M of J_2 are as follows (so the maximals of $2.J_2$ are degree-2 central extensions of M; we also list the dimensions of the Jordan-Hölder factors of $\mathbf{F}_5[2.M]$:

M	Dimension
A_5	(2,4)
$5^2 : D_6$	(1,1,2,2)
$L_3(2) : 2$	(6)
$A_5 \times D_5$	(2,2,2)
$A_4 \times A_5$	(6)
$2^{2+4} : (3 \times S_3)$	(6)
$2_-^{1+4} : A_5$	(2,4)
$3 \cdot \mathrm{PGL}_2(\mathbf{F}_9)$	(6)
$U_3(\mathbf{F}_9)$	(6)

At this point we invoke the computer-algebra package MAGMA. The following command shows that there are 293,875 fixed-point elements of $G := 2.J_2$ and have orders 1, 2, 3, 4, 5, and 10:

```
C:=ConjugacyClasses(G);
for i:=1 to #C do;
print FactoredCharacteristicPolynomial(C[i][3]);
end for;
C;
```

We are therefore searching for irreducible fixed-point subgroups of $2.J_2$ with trivial center and that can only contain elements of orders 1, 2, 3, 4, 5, or 10. Another search using MAGMA reveals that no such groups exist.

4. Endomorphism Rings – Examples and Future Work

The classification above and in [3, 4] give examples of representations with interesting properties. A natural question is: how much information about the abelian varieties can be deduced purely from the mod ℓ representation? In particular, can the endomorphism ring of an abelian variety A/K be computed from $\operatorname{im}\overline{\rho_\ell}$? In certain instances, the answer follows readily from a construction of Zarhin [14, 15]. In particular, when $\overline{\rho_\ell}$ is *very simple* then $\mathrm{End}(A) \simeq \mathbf{Z}$. In some cases, this allows for the endomorphism rings of these abelian varieties which are counterexamples to be determined.

As a first example, we recall the main counterexample of [4]. There it was shown that if $\dim A = 3$ and the number of $\mathbf{F}_\mathfrak{p}$-rational points is even for almost all $\mathfrak{p}$, then there exists a K-isogenous A' such that $\#A'(K)_{tor}$ is even. In other words, there do not exist any counterexamples to the local-to-global divisibility problem in dimension 3. In dimension 4 however, one can check that the irreducible, symplectic 8-dimensional representation of $\mathrm{SL}_3(\mathbf{F}_2)$ (the Steinberg representation) has $\det(1-g) = 0$ for all $g \in \mathrm{SL}_3(\mathbf{F}_2)$. This representation 1) is absolutely irreducible, 2) does not decompose as the tensor product of two (non-trivial) representations of smaller degree, and 3) is not induced from a representation of a proper subgroup. Zarhin calls this a *very simple* representation and in [14, 15] shows that when $\overline{\rho_\ell}$ is very simple, then $\mathrm{End}(A) = \mathbf{Z}$ or $\mathrm{char}(K) > 0$ and A is supersingular. Since K is a number field, we conclude that for our four-dimensional example, $\mathrm{End}(A) = \mathbf{Z}$.

Similarly, the 14-dimensional representation of $\mathrm{PSL}_2(\mathbf{F}_{13})$ and the 20-dimensional representation of A_7 over the field $\mathbf{F}_2$ are very simple representations of fixed-point subgroups of $\mathrm{Sp}_{14}(\mathbf{F}_2)$ and $\mathrm{Sp}_{20}(\mathbf{F}_2)$. These give rise to counterexamples in dimensions 7 and 10. A non-example is the irreducible fixed-point subgroup A_6 of $\mathrm{GSp}_{16}(\mathbf{F}_2)$. This representation is not absolutely irreducible and splits into two 8-dimensional representations over its splitting field $\mathbf{F}_4$.

The counterexamples in this paper do not lend themselves immediately to this theorem since none of the groups act irreducibly on their underlying vector space. An interesting question is whether the theorems of [14, 15] can be extended depending on the absolute simplicity, etc. of its Jordan-Hölder factors. The following appendix provides a complement to the theorems of [14, 15].

Appendix A. Endomorphisms of low-dimensional abelian varieties and points of order 2

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Let K be a field, $\overline{K}$ its algebraic closure, $\mathrm{Gal}(K) = \mathrm{Aut}(\overline{K}/K)$ its absolute Galois group. Let ℓ be a prime different from $\mathrm{char}(K)$ and $\mathbf{F}_\ell$ the finite prime field of characteristic ℓ. Let X be an abelian variety over K of positive dimension g. We write X_ℓ for the kernel of multiplication by ℓ in $X(\overline{K})$; it is well known that X_ℓ is a $2g$-dimensional $\mathbf{F}_\ell$-vector space and a Galois submodule of $X(\overline{K})$. We write

$$\tilde{\rho}_\ell : \mathrm{Gal}(K) \to \mathrm{Aut}(X_\ell) = \mathrm{Aut}_{\mathbf{F}_\ell}(X_\ell)$$

for the corresponding Galois structure homomorphism and denote by

$$\tilde{G}_\ell \subset \mathrm{Aut}_{\mathbf{F}_\ell}(X_\ell)$$

its image. Clearly, X_ℓ carries the natural structure of a faithful $\tilde{G}_\ell$-module. We write $\mathrm{End}(X)$ for the ring of all $\overline{K}$-endomorphisms of X.

