Thermality of Spherical Causal Domains & the Entanglement Spectrum

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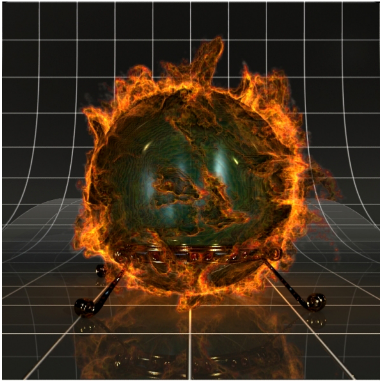
relativity.phys.lsu.edu/ilqgs/

Entanglement thermality

Entanglement leads to even finite regions of spacetime being hot.



We illustrate this claim with a spherical entangling surface.

Entanglement insights

Casini, Huerta & Myers have extensively studied the entanglement entropy of spherical causal domains. [Casini, Huerta & Myers '11]

Bianchi has been shedding interesting light on black holes through entanglement. [Bianchi 1-'12, 2-'12]

Marolf is introducing holography without strings. [Marolf '13]

Disappearance of distinction between statistical and quantum fluctuations. [Bianchi, HMH, Rovelli '13]

♦ Focus on QFT while aiming for spin networks and loop gravity.

Boundary conditions for talk

For simplicity I restrict to:

Scalar field $\varphi(x)$,

with $m = 0$

on flat $D = 3 + 1$ spacetime

unless otherwise stated. Metric signature $(-, +, +, +)$.

- Results apply to any conformal field theory.

Outline

1. What kind of entanglement?
2. Where (and when) is the region?
3. Why the entanglement spectrum?

Entanglement

Pure state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

A	B

$\mathcal{H}_A \otimes \mathcal{H}_B$

Schmidt decomposition:

$$|\Psi\rangle = \sum_i \lambda_i |i_A\rangle \otimes |i_B\rangle$$

with $|i_A\rangle$ and $|i_B\rangle$ orthonormal bases in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

Leads to the reduced density matrix

$$\begin{aligned} \rho_B &= \mathrm{Tr}_A |\Psi\rangle\langle\Psi| \\ &= \sum_i \lambda_i^2 |i_B\rangle\langle i_B| \end{aligned}$$

Entanglement entropy

$$S_E \equiv -\mathsf{Tr}\,\rho_B \log \rho_B = -\sum_i \lambda_i^2 \log \lambda_i^2$$

For example,

$$|\Psi\rangle = \sum_i^D \frac{1}{\sqrt{D}} |i_A\rangle \otimes |i_B\rangle$$
$$\implies S_E = \log D$$

- Interested in *bipartite entanglement of pure states*.

But, S_E is a single number $\rightsquigarrow$ encodes limited information...

Entanglement spectrum

Can always write

$$\rho_B = e^{-H_E}, \quad \text{i.e.} \quad H_E \equiv -\log \rho_B,$$

the “entanglement Hamiltonian".

Already diagonalized H_E:

$$\rho_B = \sum_i e^{-\epsilon_i} |i_B\rangle\langle i_B|$$

with ϵ_i $(i = 1, 2, \dots)$ $[\lambda_i^2 = e^{-\epsilon_i}]$ the “entanglement spectrum".

[Li & Haldane '08]

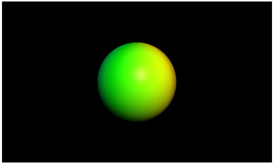
Provides thorough understanding of entanglement.

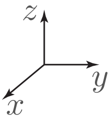
- Study for spacetime fields.

Outline

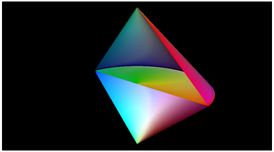
Spherical causal domain

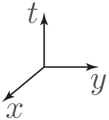
Cauchy development of 3-ball with boundary 2-sphere of radius R:





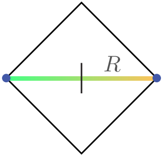
Spatial 3-ball B $\rightsquigarrow$ Cauchy development D(B).

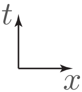


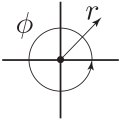


Entangling surface the boundary 2-sphere $\partial B = S^2$.

Choose adapted coordinates that preserve S^2: similar to how polar coords fix $(0, 0)$...







