Maslov indices, Poisson brackets, and singular differential forms

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Abstract – Maslov indices are integers that appear in semiclassical wave functions and quantization conditions. They are often notoriously difficult to compute. We present methods of computing the Maslov index that rely only on typically elementary Poisson brackets and simple linear algebra. We also present a singular differential form, whose integral along a curve gives the Maslov index of that curve. The form is closed but not exact, and transforms by an exact differential under canonical transformations. We illustrate the method with the $6j$-symbol, which is important in angular momentum theory and in quantum gravity.

Introduction. – Maslov indices are integers representing phase shifts in semiclassical expressions for wave functions, matrix elements, and position-space representations of operators in quantum mechanics [1–3]. They are essential to deriving correct quantization conditions and for obtaining the correct interference patterns in sums over classical paths, for example, in periodic orbit expansions [4–15]. Maslov indices are responsible for the zero-point energy of oscillators and are useful in applications from quantum optics to quantum gravity.

In this paper we present new techniques for calculating the Maslov index that involve only simple Poisson brackets and linear algebra. For example, in the analysis of spin networks we need just the standard Poisson brackets of the components of angular momenta among themselves. Further, a linear dependency between the differentials of the angular momenta appears at caustics and allows the calculation to proceed without reference to conjugate angles, a significant simplification. This article deals with the Maslov index along open or closed paths on a Lagrangian manifold; the special case of closed paths, for which the Maslov index is a winding number in the complex plane, is significantly simpler [7–10].

Our techniques allow us to put the transformation properties of the Maslov index under canonical transformations into a neat form, including the invariance for the case of a closed path. The notion that quantization conditions should be invariant under canonical transformations goes back to Einstein [16], and has been an important part of the mathematical literature on the Maslov index in recent years [17].

The mathematical literature on the Maslov index is extensive [17–21] but difficult to use for computational purposes in physical problems. This is the case, for example, in the asymptotics of the Wigner $6j$-symbol [22], which plays an important role in the setting for the volume operator in loop gravity [23]. The $6j$-symbol has played a central role in the road to [24–26] and conceptual development of [27, 28] loop gravity, but several authors who have studied its asymptotics have had to appeal to numerical methods to compute the Maslov index [29, 30]. The only successful calculations of the Maslov index for the $6j$-symbol have been those that reduced the problem to a one-dimensional system [31–33], an option not available in problems that are intrinsically multidimensional, such as the $9j$-symbol [34].

We also present several relations satisfied by the Maslov index, including an expression for it in terms of a singular differential form that is closed but not exact. We refer to this differential form as singular because it is expressed in terms of Dirac delta functions times the differentials of smooth functions. This approach unifies the phase contributions of the action and the Maslov phase into a single differential form.

We work in the phase space $\mathbb{R}^{2n}$ with coordinates $(\mathbf{x}, \mathbf{p})$ (but see the $6j$-example below for other cases). A wave function $\psi(\mathbf{x})$ has a semiclassical representation as a sum over branches; each branch has a phase $S(\mathbf{x})/\hbar$. With the right understandings, this notation covers energy and other eigenfunctions, time-dependent wave functions, kernels of operators such as $\langle \mathbf{x}|A|\mathbf{x}'\rangle$, matrix elements in angular momentum theory, periodic orbit contributions in the Gutzwiller trace formula, and other cases.

Derivation and Results. – The n-dimensional manifold $\mathbf{p} = \nabla S(\mathbf{x})$ in phase space is a Lagrangian manifold [35], call it L. It is the level set $H_i(\mathbf{x}, \mathbf{p}) = h_i$, $i = 1, \ldots, n$, where the H_i are a set of functions and the h_i their values, and where $\{H_i, H_j\} = 0$ on L. (For energy eigenfunctions, one of the H_i is the Hamiltonian.) These Poisson brackets are required to vanish only on L, not necessarily elsewhere in phase space; this means that conjugate (e.g., angle) variables on L may not exist, a case that must be covered for applications to angular momentum theory.

