Death and resurrection of the zeroth principle of thermodynamics

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Abstract

The zeroth principle of thermodynamics in the form "temperature is uniform at equilibrium" is notoriously violated in relativistic gravity. Temperature uniformity is often derived from the maximization of the total number of microstates of two interacting systems under energy exchanges. Here we discuss a generalized version of this derivation, based on informational notions, which remains valid in the general context. The result is based on the observation that the time taken by any system to move to a distinguishable (nearly orthogonal) quantum state is a universal quantity that depends solely on the temperature. At equilibrium the net information flow between two systems must vanish, and this happens when two systems transit the same number of distinguishable states in the course of their interaction.

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According to non-relativistic thermodynamics, a thermometer (say, a line of mercury in a glass tube), moved up and down a column of gas at equilibrium in a constant gravitational field, measures a *uniform* temperature.[1] But this prediction is wrong. Relativistic effects make the gas warmer at the bottom and cooler at the top, by a correction proportional to c^{-2}, where c is the speed of light. This is the well known Tolman-Ehrenfest effect, discovered in the thirties [2, 3] and later derived in a variety of different manners [4–12]. The temperatures T_1 and T_2 measured by the same thermometer at two altitudes h_1 and h_2 in a Newtonian potential $\Phi(h)$ are related by the Tolman law

$$T_1 \left(1 + \frac{\Phi(h_1)}{c^2}\right) = T_2 \left(1 + \frac{\Phi(h_2)}{c^2}\right). \tag{1}$$

This law can also be expressed in a general-covariant fashion

$$T|\xi| = \text{const.}, \tag{2}$$

where $|\xi|$ is the norm of the timelike Killing field on a stationary spacetime.

A violation of the uniformity of temperature seems counterintuitive at first, especially if one has in mind a definition of "temperature" as a label of the equivalence classes of all systems in equilibrium with one another. In a relativistic context a physical thermometer does not measure this label and we must therefore distinguish two notions: *(i)* a quantity τ_o defined as this label (proportional to the constant in (2)), and *(ii)* the temperature T measured by a standard thermometer.

In the micro-canonical framework maximizing the total number of states $N = N_1 N_2$ under an energy transfer dE between two systems gives $T_1 = T_2$. In the presence of relativistic gravity, this derivation fails because conservation of energy becomes tricky: intuitively speaking, the energy dE reaching the upper system is smaller than the one leaving the lower system because "energy weighs".

Is there a more general statistical argument that governs equilibrium in a relativistic context? Can the Tolman law be derived from a principle generalizing the maximization of the number of microstates, without recourse to specific models of energy transfer, as is commonly done in the derivations of the Tolman-Ehrenfest effect? In this essay we show that the answer to these questions is positive, and provide a generalization of the statistical derivation of the uniformity of temperature, which remains valid in a relativistic context.

[1] A longer version of this essay has also appeared in PRD [1].

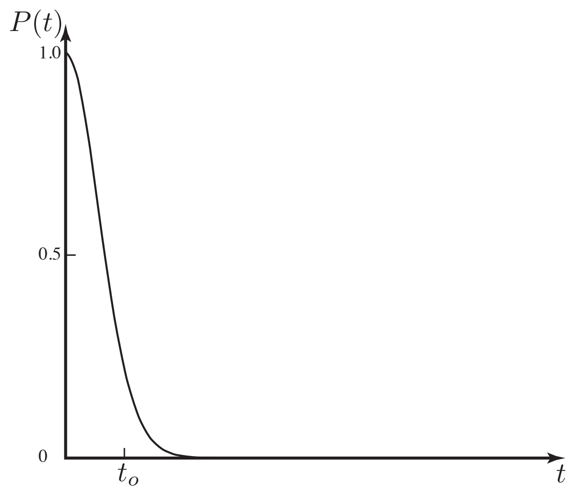

FIG. 1: Typical overlap between $\psi(0)$ and $\psi(t)$ as a function of time.

The core idea is to focus on *processes*, or histories, rather than *states* [13]. We demonstrate that one can assign an information content to a history, and two systems are in equilibrium when their interacting histories have the same information content. Equilibrium in a stationary spacetime, namely the Tolman law, is our short-term objective, but our long-term aim is understanding equilibrium in a fully generally covariant context, where thermal energy can flow also to gravity [14–16], therefore we look for a general principle that retains its meaning also in the absence of a background spacetime.

Consider a conventional hamiltonian system with hamiltonian operator H. Let $\psi(0)$ be the state at time zero and $\psi(t)$ its evolution. What is the time scale for $\psi(t)$ to become significantly distinct from $\psi(0)$? The separation of the state from its initial position is given by the overlap between $\psi(0)$ and $\psi(t)$, namely $P(t) = |\langle\psi(0)|\psi(t)\rangle|^2$. The typical behavior of $P(t)$, for instance in the case of a semiclassical wave packet, is as in Figure 1. Let us call t_o the characteristic decay time for the system self overlap. What is its value? The time t_o can be estimated by Taylor expanding $P(t)$ for small times. The first time derivative of $P(t)$ clearly vanishes at $t = 0$ which is a maximum, therefore we get the time scale from the second derivative. A straightforward calculation gives

$$\frac{d^2P(t)}{dt^2} = -\frac{1}{\hbar^2}(\langle H^2\rangle - \langle H\rangle^2) = -\frac{(\Delta E)^2}{\hbar^2}, \tag{3}$$

which implies a characteristic decay time $t_o = \hbar/\Delta E$, in accord with the time-energy Heisenberg principle, and with the fact that energy eigenstates “do not change”.

