Symmetry in Spacetimes with a Cosmological Constant

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General Relativity and Gravitation: a Centennial Perspective
Pennsylvania State University

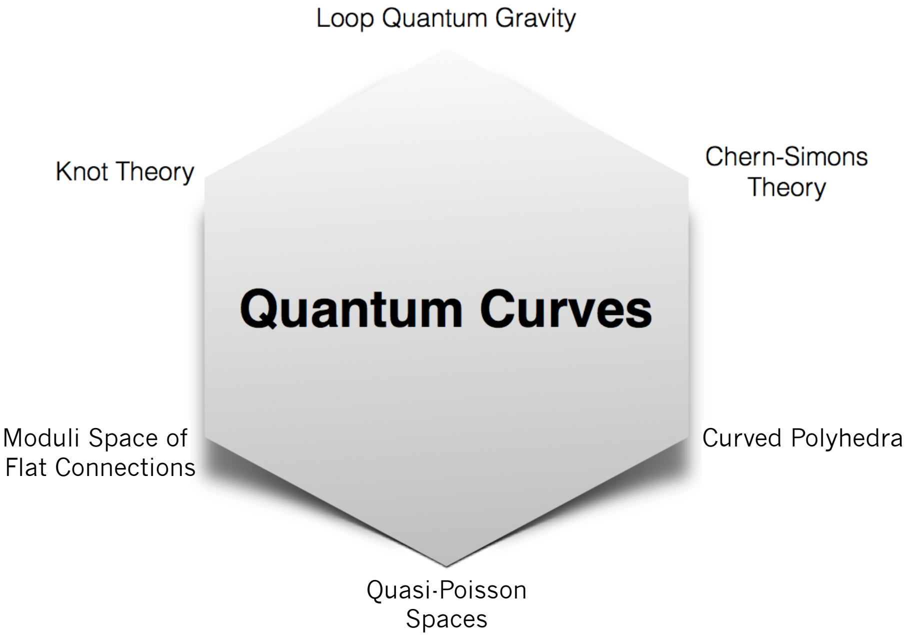
math-ph/1506.03053

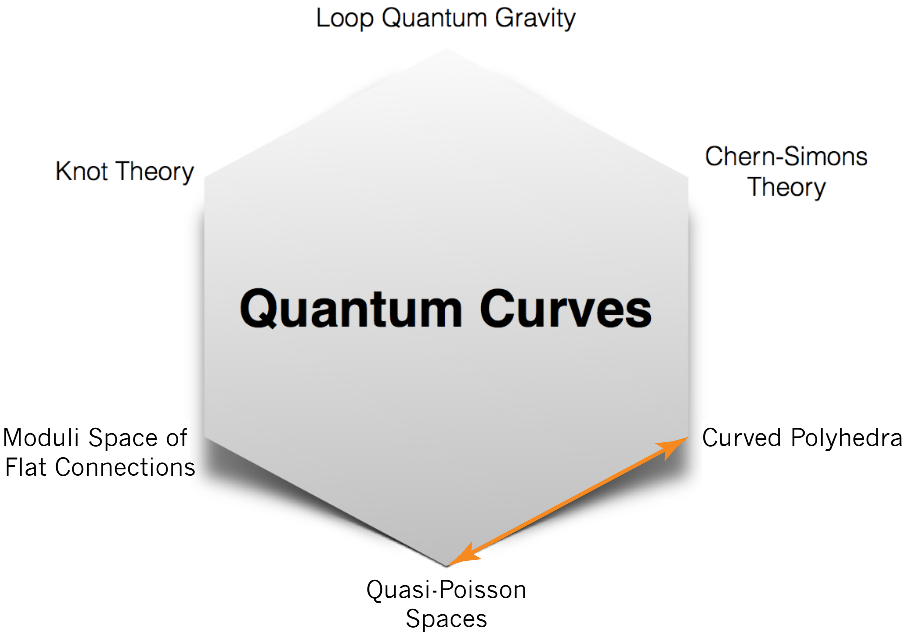
"Even a tiny cosmological constant casts a long shadow"