We refer the reader to [14, 15] for the definition and basic properties of *very simple* representations. We will need the following two assertions.

Lemma 4 (Lemma 2.3 of [14])**.** *If the Galois module X_ℓ is very simple then either $\mathrm{End}(X) = \mathbf{Z}$ or $\mathrm{char}(K) > 0$ and X is a supersingular abelian variety.*

Theorem 2. *Let V be a finite-dimensional $\mathbf{F}_2$-vector space of positive dimension n and let $G \subset \mathrm{Aut}_{\mathbf{F}_2}(X_2)$ be a perfect subgroup such that the G-module V is absolutely simple.*
If $3 \le n \le 8$ then the G-module V is very simple.

Proof. Clearly, $G \ne \{1\}$. It is well known that if G has a subgroup of index $m > 1$ then there is a *nontrivial* homomorphism from G to the full symmetric group S_m and the perfectness of G implies that S_m is *not* solvable, i.e., $m \ge 5$. It follows that if G has a proper subgroup of index m and m divides n then $5 \le m \le 8$ and therefore

$$n = m = 8.$$

Assume that the absolutely simple G-module V is *not* very simple. It follows from [14, Corollary 4.2] that one of the following two conditions holds.

(i) The G-module V splits into a tensor product $V = V_1 \otimes_{\mathbf{F}_2} V_2$ of two absolutely simple G-modules V_1 and V_2 with

$$\dim_{\mathbf{F}_2}(V_1) > 1,\ \dim_{\mathbf{F}_2}(V_2) > 1.$$

(ii) The G-module V is induced from a representation of a *proper* subgroup H of G.

If case (i) holds then

$$8 \ge n = \dim_{\mathbf{F}_2}(V) = \dim_{\mathbf{F}_2}(V_1) \cdot \dim_{\mathbf{F}_2}(V_2).$$

It follows easily that $\dim_{\mathbf{F}_2}(V_i) = 2$ for (at least) one of indices i. Then the corresponding structure homomorphism

$$G \to \mathrm{Aut}_{\mathbf{F}_2}(V_i) \cong \mathrm{GL}(2, \mathbf{F}_2)$$

is trivial, because G is perfect and $\mathrm{GL}(2, \mathbf{F}_2) \cong S_3$ is solvable. This contradicts the absolute simplicity of V_i and rules out case (1). This implies that there exists a proper subgroup $H \subset G$ of index $m > 1$ and an H-module W such that V is induced from W. It follows that

$$n = \dim_{\mathbf{F}_2}(V) = m \cdot \dim_{\mathbf{F}_2}(W).$$

In particular, m divides n and therefore (as we have seen above) $n = m$. This implies that $\dim_{\mathbf{F}_2}(W) = 1$ and the corresponding structure homomorphism

$$H \to \mathrm{Aut}_{\mathbf{F}_2}(W) = \mathbf{F}_2^* = \{1\}$$

is trivial. But then the induced G-module V is *not* simple [15, Example 3.4] [1]. The obtained contradiction proves the very simplicity of V. $\square$

Theorem 3. *Suppose that* $\ell = 2$ *and* $g = \dim(X)$ *is either* 2 *or* 3 *or* 4. *Assume that* $\tilde{G}_2$ *contains a perfect subgroup* G *such that the* G*-module* X_2 *is absolutely simple. Then either* $\mathrm{End}(X) = \mathbf{Z}$ *or* $\mathrm{char}(K) > 0$ *and* X *is a supersingular abelian variety.*

Proof. Enlarging K if necessary, we may and will assume that $G = \tilde{G}_2$, i.e.,

$$\tilde{\rho}_2(\mathrm{Gal}(K)) = G \subset \mathrm{Aut}_{\mathbf{F}_2}(X_2).$$

Applying Theorem 2 to $V = X_2$ and $n = 2g$, we conclude that the G-module X_2 is very simple. Since $\tilde{\rho}_2(\mathrm{Gal}(K)) = G$, the very simplicity of the G-module X_2 implies that the $\mathrm{Gal}(K)$-module X_2 is also very simple. Now the result follows from Lemma 4. $\square$

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[1] The V' in the first line of second paragraph of Example 3.4 in [15, p. 156] should be W.