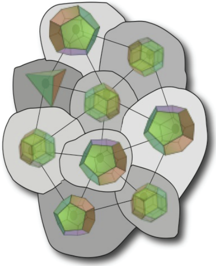
New Perspectives on Polyhedra

Hal Haggard with Simone Speziale
International Loop Quantum Gravity Seminar
November 22nd, 2016

The polyhedral picture is important for understanding loop gravity

- crucial to the dynamics of spin foams



- for fixed-graph Hilbert space the semiclassical limit is a collection of flat polyhedra

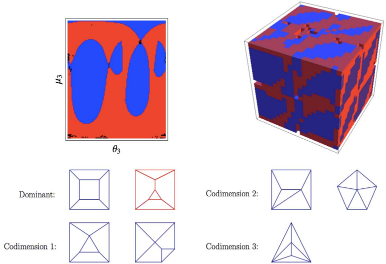
&

their extrinsic geometry is encoded in the Ashtekar-Barbero holonomy

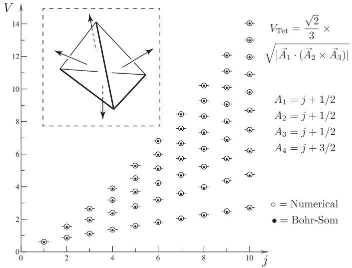
- generalizes Regge calculus: tetrahedra $\rightsquigarrow$ polyhedra, & discontinuity is allowed; $\exists$ shape mismatch in the general case

Many interesting directions to explore...

Bianchi, Doná, and Speziale numerical reconstruction '11

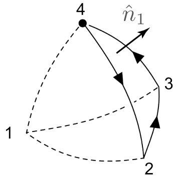


Bianchi and HMH Bohr-Sommerfeld tetrahedral volume spectrum '11

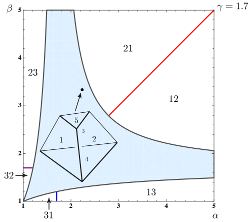


HMH, Han, Kamiński, Riello curved tetrahedron reconstruction '16

$$O_4 O_3 O_2 O_1 = \mathbb{1}, \quad O_i \in SO(3) \iff$$



HMH described full phase diagram of pentahedral adjacencies '13



Beautiful connection with quantum groups, Λ, and CS theory, maybe important for renormalization [Dittrich]

Projective perspective may be key to proving a generalized Minkowski theorem for curved polyhedra [Dupuis, Girelli, Livine, HHKR]

Volumes from fluxes:
Lasserre's algorithm allows reconstruction of polyhedra (adjacency, edge lengths, volume) from the normals and heights wrt a ref pt;

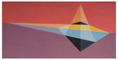
Bianchi-Doná-Speziale: heights can be inverted for the areas numerically, to complete the reconstruction from the fluxes:

$$A_i(h,n) = \sum_{j,k=1}^{F} M_i^{jk}(n_1,\ldots,n_F)h_j h_k$$

Can we do better, and have an analytic reconstruction procedure?
Our central observation: adjacency is projective

Outline

I. Classical Results in Projective Geometry



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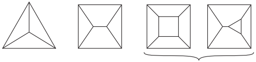
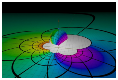


III. Projective Varieties and Spaces



Projective geometry uses only a straightedge in constructions

Girard Desargues' founding work will be important to us

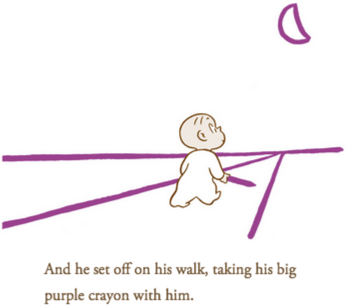
Detail from Raphael's fresco with Euclid using a compass

The heart of theorems is thus about incidence:
points lying on lines and lines intersecting in points

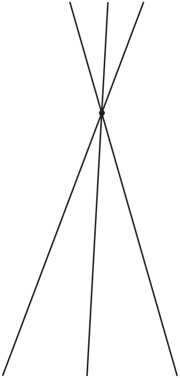
All pairs of lines in a projective plane meet in a point;

sometimes a point at infinity

Harold and the Purple Crayon



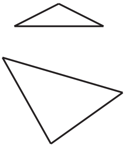
The line of points at infinity
we call the horizon

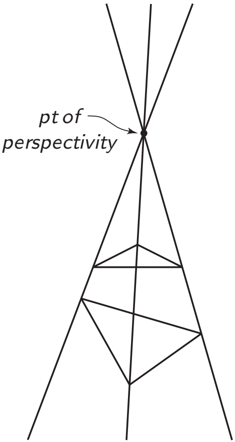


A special circumstance:
three lines incident on a point

Two triangles in the plain are perspective from a point if...

