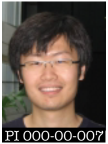
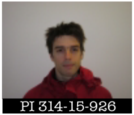
Curved Polyhedra

Hal Haggard
with Muxin Han, Wojciech Kaminski, and Aldo Riello



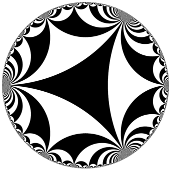


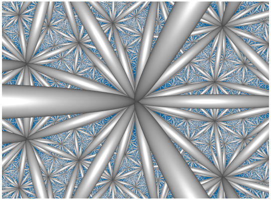
July 18th, 2014

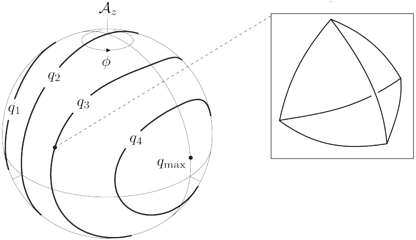
Frontiers of Fundamental Physics

Three streams of motivation

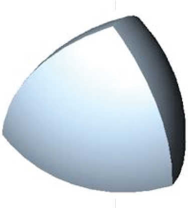
Build a 4D spinfoam with
cosmological constant...

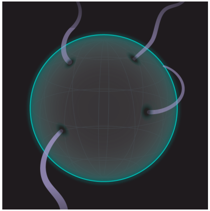






...explore curved, dynamically
evolving discrete geometries...



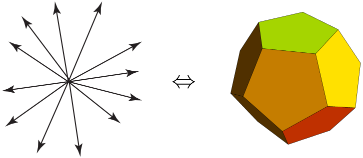


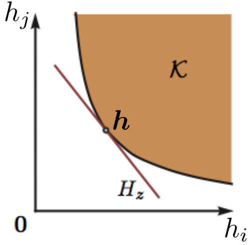
... connect loop gravity to
knot & Chern-Simons theory.

A set of N area vectors that closes uniquely determines a convex Euclidean polyhedron of N faces. (Minkowski 1897)

$$\vec{a}_1 + \cdots + \vec{a}_N = 0$$

Non-constructive proof: an existence and uniqueness result that relies on convexity.





Major difficulty in constructive approach is determining the adjacency ahead of time.

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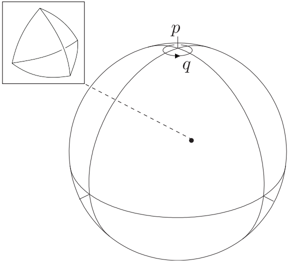
Minkowski theorem
for curved tetrahedra



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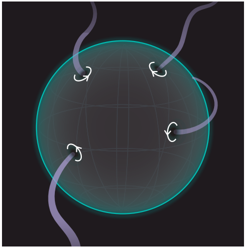
Phase space of shapes



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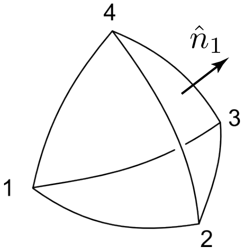
Connection with knot
& Chern-Simons theory

A spherical tetrahedron is 4 points of S^3 connected by geodesics

Each face is a triangular portion of a great 2-sphere!

- Great spheres are flatly embedded in S^3 (i.e. $K_{ij} = 0$)

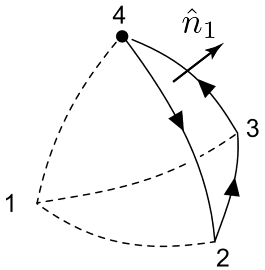


Hence, the normal to a face is well-defined and invariant under parallel transport

Holonomies are the crown jewel of gravitational observables: convert flux variables to ‘transverse’ holonomies

A fun calculation shows the holonomy has angle the face area:

$$O = \exp\left(\frac{a}{R^2}\,\hat{n}\cdot\vec{J}\right), \qquad O \in SO(3)$$



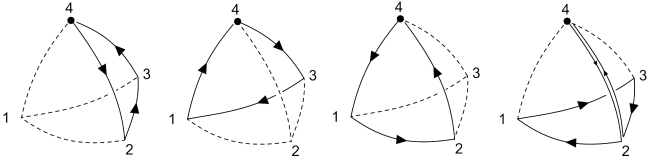
Idea: the closure relation should be replaced by the automatic homotopy constraint [Bonzom, Charles, Dupuis, Girelli, Livine]

$$\boxed{O_4\,O_3\,O_2\,O_1 = \mathbb{1}}$$

For $R \to \infty$

$$O_4\,O_3\,O_2\,O_1 = \mathbb{1} + R^{-2}(a_1\hat{n}_1 + a_2\hat{n}_2 + a_3\hat{n}_3 + a_4\hat{n}_4)\cdot\vec{J} + \cdots = \mathbb{1}$$

How do you access the global geometry? We use ‘simple’ paths.