Consider a point $(\mathbf{x}, \mathbf{p})$ on L, and let the Hamiltonian vector fields generated by the H_i at this point be

$$X_i = \sum_j E_{ji} \frac{\partial}{\partial x_j} + F_{ji} \frac{\partial}{\partial p_j}, \tag{1}$$

where $E_{ij} = \{x_i, H_j\} = \partial H_j/\partial p_i$, $F_{ij} = \{p_i, H_j\} = -\partial H_j/\partial x_i$. Vectors X_i are tangent to L and span its tangent space, since $\{H_i, H_j\} = X_j(H_i) = 0$ (the final expression is the vector field X_j acting on the scalar H_i). Matrix E_{ij} is the Jacobian of the projection π_x from L to $\mathbf{x}$-space, in the bases X_i and $\partial/\partial x_i$; and F_{ij} is that of the projection π_p from L to $\mathbf{p}$-space, in the bases X_i and $\partial/\partial p_i$. Caustics in $\mathbf{x}$-space, that is, of the wave function $\psi(\mathbf{x})$, occur where E_{ij} is singular; here the semiclassical wave function suffers a phase shift given by $e^{-im\pi/2}$, where the integer m is the Maslov index. We view m as a function of a directed path γ on L, which passes through an $\mathbf{x}$-space caustic. Similarly, the matrix F_{ij} is singular at caustics in $\mathbf{p}$-space (that is, caustics of the momentum space wave function, the Fourier transform of $\psi(\mathbf{x})$). In one dimension, $\mathbf{p}$-space caustics never occur at an $\mathbf{x}$-space caustic. In higher dimensions, a $\mathbf{p}$-space caustic can occur on top of an $\mathbf{x}$-space caustic, that is, F can be singular when E is singular, a case that must be covered in practice. Initially, however, we assume that F is nonsingular in a neighborhood of an $\mathbf{x}$-space caustic, in which our curve γ lies.

Maslov's method [19] for computing his index involves switching to the momentum representation in a neighborhood of the $\mathbf{x}$-space caustic. Maslov only carried out the Fourier transform in a single variable, all that is needed for a generic, first-order caustic. Here we use a slight variation on the method, in which one Fourier transforms in all the configuration space variables. The phase of the momentum-space wave function is $\tilde{S}(\mathbf{p})/\hbar$, where $\tilde{S}(\mathbf{p}) = S(\mathbf{x}) - \mathbf{x} \cdot \mathbf{p}$. Here $\mathbf{x}$ is understood to be a function of $\mathbf{p}$ by restricting $(\mathbf{x}, \mathbf{p})$ to be on L. The momentum-space action satisfies $\partial \tilde{S}/\partial p_i = -x_i$ and $T_{ij} = \partial^2 \tilde{S}/\partial p_i \partial p_j = -(\partial x_i/\partial p_j)_H = T_{ji}$, where the subscript H indicates that the H_i are held constant, that is, the derivative is taken on L. The momentum-space wave function is nonsingular (it has no caustics) in the neighborhood of the $\mathbf{x}$-space caustic, but when we Fourier transform back to the $\mathbf{x}$-representation, there is a phase difference when the integral is evaluated on the two sides of the $\mathbf{x}$-space caustic. This gives rise to a relative phase shift in the $\mathbf{x}$-space wave function of $e^{-im\pi/2}$, where m is related to the change in the signature of matrix T by $m = -(1/2)\Delta \operatorname{sgn} T$. (The signature is the number of positive minus the number of negative eigenvalues; T is symmetric and has real eigenvalues.)