Two triangles





Two triangles perspective
from a point

...$\exists$ a pairing of vertices s.t. lines through the pairs are incident.

Desargues' Theorem: Two triangles perspective from a point are also perspective from a line.

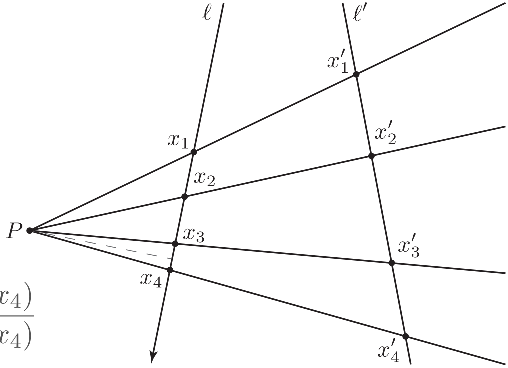
(See animation)

The fundamental projective invariant is the cross ratio

$$\rho = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_2 - x_3)(x_1 - x_4)}$$

Using triangle areas, can prove

$$\rho = \frac{\sin(x_1 P x_3)\sin(x_2 P x_4)}{\sin(x_2 P x_3)\sin(x_1 P x_4)}$$

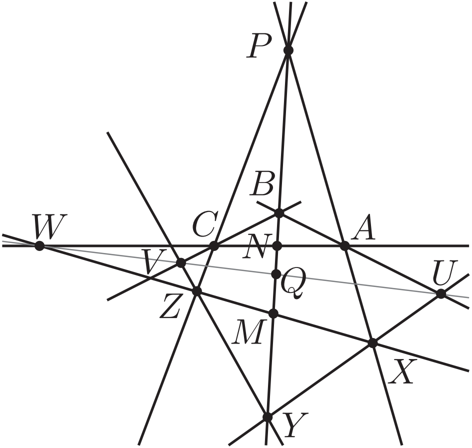


The latter implies immediately that $\rho = \rho'$. Convenient to write

$$\rho = (x_1, x_2; x_3, x_4) \stackrel{P}{=} (x'_1, x'_2; x'_3, x'_4)$$

Proof of Desargues' theorem using cross ratios

Let $U = BA \cap YX$ etc.



$$(W, V; Q, BA \cap WV) \overset{B}{=} (W, C; N, A)$$

$$\overset{P}{=} (W, Z; M, X) \overset{Y}{=} (W, V; Q, YX \cap WV)$$

[Brief aside...

Some permutations of the four points change the cross ratio

$$(x_1, x_2; x_3, x_4) = \rho \qquad (x_1, x_2; x_4, x_3) = \frac{1}{\rho}$$

$$(x_1, x_3; x_4, x_2) = \frac{1}{1-\rho} \qquad (x_1, x_3; x_2, x_4) = 1 - \rho$$

$$(x_1, x_4; x_3, x_2) = \frac{\rho}{\rho - 1} \qquad (x_1, x_4; x_2, x_3) = \frac{\rho - 1}{\rho}$$

These permutations form a group called the **anharmonic group**
—the other permutations leave it invariant

...]

For some time no synthetic, constructive proof was known...