The Gram matrix

$$G = \begin{pmatrix} 1 & \hat{n}_1 \cdot \hat{n}_2 & \hat{n}_1 \cdot \hat{n}_3 & \hat{n}_1 \cdot \hat{n}_4 \\ * & 1 & \hat{n}_2 \cdot \hat{n}_3 & \hat{n}_2 \cdot \mathbf{O}_1 \hat{n}_4 \\ * & * & 1 & \hat{n}_3 \cdot \hat{n}_4 \\ sym & * & * & 1 \end{pmatrix}$$

is geometrically meaningful.

Get by tracing: $\langle O_\ell O_m \rangle_C = \frac{1}{2}\mathrm{Tr}\,(\mathbf{O}_\ell \mathbf{O}_m) - \frac{1}{4}\mathrm{Tr}\,(\mathbf{O}_\ell)\mathrm{Tr}\,(\mathbf{O}_m)$,

$$\hat{n}_\ell \cdot \hat{n}_m = \frac{\langle O_\ell O_m \rangle_C}{\sqrt{1 - \langle O_\ell \rangle}\sqrt{1 - \langle O_m \rangle}}$$

A wealth of troubles; and the new insights they bring.

- What determines the sign of the curvature?

A wealth of troubles; and the new insights they bring.

- What determines the sign of the curvature?
  - ⇝ The holonomies do it directly, through G.

$$\begin{cases} \det G > 0 & \text{spherical geometry} \\ \det G < 0 & \text{hyperbolic geometry} \end{cases}$$

There is no need for another group.

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  - ⇝ Convexity is essential; encoded in triple products

A wealth of troubles; and the new insights they bring.

♦ What determines the sign of the curvature?

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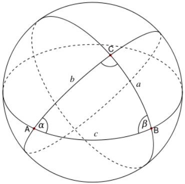
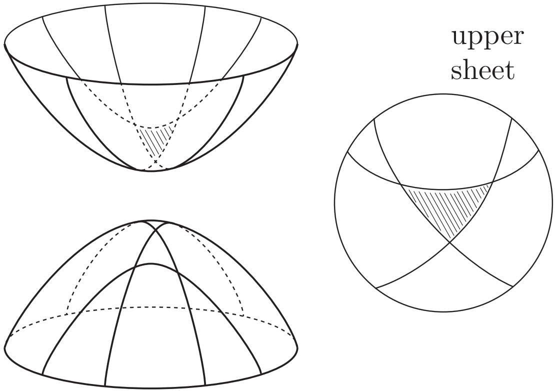
♣ Geometrical counterpart:
which tetrahedron and why?

⇝ Convexity is essential;
encoded in triple products



2D analogy

The new hyperbolic triangle



Continue path past hyperbolic ∞, assuming zero added holonomy

Generalized triangles have a full $[0, 2\pi]$ range of 'holonomy' areas

♣ Finally we introduce a spin lift,

$$O_\ell \longrightarrow H_\ell, \quad H_\ell \in SU(2)$$

$\rightsquigarrow$ H_ℓ from spin connection it's automatic; can be constructed

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Result: a full constructive proof of the Minkowski theorem for all curved tetrahedra

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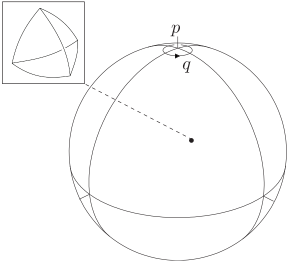
Minkowski theorem
for curved tetrahedra



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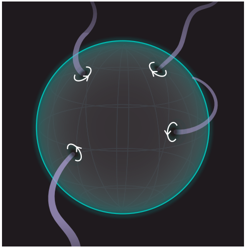
Phase space of shapes