Let us now consider a system in thermal equilibrium with a thermal bath at temperature

T. Its mean energy is going to be kT and the variance of the energy is also kT. Thus we have $\Delta E \sim kT$. At a given temperature T, the time step $t_o = \hbar/kT$ is, according to the previous discussion, the average time the system takes to move from a state to the next (distinguishable) state. This average time step is therefore universal: it depends only on the temperature, and not on the properties of the system.

The dimensionless quantity $\tau = t/t_o$ measures time in units of the time step t_o, that is, it estimates the number of distinguishable states the system has transited during a given interval. For a system in thermal equilibrium, $t_o = \hbar/kT$ gives

$$\tau = \frac{kT}{\hbar}t. \tag{4}$$

The quantity τ was introduced in [14, 15] and called *thermal time*. It is the parameter of the Tomita flow on the observable algebra generated by the thermal state.

The argument above unveils the physical interpretation of thermal time: thermal time, which is dimensionless, is simply the number of distinguishable states a system has transited during an interval. Notice also that temperature is the ratio between thermal time and (proper) time, $T = \hbar\tau/kt$ [17]. Accordingly, in $\hbar = k = 1$ units temperature is measured in "states per second" and is precisely the number of states transited by the system per unit of (proper) time. This is the general informational meaning of temperature.

Let us come to the notion of equilibrium. Consider two systems, System 1 and System 2, that are in interaction during a certain interval. This interaction can be quite general but should allow exchange of energy between the two systems. During the interaction interval the first system transits N_1 states, and the second N_2, in the sense illustrated above. Since an interaction channel is open, each system has access to the information about the states the other has transited through the physical exchanges of the interaction. Recall that information, as defined by Shannon [18], is simply a measure of a *number of states*.

System 2 has access to an amount of information $I_1 = \log N_1$ about System 1, and similarly in reverse. Let us define the net flow of information in the course of the interaction as $\delta I = I_2 - I_1$. Equilibrium is by definition invariant under time reversal, and therefore any flow must vanish. It is therefore interesting to *postulate* that the information flow δI vanishes at equilibrium. Let us do so, and study the consequences. That is, we consider the possibility of taking the vanishing of the information flow

$$\delta I = 0 \tag{5}$$

as a general condition for equilibrium, generalizing the maximization of the number of microstates of the non-relativistic formalism.

Let us see what this implies. At equilibrium $N_1 = N_2$. Since the rate that states are transited is given by τ and we assume a fixed interaction interval, the equilibrium condition also reads $\tau_1 = \tau_2$. Now, consider a non-relativistic context where two systems are in equilibrium states at temperatures T_1 and T_2, respectively. In the non-relativistic limit, time is a universal quantity, which we call t. Therefore the condition $\tau_1 = \tau_2$ together with (4) implies that $T_1 = T_2$, which is the standard non-relativistic condition for equilibrium: temperature is uniform at equilibrium. On a curved spacetime, contrariwise, (proper) time is a local quantity ds that varies from one spatial region to another. Therefore thermal time is given by $d\tau = (kT/\hbar)ds$. In order for equilibrium to exist on a given spacetime, spacetime itself must be stationary [19, 20], namely have a timelike Killing field ξ, and equilibrium will be ξ invariant. Proper time along the orbits of ξ is $ds = |\xi|dt$ where t is an affine parameter for ξ. Therefore thermal time is now

$$d\tau = \frac{kT}{\hbar}|\xi|dt. \tag{6}$$

If two systems located in regions with different $|\xi|$ are in thermal contact for a finite interval Δt, then they are in equilibrium if $|\xi|T$ has the same value. This is precisely the Tolman law (2). Therefore the generalized first principle (5) gives equality of temperature in the non relativistic case and the Tolman law in the general case.

In static coordinates, $ds^2 = g_{00}(\vec{x})dt^2 - g_{ij}(\vec{x})x^i x^j$ and thermal time is proportional to coordinate time. The Killing vector field is $\xi = \partial/\partial t$ and $|\xi| = \sqrt{g_{00}}$. In the Newtonian limit $g_{00} = 1 + 2\Phi/c^2$ and we recover (1).

Returning to the cylinder of gas in a constant gravitational field we see that during a coordinate-time interval Δt the proper times lapsed in the upper and lower systems are different: identical clocks at different altitudes run at different rates. But the lower system is hotter, its degrees of freedom move faster in clock time from one state to the next. This faster motion compensates exactly the slowing down of proper time, so that *upper and lower systems transit the same number of states during a common interaction interval* Δt. This result provides a simple and intuitive interpretation of the Tolman effect.

We have suggested a generalized principle for equilibrium in statistical mechanics, formulated in terms of histories rather than states and expressed in terms of information. It reads: *Two histories are in equilibrium if the net information flow between them vanishes, namely if they transit the same number of states during the interaction period.* This is equivalent to saying that the thermal time τ elapsed for the two systems is the same.

In non-relativistic physics, time is universal and the above principle implies that temperature is uniform at equilibrium. On a curved spacetime, proper time varies locally and what is constant is the product of temperature and proper time. We have seen that temperature admits an informational interpretation as states transited per second, consistent with its units ($second^{-1}$ if $\hbar = k = 1$). Temperature is the rate at which systems move from state to state.

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