...brings us back to our story and loop gravity

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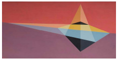
I. Classical Results in Projective Geometry



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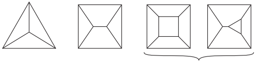

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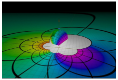
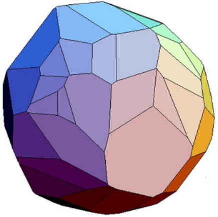


Minkowski's theorem: The areas A_l and the unit-normals $\vec{n}_l$ to the faces of a convex polyhedron fully characterize its shape. Let $\vec{A}_l = A_l \vec{n}_l$, then the *space of shapes of polyhedra* with F faces of given areas A_l is

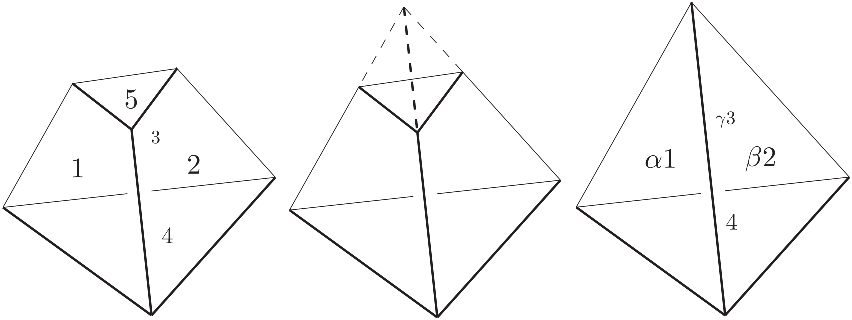
$$\mathcal{S}_{\mathsf{KM}} = \{\vec{A}_l,\, l = 1\,..\,F \mid \sum_l \vec{A}_l \,= 0\,,\, \|\vec{A}_l\| = A_l\}/ISO(3)$$

$$\vec{A}_1 + \cdots + \vec{A}_n = 0 \longleftrightarrow$$



Existence and uniqueness thm $\rightsquigarrow$ nothing to say about construction

A pentahedron can be completed to a tetrahedron

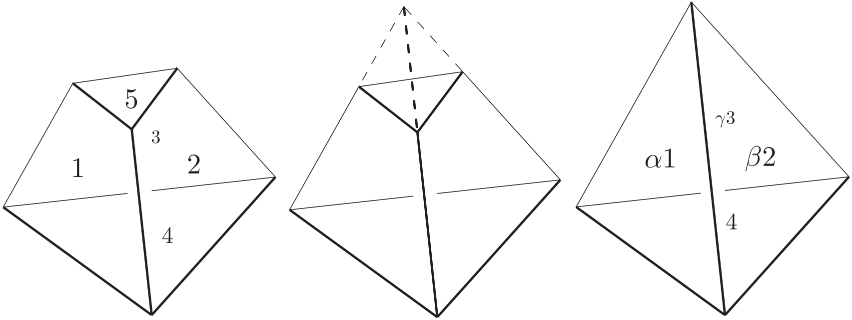


Define α as the ratio of the tetrahedron's first face area, $A_{1\text{tet}}$, to the pentahedron's first face area A_1, i.e.

$$A_{1\text{tet}} = \alpha A_1$$

and similarly for β and γ

A pentahedron can be completed to a tetrahedron



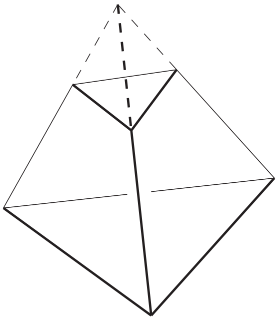
$\alpha, \beta, \gamma > 1$ found from,

$$\alpha \vec{A}_1 + \beta \vec{A}_2 + \gamma \vec{A}_3 + \vec{A}_4 = 0$$

e.g. $\implies \alpha = -\vec{A}_4 \cdot (\vec{A}_2 \times \vec{A}_3)/\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)$

Let $W_{ijk} = \vec{A}_i \cdot (\vec{A}_j \times \vec{A}_k)$. Different closures imply,

- Case 1: 54-pentahedron
  $\alpha_1 \vec{A}_1 + \beta_1 \vec{A}_2 + \gamma_1 \vec{A}_3 + \vec{A}_4 = 0,$
  $$\gamma_1 = -\frac{W_{124}}{W_{123}}$$
- Case 2: 53-pentahedron
  $\alpha_2 \vec{A}_1 + \beta_2 \vec{A}_2 + \vec{A}_3 + \gamma_2 \vec{A}_4 = 0,$
  $$\gamma_2 = -\frac{W_{123}}{W_{124}} = \frac{1}{\gamma_1}$$



Require $\alpha, \beta, \gamma > 1$: These cases are mutually incompatible!