Diamond coordinates

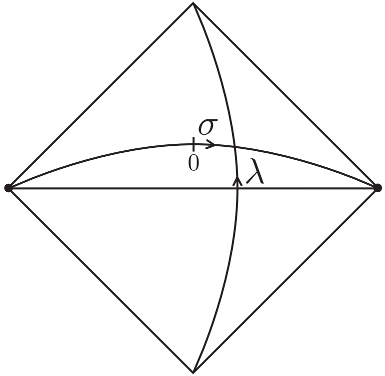
Use hyperbolas

Diamond coords $(\lambda, \sigma, \theta, \phi)$:

$$t = R\frac{\mathsf{sh}\,\lambda}{\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma}$$

$$r = R\frac{\mathsf{sh}\,\sigma}{\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma}$$

with $\lambda \in (-\infty, \infty)$, $\sigma \in [0, \infty)$.



The Minkowski metric becomes

$$ds^2 = \frac{R^2}{(\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma)^2}[-d\lambda^2 + d\sigma^2 + \mathsf{sh}^2\sigma\, d\Omega^2],$$

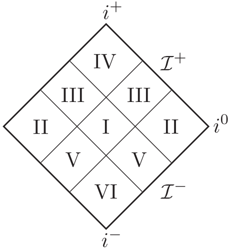
a conformal rescaling of static $\kappa = -1$ FRW.

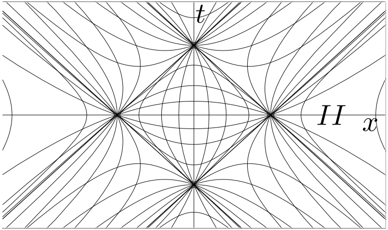
Conformal completion

Diamond coordinates can be extended to all of Minkowski space

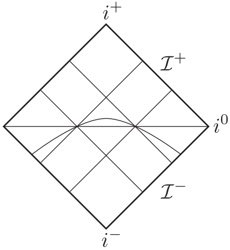
E.g. region II:

$$\left.\begin{aligned} t &= R\frac{\mathrm{sh}\,\tilde{\lambda}}{\mathrm{ch}\,\tilde{\sigma} - \mathrm{ch}\,\tilde{\lambda}}, \\ r &= R\frac{\mathrm{sh}\,\tilde{\sigma}}{\mathrm{ch}\,\tilde{\sigma} - \mathrm{ch}\,\tilde{\lambda}} \end{aligned}\right\} \qquad |\tilde{\lambda}| \leq \tilde{\sigma}.$$





All hyperbolas asymp. null:



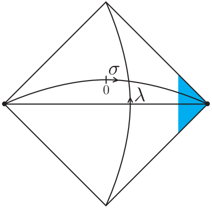
Limiting spacetimes

Large σ limit:

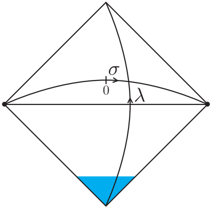
$$t = 2Re^{-\sigma}\mathsf{sh}\,\lambda = \ell\,\mathsf{sh}\,\lambda$$
$$R - r = 2Re^{-\sigma}\mathsf{ch}\,\lambda = \ell\,\mathsf{ch}\,\lambda$$

coord transformation to (left) Rindler wedge.

The proper distance from right corner is $\ell = 2Re^{-\sigma}$.

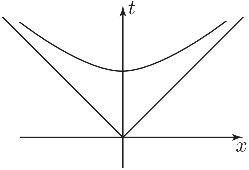


$\lambda \to -\infty$ limit:



$$ds^2 \approx 4R^2 e^{2\lambda}(-d\lambda^2 + d\sigma^2 + \mathsf{sh}^2\sigma\, d\Omega^2)$$
$$= -dT^2 + T^2(d\sigma^2 + \mathsf{sh}^2\sigma\, d\Omega^2),$$

with $T = 2Re^{\lambda}$. The Milne universe:



Current

Congruence $\xi^\mu = \left(\frac{\partial}{\partial\lambda}\right)^\mu$ with current $J^\mu = T^{\mu\nu}\xi_\nu$,

$$\nabla_\mu J^\mu = \cancelto{0}{(\nabla_\mu T^{\mu\nu})}\xi_\nu + T^{\mu\nu}\nabla_\mu\xi_\nu = T^{\mu\nu}\nabla_{(\mu}\xi_{\nu)}.$$

▲ ξ^μ is a conformal Killing $\implies \nabla_\mu J^\mu = \frac{1}{2}\theta\, T^\mu{}_\mu \quad (\theta = \nabla_\rho\xi^\rho)$.