A. Ashtekar

The kinematical structure of discrete geometries is strongly impacted by Λ

Symmetries have a richer structure that is prime for new ideas & insights


Loop Quantum Gravity
Knot Theory
Chern-Simons Theory
Quantum Curves
Moduli Space of Flat Connections
Curved Polyhedra
Quasi-Poisson Spaces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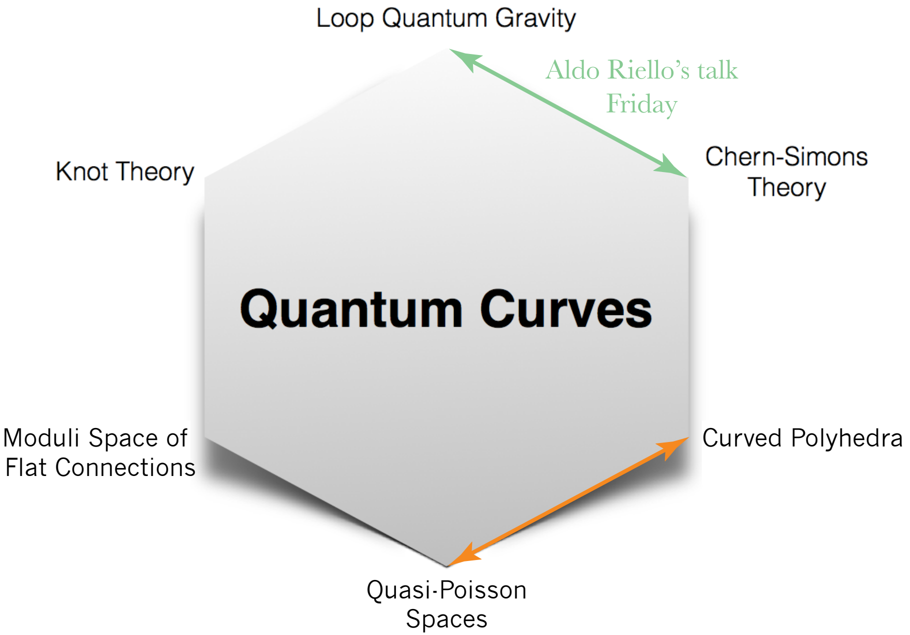
Loop Quantum Gravity
Aldo Riello's talk
Friday
Chern-Simons
Theory
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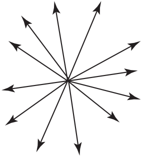
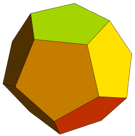
In 1897 H. Minkowski proved

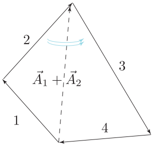


$\Leftrightarrow$



$\vec{A}_1 + \cdots + \vec{A}_N = 0,$ with $\vec{A}_i = A_i \hat{n}_i,$ $A_i =$ area face i, $\hat{n}_i = \perp$ to face i

99 years later, in 1996, M. Kapovich and J. J. Millson found a phase space for the space of polygons with fixed edge lengths



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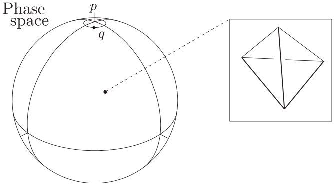
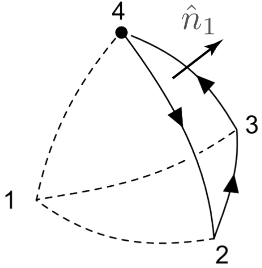


We have proved that

$$O_4\,O_3\,O_2\,O_1 = \mathbb{1} \qquad \Longleftrightarrow$$



Here

$$O = \exp\left(\frac{a}{r^2}\,\hat{n}\cdot\vec{J}\right), \quad O \in SO(3)$$

The closure relation is the automatic homotopy constraint

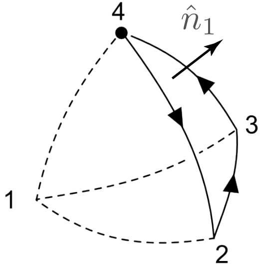
Two immediate checks:

■ For $r \to \infty$

$$O_4 O_3 O_2 O_1 = \mathbb{1} + r^{-2}(a_1 \hat{n}_1 + a_2 \hat{n}_2 + a_3 \hat{n}_3 + a_4 \hat{n}_4) \cdot \vec{J} + \cdots = \mathbb{1}$$

and we recover the flat space Minkowski theorem.

♦ Normals make sense because the faces are flatly embedded (completely geodesic).