One or more of the eigenvalues of T pass through 0 at the caustic, that is, T is singular at the caustic. This can be seen by expressing T in terms of matrices E and F; the relation is $T = -EF^{-1}$, as can be proved by manipulating partial derivatives. Since F is nonsingular in the neighborhood of the $\mathbf{x}$-space caustic (by our assumptions) and E is singular at the caustic, T is singular at the caustic. Thus we have $m = (1/2)\Delta \operatorname{sgn}(EF^{-1}) = (1/2)\Delta \operatorname{sgn}(F^T E)$, where in the final expression we have used Sylvester's theorem on the invariance of the signature under congruency transformation by a nonsingular matrix (in this case, F) and where F^T is the transpose of F. Note that $F^T E$, like T, is symmetric.

For simplicity we assume that only one eigenvalue of T (hence of $F^T E$) changes sign at the caustic; this is the generic situation. Let λ be this eigenvalue of $F^T E$, and let v be the corresponding (nonzero) eigenvector, so that $F^T E v = \lambda v$. Also let $u = Fv$; since F is nonsingular, $u \neq 0$. We consider F, E, λ, u and v to be functions of a parameter t (not necessarily time) along the curve γ, and we let $t = 0$ at the caustic, so that $\lambda(0) = 0$. Then $m = \operatorname{sgn} \dot{\lambda}(0)$; we assume $\dot{\lambda}(0) \neq 0$ (the generic situation).

At $t = 0$, $F^T E v = 0$; but since F is nonsingular, this implies $Ev = 0$, and v spans the kernel of E at the caustic. Matrix E is not symmetric, so its left and right eigenvectors are not transposes of each other, but since $F^T E$ is symmetric, we have $v^T F^T E = u^T E = 0$ at $t = 0$, so u spans the (left) kernel of E at $t = 0$. Now by differentiating $u^T E v = \lambda v^T v$ with respect to t and using $Ev = 0$, $u^T E = 0$ and $\lambda = 0$ at $t = 0$, we find

$$m = \operatorname{sgn} u^T \dot{E} v, \tag{2}$$

evaluated at $t = 0$. This is our main result for a local calculation of the Maslov index, that is, in a neighborhood of a caustic.

To calculate m we first find the caustics, which are the places where $Ev = 0$ has a solution $v \neq 0$; these are the places where $\det E = 0$. The matrix E is needed for the amplitude of the semiclassical wave function, which can be expressed as $|\det E|^{-1/2}$ [36, 37]; the amplitude diverges at the caustics. At the caustic we find vector v with an arbitrary normalization and phase; as mentioned, we are assuming that the kernel of E is one-dimensional.

Next we must find the vector $u = Fv$, which spans the left kernel of E. If we have the matrix F we can just do the matrix multiplication, but in many applications the Poisson brackets in F involve angle variables and are not easy to compute. Moreover, in some cases the Lagrangian manifold is not a member of a foliation and angle variables do not exist.

A different approach that avoids these difficulties is based on the geometrical meaning of the vectors v and u, which emerges if we multiply (1) by v_i and sum on i. At the caustic, where $\sum_i E_{ji}\, v_i = 0$, this gives $\sum_i v_i X_i = \sum_j u_j \partial/\partial p_j$, where we have used $u = Fv$. Recalling that the X_i are the Hamiltonian vector fields generated by the H_i, we let Y_i be the Hamiltonian vector fields generated by the x_i, that is, we let $Y_i = -\partial/\partial p_i$. Then we have $\sum_i v_i X_i = -\sum_i u_i Y_i$. We see that there is a linear combination of the X_i, that is, a vector tangent to L, that is equal to a linear combination of the Y_i, that is, a vector tangent to the vertical Lagrangian manifold $\mathbf{x} = \text{const}$. The two Lagrangian planes tangent to the two manifolds at the caustic have a nontrivial (one-dimensional) intersection.

