Asymptotics of the 9j-symbol

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Introduction

What characterizes physical states in a quantum theory of gravity? One sense of this question that would be interesting to understand is how incorporating quantum effects discretizes the geometry of gravitation. An early, and elegant, understanding of a discrete approach to classical gravity was provided by Tullio Regge. In Regge's calculus the Einstein-Hilbert action in **3*D***, on a manifold M with boundary,

$$S_{EH} = \int_M d^3x\sqrt{g}R + \int_{\partial M} d^2x\sqrt{h}K$$

is converted into a discrete sum over a decomposition of the manifold into tetrahedra,

$$S_R = \sum_{e\in int(M)} J_e\delta_e + \sum_{e\in\partial M} J_e\theta_e.$$

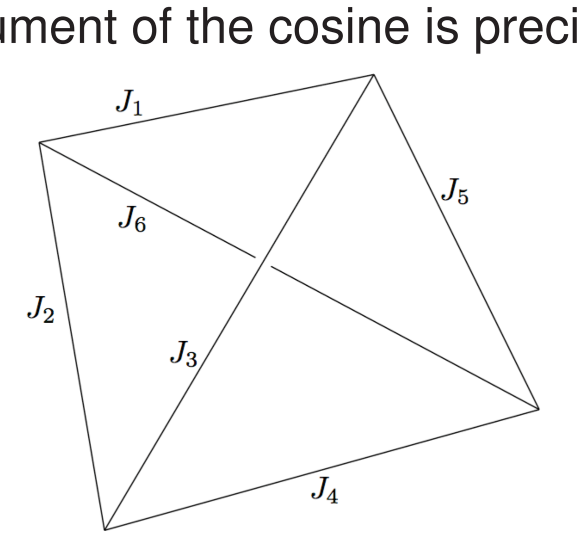
Here the J_e are the lengths of the edges of the tetrahedral decomposition, the δ_e are the deficit angles about internal edges, and the θ_e are the external dihedral angles between faces of the boundary of the decomposition. If the interior of the manifold is flat space then the deficit angles δ_e are zero and the only contribution to the Regge action is from the boundary. This poster describes a set of geometries that arise in the classical limit of the **9*j***-symbol that are closely related to the Regge boundary action $S_R = \sum_{e\in\partial M} J_e\theta_e$.

History

Just one year after Regge's work on discrete gravity, Ponzano and Regge found a remarkable semiclassical formula for the **6*j***-symbol. They considered the **6*j***-symbol – originally defined in angular momentum theory – and looked at its asymptotics in the large j_i limit. They found an asymptotic formula intimately related to the geometry of a tetrahedron:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \sim \frac{1}{\sqrt{12\pi V}}\cos\Big(\sum_i J_i\theta_i\Big).$$

The j_i are related to the edge lengths of the tetrahedron by $J_i = j_i + \frac{1}{2}$. The amplitude is expressed in terms of the volume V of the tetrahedron and the phase is given by the edge lengths times the external dihedral angles θ_i of the tetrahedron. The argument of the cosine is precisely the Regge action!

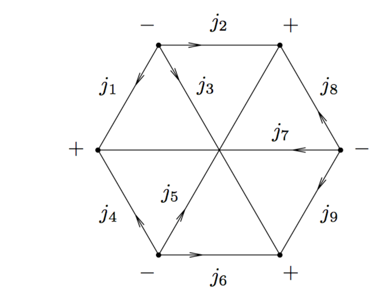


This discovery led them to propose the first sum over histories theory for quantum gravity.

A new asymptotics: the 9j-symbol

A number of technical difficulties prevented Ponzano and Regge and subsequent researches from extending the **6*j*** result to more complicated symbols. The **9*j***-symbol is the next most complicated symbol after the **6*j***-symbol. Here are the symbolic and graphical (usually called a spin network) representations:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}$$



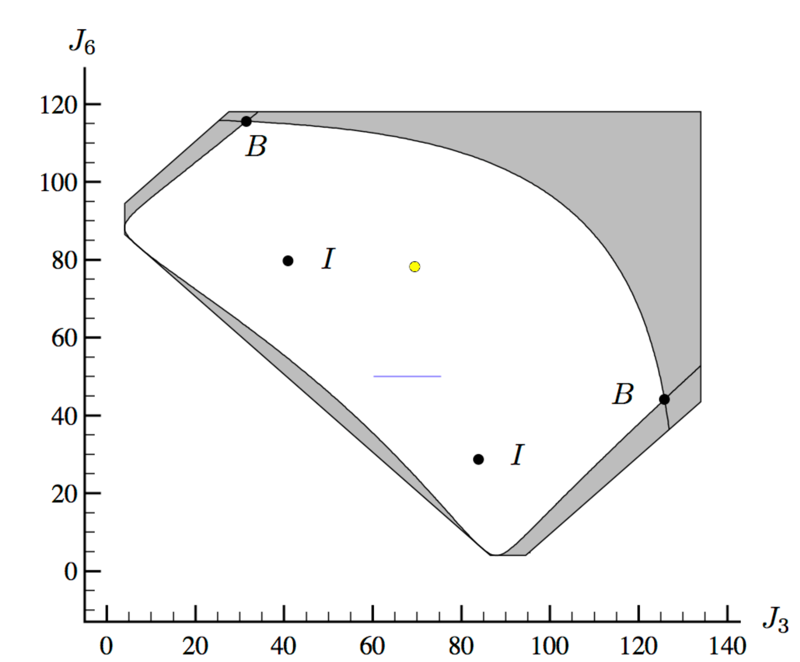
We have studied the asymptotics of this symbol using a classical Schwinger oscillator phase space and surmounted all of the technical difficulties mentioned above. This leads to an intriguing generalization of the Ponzano and Regge result, once again valid in the classical (large j_i) limit:

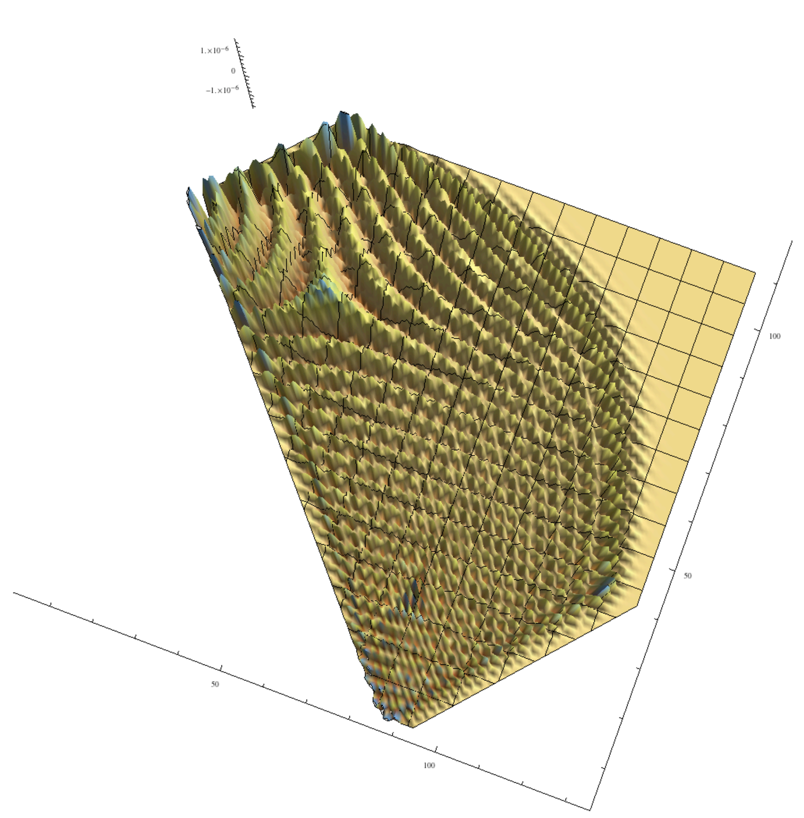
$$\{9j\} \sim A_1\cos\Big(\sum_i J_i\theta_i\Big) + A_2\sin\Big(\sum_i J_i\theta_i\Big).$$

This time there are two classical geometries (see vector diagrams at right) associated to each classically allowed symbol and these correspond precisely to the two terms in the formula above.

Configuration Space

The large number of parameters in a **9*j***-symbol complicate its study. We are rescued from this complexity by performing a symmetry reduction of the full classical phase space. At *right* is a plot of a configuration space slice of this symmetry reduced phase space with coordinates J_3 and J_6. The convex bounding region is determined by the triangle inequalities.