For dilatation invariant field theory $T^\mu{}_\mu = 0$ (on shell) so

$$\boxed{\nabla_\mu J^\mu = 0,}$$

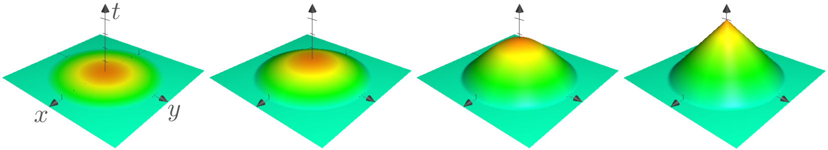
and J^μ is a Nöther current.

Entanglement (or foliation) Hamiltonian

The conserved charge is

$$C = \int T_{\mu\nu} \xi^\mu d\Sigma^\nu$$

with $T_{\mu\nu}$ the stress-tensor. It generates the spatial foliation discussed above:



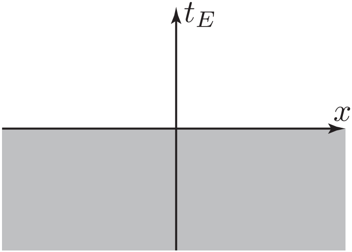
Explicitly this charge is,

$$C_{in} = \int_B r^2 dr\, d\tilde{\Omega} \frac{(R^2 - r^2)}{4R} (\dot{\varphi}^2 + \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi - \frac{1}{3}\Delta\varphi^2).$$

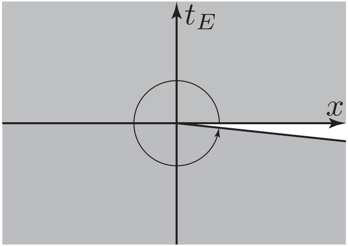
Outline

Density matrix from Euclidean path integral

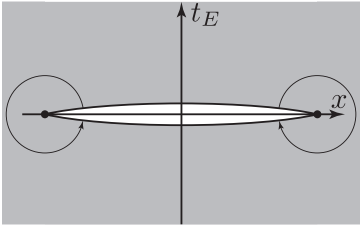
Minkowski vacuum:



Rindler density matrix:



Spherical density matrix:



$$\left.\begin{aligned} t_E &= R\frac{\sin\lambda_E}{\cos\lambda_E + \mathrm{ch}\,\sigma} \\ r &= R\frac{\mathrm{sh}\,\sigma}{\cos\lambda_E + \mathrm{ch}\,\sigma} \end{aligned}\right\}\ \text{Bipolar coords}$$

$$\boxed{\rho_B = \int \mathcal{D}\varphi\, e^{-S_E} = e^{-2\pi C_{in}}}$$

Temperatures

$\rho_B = e^{-2\pi C_{in}}/Z$: thermal density matrix satisfying KMS condition.

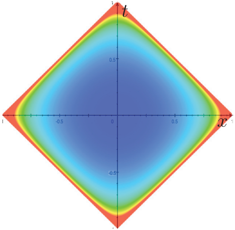
1) Geometric temperature

$$T_G = \frac{1}{2\pi}$$

$$ds^2 = \frac{R^2}{(\text{ch}\,\lambda + \text{ch}\,\sigma)^2}[-d\lambda^2 + d\sigma^2 + \text{sh}^2\sigma\, d\Omega^2]$$

2) Martinetti-Rovelli temperature

$$T_{MR} = \frac{1}{2\pi}\frac{d\lambda}{d\tau} = \frac{1}{2\pi}\frac{(\text{ch}\,\lambda + \text{ch}\,\sigma)}{R}$$

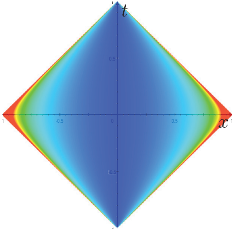


[M & R '02, Wong et al '13]

Thermometer measures $T_{\mu\nu}\xi^\mu u^\nu$.