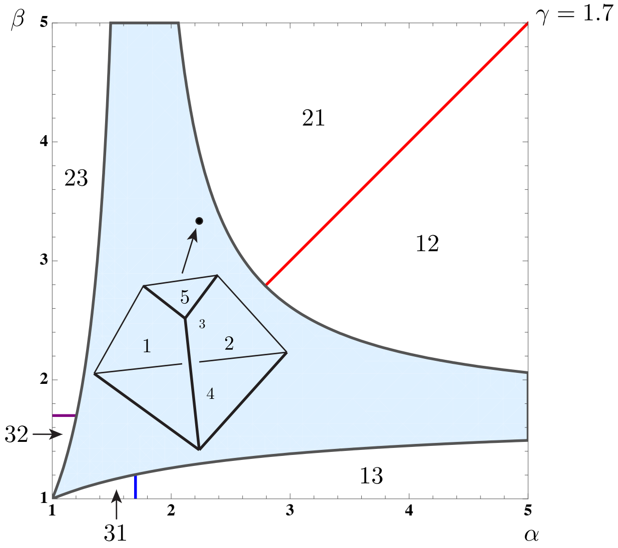
A representative sample of the 20 cases:

1. $\alpha_1 \equiv \alpha \quad \beta_1 \equiv \beta \quad \gamma_1 \equiv \gamma$
2. $\alpha_2 = \frac{\alpha}{\gamma} \quad \beta_2 = \frac{\beta}{\gamma} \quad \gamma_2 = \frac{1}{\gamma}$
3. $\alpha_3 = \frac{\alpha}{\beta} \quad \beta_3 = \frac{\gamma}{\beta} \quad \gamma_3 = \frac{1}{\beta}$
4. $\alpha_4 = \frac{\beta}{\alpha} \quad \beta_4 = \frac{\gamma}{\alpha} \quad \gamma_4 = \frac{1}{\alpha}$
5. $\alpha_5 = 1 - \alpha \quad \beta_5 = 1 - \beta \quad \gamma_5 = 1 - \gamma$
6. $\alpha_6 = \frac{1-\alpha}{1-\gamma} \quad \beta_6 = \frac{1-\beta}{1-\gamma} \quad \gamma_6 = \frac{1}{1-\gamma}$
7. $\alpha_7 = \frac{1-\alpha}{1-\beta} \quad \beta_7 = \frac{1-\gamma}{1-\beta} \quad \gamma_7 = \frac{1}{1-\beta}$
8. $\alpha_8 = \frac{1-\beta}{1-\alpha} \quad \beta_8 = \frac{1-\gamma}{1-\alpha} \quad \gamma_8 = \frac{1}{1-\alpha}$
9. $\alpha_9 = \frac{\gamma-\alpha}{\gamma} \quad \beta_9 = \frac{\gamma-\beta}{\gamma} \quad \gamma_9 = \frac{\gamma-1}{\gamma}$
10. $\alpha_{10} = \frac{\gamma-\alpha}{\gamma-1} \quad \beta_{10} = \frac{\gamma-\beta}{\gamma-1} \quad \gamma_{10} = \frac{\gamma}{\gamma-1}$
11. ... 20.

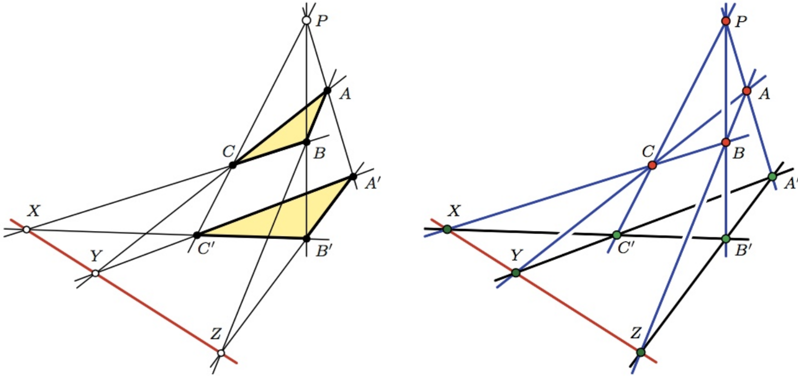
N.B. anharmonic group appears for γ in cases 1, 2, 5, 6, 9, & 10

[For defs of cases see arXiv:1211.7311]