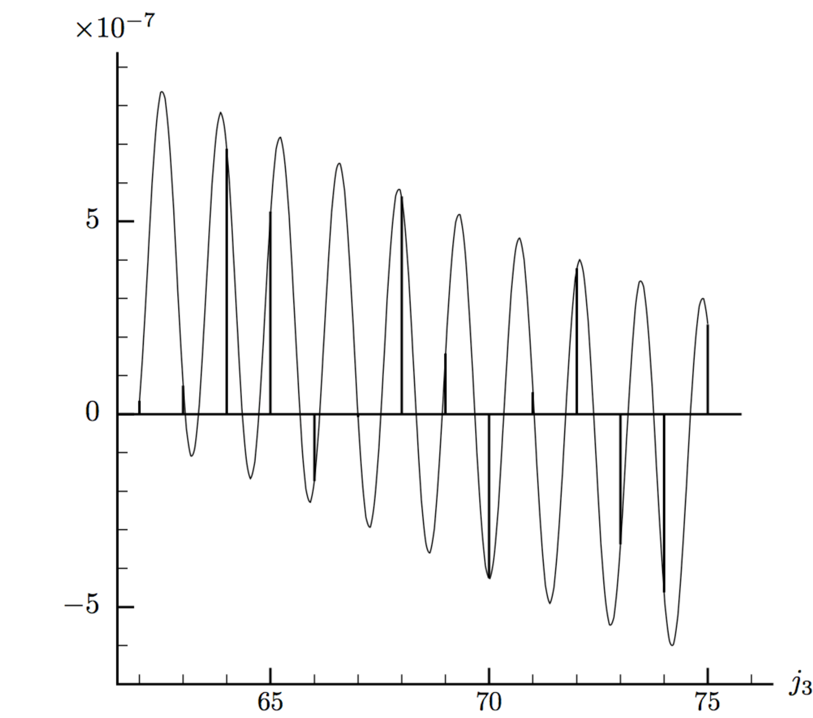


Interior to this boundary is a white region where the solutions are classically allowed and a grey region where they are classically forbidden. At *left* is a plot of the exact values of the **9*j***-symbol over the same region of parameters. The symbol is only defined on a lattice of points in this plane, for ease of visualization the function has been interpolated between these points. Note the caustic amplification of the symbol at the points I and B.

Comparison

At right the exact values (thick bars) and asymptotic formula (thin line) for the **9*j***-symbol are compared. Here, as in all the other plots on this poster we are using the following values for the 9 ***j***s:

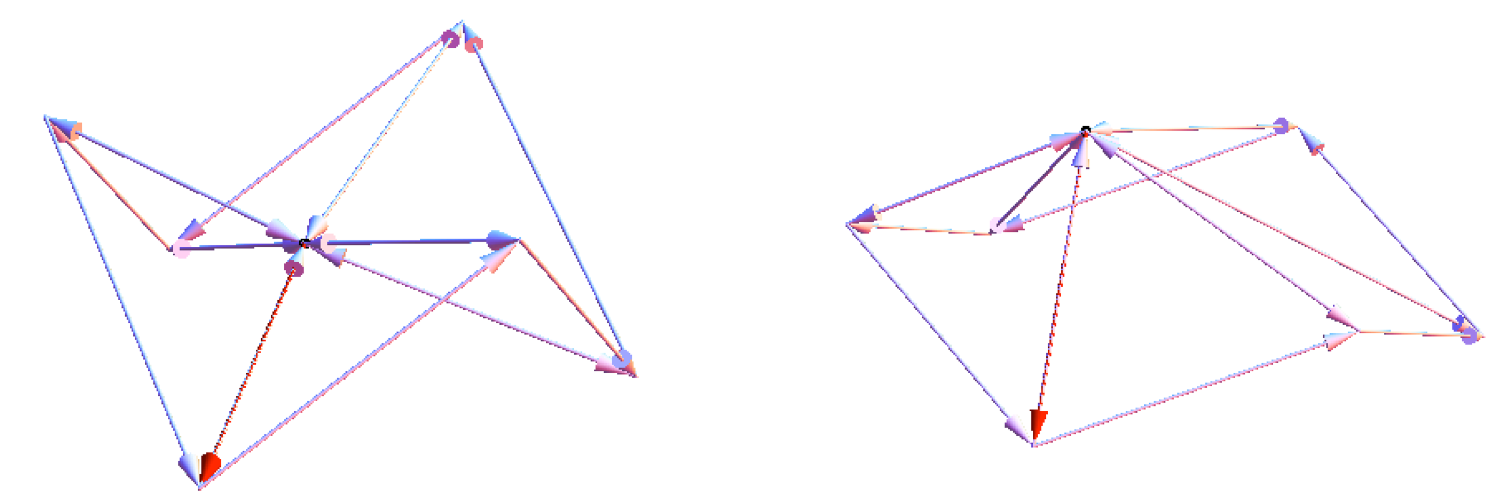
$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix} = \begin{Bmatrix} 129/2 & 137/2 & j_3 \\ 113/2 & 121/2 & j_6 \\ 64 & 108 & 90 \end{Bmatrix}$$



The above plot cuts through the configuration space along the value $j_6 = 50$, see the blue line in the configuration space plane depicted middle left.

Classical geometry of the 9j-symbol

The classically allowed region is characterized by the fact that there are two geometries that can be constructed out of vectors whose lengths are give by the 9 J_i and which can be embedded in flat space. These two geometries are depicted *below* for the yellow point of the classically allowed region shown at left.



Remarkably, a non-trivial symmetry of the **9*j***-symbol interchanges these two classical geometries, exchanging the sine and cosine terms of the asymptotic formula.

Conclusion

The asymptotic result for the **9*j***-symbol extends the classic result of Ponzano and Regge. An interesting comparison can be made with boundary value problems in particle mechanics. If you solve a particle mechanics problem with the configuration at both ends of the path specified, the result is sensitive to the time interval between the specified boundary values. For short time intervals there is a unique path connecting the boundary values, if one exists at all. For longer time intervals there can be many trajectories, having different initial speeds, that connect the two boundary values. Here we have shown that similar results hold for the boundary value problem of Regge calculus; the specification of the 9 j_is specifies the boundary conditions and more than one classical geometry fills in this boundary data.

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