3) Thermometer (Unruh) temp.

$$T_U = \frac{a}{2\pi} = \frac{1}{2\pi}\frac{\text{sh}\,\sigma}{R}$$



[Chirco, HMH & Rovelli '13]

- The region is not clearly in equilibrium.

Conformal symmetry

Curved spacetime Lagrangian density

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}[g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + (m^2 + \xi\mathcal{R}(x))\varphi^2].$$

For $\bar{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}$ and $\bar{\varphi} = \Omega(x)^{-1}\varphi$ the $m = 0$ action is invariant if

$$\xi = \xi(D) = \frac{(D-2)}{4(D-1)} \qquad \text{with} \qquad \xi(D=4) = \frac{1}{6}.$$

With $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ the EOM transform as

$$\bar{\Box}\bar{\varphi} = \Omega^{-3}[\Box - \frac{1}{6}\mathcal{R}]\varphi,$$

with $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$.

Modes

EOM: $\bar{\Box}\bar{\varphi} = \Omega^{-3}[\Box - \frac{1}{6}\mathcal{R}]\varphi = 0$ with barred metric $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, and unbarred $g_{\mu\nu}\,dx^\mu\,dx^\nu = -d\lambda^2 + d\sigma^2 + \mathrm{sh}^2\sigma\,d\Omega^2$.

Solve for $\kappa = -1$ FRW modes: $u_{\mathbf{k}}(x) = \chi_k(\lambda)\Pi^-_{kJ}(\sigma)\,Y^M_J(\theta,\phi)$ with

$$\Pi^-_{kJ}(\sigma) = N(k,J)\,\mathrm{sh}^J\sigma\left(\frac{d}{d\,\mathrm{ch}\,\sigma}\right)^{1+J}\cos(k\sigma)$$

$$\chi_k(\lambda) = (2k)^{-\frac{1}{2}}\,e^{-ik\lambda}$$

$M = -J, -J+1, \ldots, J; \qquad J = 0, 1, \ldots; \qquad 0 < k < \infty.$

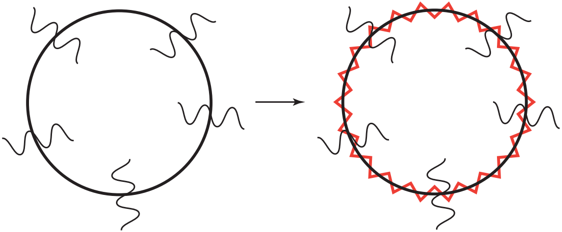
Sphere modes: $\bar{\varphi} = \Omega(x)^{-1}\varphi$

$$\bar{u}^I_{\mathbf{k}}(x) = \frac{(2k)^{-\frac{1}{2}}}{R}(\mathrm{ch}\,\lambda + \mathrm{ch}\,\sigma)\Pi^-_{kJ}(\sigma)\,Y^M_J(\theta,\phi)\,e^{-ik\lambda}.$$

Sphere vacuum and spherons

Minkowski case: $\varphi(x) = \int d^{D-1}k[a_{\mathbf{k}} u_{\mathbf{k}}(x) + a^\dagger_{\mathbf{k}} u^*_{\mathbf{k}}(x)]$
The vacuum satisfies $a_{\mathbf{k}}|0\rangle_M = 0$.

Cut out all in-out entanglement to get sphere vacuum



Sphere case: $\varphi(x) = \widetilde{\sum}_{\mathbf{k}}[s^I_{\mathbf{k}} u^I_{\mathbf{k}} + s^{I\dagger}_{\mathbf{k}} u^{I*}_{\mathbf{k}} + s^{II}_{\mathbf{k}} u^{II}_{\mathbf{k}} + s^{II\dagger}_{\mathbf{k}} u^{II*}_{\mathbf{k}}]$
The sphere vacuum satisfies $s^I_{\mathbf{k}}|0\rangle_S = s^{II}_{\mathbf{k}}|0\rangle_S = 0$.

- $s^{I\dagger}_{\mathbf{k}}$ creates *spherons*, excitations localized within the spherical entangling surface.