If we regard the symplectic form ω at a point of phase space as a linear map between vectors and covectors, then Hamilton's equations for the H_i can be written $X_i = \omega^{-1} dH_i$, and, similarly, $Y_i = \omega^{-1} dx_i$. Now multiplying the previous relation by ω, we obtain

$$\sum_i v_i\, dH_i = -\sum_i u_i\, dx_i. \tag{3}$$

This equation allows the vector u to be determined, given the vector v and the differentials dH_i and dx_i at the caustic. The calculation is just linear algebra in the cotangent space at the caustic. Finally, let the curve γ be an orbit of one of the H's, say, H_n. Then the t-derivative in $\dot{E}$ is a Poisson bracket, and the Maslov index is $m = \operatorname{sgn} u^T \{E, H_n\} v$. The calculation of the Maslov index is reduced to the calculation of Poisson brackets and linear algebra.

As an example consider the one-dimensional Hamiltonian $H = p^2/2M + V(x)$, where $H = H_1$ in the notation above. In a one-dimensional case such as this we will write e and f for matrices E and F, which now are scalars. Here $e = \{x, H\} = p/M$, $f = \{p, H\} = -V'(x)$. The caustics are where $e = 0$, that is, $p = 0$. Choosing $v = 1$, we have $u = fv = -V'(x)$. The same result is obtained from (3), that is, $v\, dH = -u\, dx$, since $dH = V'(x)\, dx$ at the caustic where $p\, dp/M = 0$. Finally, using $\dot{e} = \{e, H\} = -V'(x)/M$, we have $m = \operatorname{sgn}[V'(x)^2/M] = +1$. The Maslov index always increases by 1 at a turning point in a kinetic-plus-potential problem.

The $6j$-symbol is a less trivial example. In this case the phase space is not $\mathbb{R}^{2N}$, but the method presented applies in a symplectic chart, within which parts of the caustic set on the Lagrangian manifold lie. Maslov indices relative to such a chart are useful in applications, such as our work on the asymptotics of the $9j$-symbol [34], in which we did not have the luxury of knowing the correct answer when we started. The quantum mechanics of the $6j$ symbol [22] involves four angular momenta, $\mathbf{J}_r$, $r = 1, \ldots, 4$ that act on a product of four carrier spaces with quantum numbers j_r. Intermediate angular momenta $\mathbf{J}_{12} = \mathbf{J}_1 + \mathbf{J}_2$ and $\mathbf{J}_{23} = \mathbf{J}_2 + \mathbf{J}_3$ with quantum numbers j_{12} and j_{23} are defined. The $6j$-symbol concerns the subspace $\sum_{r=1}^4 \mathbf{J}_r = \mathbf{J}_{\text{tot}} = 0$, upon which J_{12}^2 and J_{23}^2 have eigenbases $|j_{12}\rangle$ and $|j_{23}\rangle$. The $6j$-symbol is proportional to the orthogonal matrix connecting these bases,

$$\langle j_{12} | j_{23} \rangle = \text{const} \times \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{array} \right\}. \tag{4}$$

The $6j$-symbol involves a quantum dynamical system in which a state is a vector in the subspace $\mathbf{J}_{\text{tot}} = 0$.

A state of the corresponding classical system is a quadrilateral, not necessarily planar, whose edges are four classical angular momentum vectors $\mathbf{J}_r$ of fixed lengths $J_r = |\mathbf{J}_r|$, modulo overall rotations. The vectors satisfy $\sum_r \mathbf{J}_r = 0$. If vectors $\mathbf{J}_{12} = \mathbf{J}_1 + \mathbf{J}_2$ and $\mathbf{J}_{23} = \mathbf{J}_2 + \mathbf{J}_3$ are drawn in, the quadrilateral becomes Wigner's tetrahedron [38] with edge lengths J_r, $r = 1, \ldots, 4$ and $J_{12} = |\mathbf{J}_{12}|$ and $J_{23} = |\mathbf{J}_{23}|$. The quadrilateral is flexible; changing its shape while holding J_r, $r = 1, \ldots, 4$ fixed changes the classical state, as well as the lengths J_{12} and J_{23}. The space of shapes of the quadrilateral or tetrahedron is a sphere, the phase space of the system [39–41]; the Poisson bracket of any two functions f and g of $\mathbf{J}_r$, $r = 1, \ldots, 4$ is $\{f, g\} = \sum_{r=1}^4 \mathbf{J}_r \cdot (\nabla_r f \times \nabla_r g)$, where $\nabla_r = \partial/\partial \mathbf{J}_r$; this is the standard Poisson bracket for classical angular momenta.