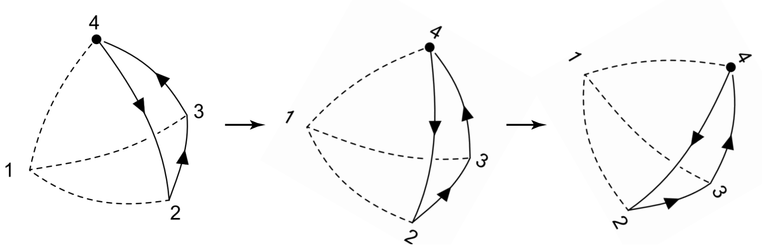


The full generalization leads to a new kind of Gram matrix:

$$\text{Gram} = \begin{pmatrix} 1 & \hat{n}_1 \cdot \hat{n}_2 & \hat{n}_1 \cdot \hat{n}_3 & \hat{n}_1 \cdot \hat{n}_4 \\ * & 1 & \hat{n}_2 \cdot \hat{n}_3 & \hat{n}_2 \cdot \mathbf{O}_1 \hat{n}_4 \\ * & * & 1 & \hat{n}_3 \cdot \hat{n}_4 \\ \text{sym} & * & * & 1 \end{pmatrix}$$

Lift the set $\{O_\ell\}$ to a set $\{H_\ell\} \subset SU(2)$.

The new closure, $H_4 H_3 H_2 H_1 = \mathbb{1}$, is a curved Gauß constraint and should again play its role as generator of gauge transfrmtns.

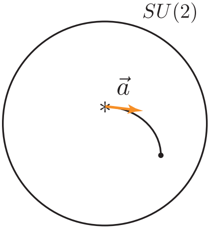


- To achieve this we must have group-valued momenta
  $\rightsquigarrow$ and thusly enters the theory of quasi-Poisson spaces

A quasi-Poisson space has a Poisson bivector, i.e. a Poisson bracket, that violates the Jacobi identity in a specific way

If we parametrize $SU(2)$ using the fundamental representation, then the holonomies around faces are

$$H = e^{-i\frac{\vec{a}}{2r^2}\cdot\vec{\sigma}}$$

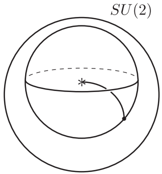


With this parametrization we can construct the quasi-Poisson brackets for the fluxes $\vec{a}$

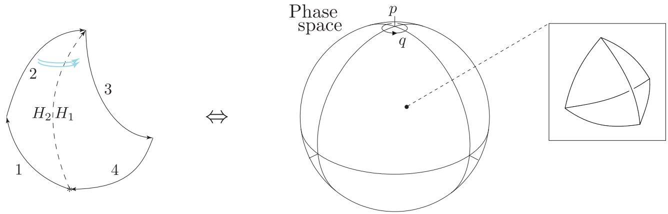
$$\boxed{\boxed{\{a^i, a^j\}_{qP} = \frac{a}{2r^2}\cot\frac{a}{2r^2}\epsilon^{ij}_k a^k \xrightarrow{r\to\infty} \epsilon^{ij}{}_k a^k.}}$$

This Poisson structure naturally foliates $SU(2)$ into leaves, these are the conjugacy classes of $SU(2)$

Fixing the area of a face gives a geodesic that sweeps out a 2-sphere in $SU(2) \cong S^3$.

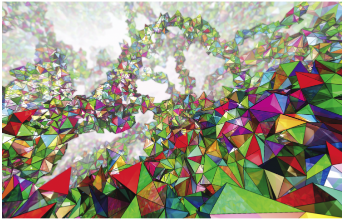


Forming the fusion product of 4 of these phase spaces and reducing by the Gauß constraint gives



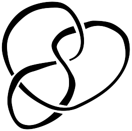
$$\omega_H = r^2 \sin \frac{a}{r^2}\, \mathrm{d}^2\Omega \qquad \text{and} \qquad \mathcal{L}_H = 2r^2 \sin \frac{a}{2r^2}\, \mathrm{d}^2\Omega.$$

■ Established a classical foundation on which to build the quantization of spacetimes with a cosmological constant.



♦ Conjecture: the curved Minkowski theorem holds in general $\rightsquigarrow$ study of flat connections on Riemann surfaces closely related to study of discrete, curved polyhedra.

♣ Provide an enriched context for understanding the role of quantum groups in cosmological spacetimes.



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