Two-point functions

Leveraging conformal symmetry we can achieve a more substantial characterization of the vacuum, finding its two-point functions:

$$D^+(x,x') = \Omega^{\frac{D-2}{2}}(x)\tilde{D}^+(x,x')\Omega^{\frac{D-2}{2}}(x')$$

- 1+1 spacetime:

$$D^+(x,x') = -\frac{1}{4\pi}\log|-\Delta\lambda^2+\Delta\sigma^2|$$

- 3+1 spacetime:

$$D^+(x,x') = \frac{(\mathsf{ch}\,\lambda+\mathsf{ch}\,\sigma)\Delta\sigma(\mathsf{ch}\,\lambda'+\mathsf{ch}\,\sigma')}{4\pi^2R^2\,\mathsf{sh}\,(\Delta\sigma)\,[-\Delta\lambda^2+\Delta\sigma^2]}$$

Sorkin-Johnston vacuum

Recently researchers working on causal sets have been investigating a remarkable state, the Sorkin-Johnston vacuum:

[Johnston '09, Sorkin '11]

$\rightsquigarrow$ proposed vacuum for a causal set within a causal diamond. It

- is defined only through referencing the diamond's interior
- has no entanglement with outside
- and so does not satisfy the Reeh-Schlieder theorem.

Afshordi et al have just investigated the Sorkin-Johnston vacuum in 2D analytically and numerically... [Afshordi et al '12]

Comparison

... and amongst many other things they found a surprise: near the L and R diamond corners the Sorkin-Johnston two-point function is that of a static mirror in Minkowski spacetime.

- How does the sphere vacuum compare?

In the limit $\lambda = \lambda' = 0$, $\sigma, \sigma' \to \infty$:

$$D^+_{2D} \sim -\frac{1}{4\pi} \log\left(\log \frac{\ell}{\ell'}\right) \qquad\qquad D^+_{4D} \sim \frac{1}{2\pi(\ell^2 - \ell'^2)\log\frac{\ell}{\ell'}}$$

The Rindler two-point functions. [Troost & Van Dam '79]
(Recall $\ell = 2Re^{-\sigma}$ is the proper distance from the right corner.)

$\implies$ the sphere and Sorkin-Johnston vacuums differ in 2D.
Is this still true in 4D?

Entanglement spectrum

Let us return to considering the Minkowski vacuum $|0\rangle_M$.

Explicitly constructed the entanglement Hamiltonian $\rightsquigarrow$ diagonalize it to recover the entanglement spectrum $\lambda_i = e^{-\epsilon_i/2}$ and

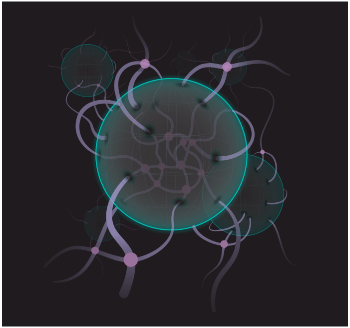
$$|0\rangle_M = \sum_i \lambda_i |i_{in}\rangle_S \otimes |i_{out}\rangle_S.$$

This shows that despite the absence of in-out entanglement you can recover the Minkowski vacuum through a well-tuned superposition of sphere states.

Spin network entanglement

This suggests an intriguing perspective on spin networks.

Individual nodes are like the interior of the sphere with no entanglement between a given node and its neighbors.



Can we choreograph entanglement to yield the Minkowski vacuum?

♦ A wealth of condensed matter research on entanglement to draw from (MPS, PEPs, etc).

Milo

Milo, who turns 2 today, has been helping with the calculations!






Conclusions

- Interesting connections to causal sets. What is the nature of the Sorkin-Johnston vacuum in 4D?
- Exhibited an example outside the hypotheses of the Reeh-Schlieder theorem.
- Looking to engineer the Minkowski vacuum and its entanglement from spin network superposition.

Numerous possibilities

- More general entangling surfaces
- Deeper insights from condensed matter
- Anomalies

Credits

Burning orb image: http://www.beautifullife.info/graphic-design/the-sphere-is-not-enough/

Spherical spin network: Z. Merali, "The origins of space and time," Nature News, Aug. 28, 2013

Milo pictures: Goffredo Chirco.

Special thanks to the Cracow School of Theoretical Physics, LIII Course, 2013 for a lovely school and setting while this work was begun. And to the Perimeter Institute for their gracious hosting of visitors and support during the continuation of this work.