Interesting classical observables on this phase space are J_{12}, J_{23} and $V = \mathbf{J}_1 \cdot (\mathbf{J}_2 \times \mathbf{J}_3)$ (this is six times the volume of the tetrahedron). The Hamiltonian flow generated by J_{12} is a rotation of vectors $\mathbf{J}_1$ and $\mathbf{J}_2$ about the axis defined by $\mathbf{J}_{12}$, while holding $\mathbf{J}_3$ and $\mathbf{J}_4$ fixed; we call the conjugate angle ϕ_{12}. Similary, J_{23} generates rotations of $\mathbf{J}_2$ and $\mathbf{J}_3$ with angle ϕ_{23} about axis $\mathbf{J}_{23}$. These rotations are rigid, relative motions of two faces of the tetrahedron about their common edge (J_{12} or J_{23}). If we denote the interior dihedral angles of the tetrahedron about edges J_{12} and J_{23} by α_{12} and α_{13}, with $0 \leq \alpha_{12}, \alpha_{23} \leq \pi$, then when $V > 0$ we have $\phi_{12} = \alpha_{12}$ and $\phi_{23} = -\alpha_{23}$; this is clear from a picture of the tetrahedron. With a change of signs for the case $V < 0$, the angles ϕ_{12} and ϕ_{23} lie in the range $-\pi \leq \phi_{12}, \phi_{23} < \pi$ on the space of all tetrahedra. For semiclassical purposes we set $J_r = j_r + 1/2$ [24,37].

In calculating Poisson brackets the vectors $\mathbf{A}_{rs} = \mathbf{J}_r \times \mathbf{J}_s$ are convenient; the magnitude $A_{rs} = |\mathbf{A}_{rs}|$ is twice the area of the face spanned by $\mathbf{J}_r$, $\mathbf{J}_s$. We find $\{J_{12}, J_{23}\} = -V/J_{12}J_{23} = dJ_{12}/d\phi_{23} = -dJ_{23}/d\phi_{12}$; $\{V, J_{12}\} = dV/d\phi_{12} = A_{34}A_{12}\cos\phi_{12}/J_{12}$; and $\{V, J_{23}\} = dV/d\phi_{23} = -A_{23}A_{14}\cos\phi_{23}/J_{23}$.

To compute the Maslov index of the $6j$-symbol we compare $\langle j_{12}|j_{23}\rangle$ with the energy eigenfunction $\psi(x) = \langle x|H\rangle$, which shows that we should identify H (or H_1) above with J_{23} and x with J_{12}. As for p, we identify it with $-\phi_{12}$ so that $\{x,p\}=1$ goes into $\{J_{12},-\phi_{12}\}=1$. The idea is that the Lagrangian manifold is specified by $J_{23}=j_{23}+1/2=\text{const}$, while J_{12} provides the representation of the wave function. The caustics occur when $e=\{J_{12},J_{23}\}=0$, that is, when $V=0$; these are the flat tetrahedra. To obtain u and v we need a relation between dJ_{12} and dJ_{23}. This may be obtained by differentiating the Cayley-Menger [24] or Gram [41] matrix, but an approach based on Poisson brackets may be given. Let V be considered a function of J_{12} and J_{23}. Then $\{V,J_{12}\}=\{J_{23},J_{12}\}\,\partial V/\partial J_{23}$, which combined with the above gives $\partial V/\partial J_{23}=A_{12}A_{34}J_{23}\cos\phi_{12}/V$. Similarly, consideration of $\{V,J_{23}\}$ gives $\partial V/\partial J_{12}$. The results are summarized by

$$V\,dV=A_{14}A_{23}\cos\phi_{23}\,J_{12}\,dJ_{12}+A_{12}A_{34}\cos\phi_{12}\,J_{23}\,dJ_{23}. \tag{5}$$