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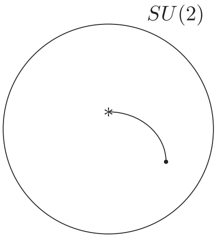
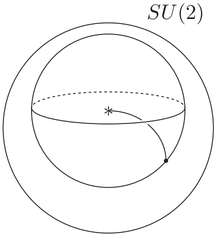
Connection with knot
& Chern-Simons theory

Transverse holonomies are, like fluxes, phase space variables



Each holonomy acts like a curved vector; really a geodesic segment from id to the group element.
$\rightsquigarrow$ $SU(2)$ essential: $H = e^{-i\frac{a}{2}\hat{n}\cdot\vec{\sigma}}$

The conjugacy class of an holonomy sweeps out a 2-sphere in $SU(2) \cong S^3$.

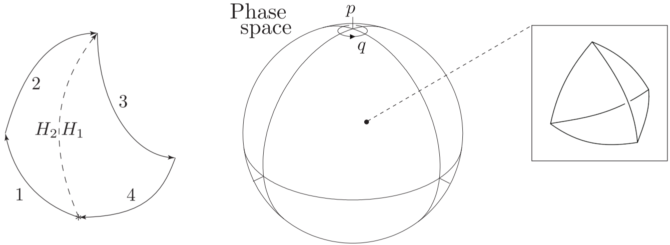


This 2-sphere is symplectic, an orbit of the dressing action of a Poisson-Lie group ($\sim$ q-def.) $\Longrightarrow$ can use symplectic tools!

[Amelino-Camelia, Freidel, Kowalsky-Glikman, Smolin]

Like the flat case, we can construct a phase space of shapes

Form product of 4 fixed conj class (const area) spheres and symplectically reduce by overall rotations [Ditrrich & Bahr, Treloar]



Distinct polyhedra (hence intertwiners for quantum theory with cosmo const) correspond to different shapes of a spherical polygon

- Immediately conclude the volume of curved tetrahedra quantized

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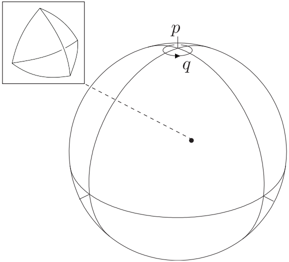
Minkowski theorem
for curved tetrahedra



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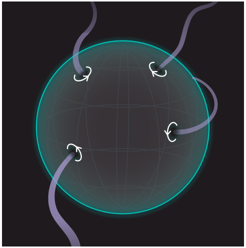
Phase space of shapes



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Connection with knot
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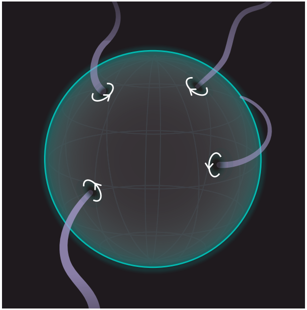
The moduli space of flat connections on a 4-punctured sphere is symplectomorphic to the phase space of shapes just described



4D: These relationships can be lifted into $SL(2,\mathbb{C})$ Chern-Simons theory. Holonomy-flux algebra encoded by transverse-longitudinal holonomy Poisson brackets on a Riemann surface arising as the knot complement of Γ_5 in S^3.

We have shown that the asymptotics of combined EPRL-CS theory is the Regge action plus the cosmo term with the curved 4-volume.

Conclusions

We have:

1. proven a constant curvature Minkowski theorem for tetrahedra
2. found the phase space of shapes for this geometry and learned that the volume spectrum for curved tetrahedra is discrete; we do not yet control the values of this spectrum
3. leveraged these constructions to build a new spin foam model including a cosmological constant. This model has elegant asymptotics, recovering the discretized Einstein-Hilbert action with *exactly* the cosmological constant term for a 4-simplex

Conjecture: There exists a unique convex constant curvature polyedron with N faces whenever [HMH, Freidel & Livine, Speziale]

$$\boxed{H_N \cdots H_1 = \mathbb{1},} \quad H_i \in SU(2)$$

Credits

Hyperbolic triangulation and honeycomb from wikipedia.

Spherical spin network: Z. Merali, "The origins of space and time," Nature News, Aug. 28, 2013

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