Now setting $V=0$ to evaluate at the caustic and writing $v\,dJ_{23}=-u\,dJ_{12}$, we find $u=A_{14}A_{23}J_{12}\cos\phi_{23}$ and $v=A_{12}A_{34}J_{23}\cos\phi_{12}$. Finally, defining $\dot{e}$ by $\{e,J_{23}\}$ (that is, evaluating the Maslov index along an orbit of J_{23}), we find $\dot{e}=A_{14}A_{23}\cos\phi_{23}/J_{12}J_{23}^2$, when evaluated at the caustic. Then the Maslov index is $m=\operatorname{sgn} u\dot{e}v=\operatorname{sgn}\cos\phi_{12}$; it is 1 when $\phi_{12}=0$, and -1 when $\phi_{12}=\pi$ (the only two possibilities for a flat tetrahedron). Notice that in this calculation we did not need any Poisson brackets involving the angles ϕ_{12} or ϕ_{23}.

The result (2) was derived under the assumption that $\det F\neq 0$ in a neighborhood of the point where $\det E=0$; but it turns out to be correct even when **x**- and **p**-space caustics coincide. Such a coincidence typically occurs on a Lagrangian manifold of dimension ≥ 2, and in cases of symmetry, such as central force motion, it may occur everywhere. When $\det F=0$ the vector v must be interpreted as any nonzero vector in the kernel of E at the caustic, not as the eigenvector of F^TE with eigenvalue 0. Vector u is still defined as Fv, and can be calculated exactly as above (without the explicit knowledge of F); although F is singular, it turns out that $Fv\neq 0$. Relevant theorems covering the case when $\det E=0$ and $\det F=0$ are the following. First, $\ker E\cap\ker F=\{0\}$; next, $\ker F^TE=\ker E\oplus\ker F$; and third, F maps $\ker E$ invertibly into $\ker E^T$. Thus, the singular F becomes nonsingular when restricted to $\ker E$.

Before turning to global considerations, we briefly note that the methods presented here are also valuable for computation of the Conley-Zehnder index [42]. The Conley-Zehnder index generalizes Morse theory to the symplectic context [43]. This index applies to closed paths and measures the winding of neighboring trajectories of energy E about a given periodic trajectory of that energy. This Morse-like index gives a new perspective on the geometry of the Maslov contributions to the Gutzwiller trace formula. Several works have demonstrated the connection between the Conley-Zehnder index and the Maslov index, see [44–49] and references therein. The formulation of [20] is particularly close to that of the present work and shows that the formulas presented here will also ease the computation of the Conley-Zehnder index.

So far we have presented local results, useful for calculating m in the neighborhood of a caustic. Now we present a global result, valid over the whole Lagrangian manifold. If we have a function f on a manifold, then the singular differential form $\delta(f)\,df=(1/2)d\operatorname{sgn}f$ is the "counting form" for the crossings of the surface $f=0$, that is, $\int_\gamma\delta(f)\,df$ counts the number of times γ crosses the surface going from negative f to positive, minus the number of crossings the other way. Since the caustic set on the Lagrangian manifold occurs where $\det E=0$, we might suspect that there is a singular differential form μ, such that the integral of μ along γ gives the Maslov index associated with the curve, and that μ involves $\delta(\det E)\,d(\det E)$. Indeed, this is the case; we find

$$\mu=\operatorname{sgn}\operatorname{tr}(C^TF)\,\delta(\det E)\ \operatorname{tr}(C^T\,dE), \tag{6}$$

where C is the cofactor matrix of E. This result applies only to first order caustics, where $\dim\ker E=1$; but higher order caustics can be perturbed into a set of first-order caustics, so they represent limiting cases of this form. Note that $\operatorname{tr}(C^T\,dE)=d(\det E)$, so the counting form for the surface $\det E=0$ is modulated by the factor $\operatorname{sgn}\operatorname{tr}(C^TF)$. It can be shown that $\operatorname{tr}(C^TF)$ is never zero when $\det E=0$, even if $\det F$ is also 0. Noting that C is proportional to $u\otimes v^T$, where $v\neq 0$ is a vector in $\ker E$ and $u=Fv$, it is easy to derive (2) from (6). We found it easiest to derive (6) itself as the limit of the differential of the phase of the complex amplitude determinant in a coherent state representation, in the limit in which the coherent state representation becomes the **x**-representation. The form μ is closed but not exact, so its integral along γ is invariant under continuous deformations of path.

The phase $S(\mathbf{x})$ is the integral of the differential form $\theta=\mathbf{p}\cdot d\mathbf{x}$ on L; this form is closed but not exact (in general) on L. By combining this form with the Maslov form μ, the Bohr-sommerfeld quantization condition can be expressed as $\oint(\theta-\frac{\pi}{2}\mu)=2n\pi$. In this way the usual action and the Maslov phase are unified in a single form.

The quantization condition cannot depend on the representation, that is, the system of canonical coordinates in which the calculation is carried out, an idea that goes back to Einstein [16]. Taking first the one-dimensional case, we let coordinates (x',p') be related to (x,p) by a linear transformation,

$$\begin{pmatrix} x' \\ p' \end{pmatrix}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}, \tag{7}$$

where A, B, C, D are constants and $AD-BC=1$. Then e and f transform into e' and f' by the same matrix as x and p. In one dimension (6) becomes $\mu=(\operatorname{sgn}f)\delta(e)\,de$. Writing $\mu'=(\operatorname{sgn}f')\delta(e')\,de'$, the Maslov differential forms in two systems of canonical coordinates are μ and μ'. Then we find that $\mu-\mu'=dK(e,e')$, where $K=(1/2)\operatorname{sgn}(e'Be)$

when $B \neq 0$, and $K = 0$ when $B = 0$. That is, μ transforms by the addition of an exact differential when the system of canonical coordinates is changed, so that $\oint \mu$ is invariant. In these calculations we use $(d/dx)\,\mathrm{sgn}(x) = 2\delta(x)$.

We will just cite the analogous results in the multidimensional case. The Maslov forms in the two systems of canonical coordinates are a primed and unprimed version of (6). In the multidimensional case a linear canonical transformation is still specified by (7), where now A, B, C and D are $n \times n$ matrices such that the whole $2n \times 2n$ matrix is symplectic (see Appendix A of [50]). Under the linear canonical transformation (7) μ transforms by an exact differential, $\mu - \mu' = dK$, where now $K = (1/2)\,\mathrm{sgn}(E^T B^{-1} E')$ when B is nonsingular. Thus $\oint \mu$ around a closed loop is independent of the canonical coordinates. Function K is a kind of F_1-type generating function [51]. Knowledge of K allows one to easily switch the Maslov phase from one representation to another. With slight changes, it can be used to switch to the coherent state representation, which is popular in recent applications [52,53] and in approaches based on geometric quantization.

The results presented in this article are of great assistance in computing the Maslov index in various applications, including the $9j$-symbol [34], which is intrinsically 2-dimensional. The strength of this method is that it reduces what is usually a delicate and lengthy tracking of signs to just two ingredients: the calculation of Poisson brackets of the observables that directly define the wave function and the corresponding Lagrangian manifolds (these are also necessary for computing the amplitude [36, 37]); and a linear algebra calculation that alleviates any need for the introduction of angle coordinates. We will report on details, extensions, and applications of the results presented here in future publications.

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