Requiring: $\beta > \alpha > \gamma > 1$ *and* $\gamma \geq \alpha\beta/(\alpha+\beta-1)$
guarantees that the 54 pentahedron is constructible

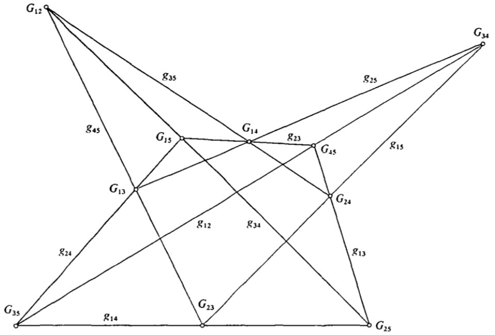


Closely related to a synthetic proof of Desargues' theorem



Corresponding to every pentahedron is a unique Desargues configuration up to projective transformations

Desargues configurations are quite symmetrical—every pt of the config is a pt of perspectivity for a pair of triangles in the config



Coxeter's Theorem: The moduli space of Desargues configurations is captured by five parameters λ_α such that

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 0, \qquad \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 \neq 0$$

with all 20 cross ratios of the configuration given by $\rho_{\alpha\beta} = -\frac{\lambda_\alpha}{\lambda_\beta}$

A small gap in pentahedral adjacency $\leftrightarrow$ Desargues remains

Certainly we can take

$$\lambda_1 = W_{234},\ \lambda_2 = -W_{134},\ \lambda_3 = W_{124},\ \lambda_4 = W_{123}, \\ \text{and } \lambda_5 = -\sum_{\alpha=1}^{4} \lambda_\alpha$$

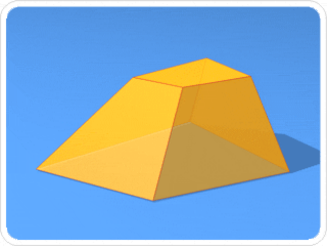
and we will have a Desargues configuration with the same cross ratios as in the 54-pentahedral construction...

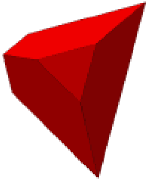
...but, we have not yet succeeded in proving that this is *the* Desargues config resulting upon projection of the pentahedron

We expect a clear correspondence and are working to close this gap

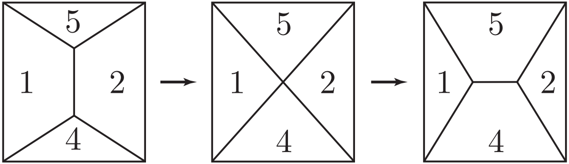
What about polyhedra with more faces?

Two dominant hexahedral classes: cuboids and pentagonal wedges

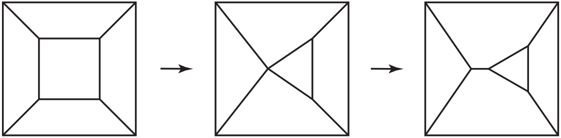




Can draw several lessons from the pentahedral case:

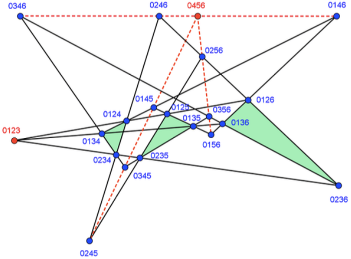


Cuboids occupy isolated islands surrounded by pentagonal wedges



cuboid adj $= 5!! = 15$ and pent wedge adj $= \binom{6}{2}\binom{4}{2} \cdot 2 = 15 \cdot 12$

Scaling procedures from the pentahedral work of '13 apply, but also there is a projective configuration, the Steiner-Plücker config



But how general is this really? We do not know yet, but...

Collect the F area vectors $\vec{A}_i$ into a $3 \times F$ matrix

$$A = \begin{pmatrix} | & \cdots & | \\ \vec{A}_1 & \cdots & \vec{A}_F \\ | & \cdots & | \end{pmatrix}$$

Weyl's Theorem: Every projectively invariant polynomial of the components of A can be expressed as a polynomial in the 3×3 sub-determinants of A.

Lasserre's algorithm is not polynomial in the components of the area vectors—our approach is in the case of pentahedra

Motivates looking for a general argument that adjacency is polynomial in the area vector components $\rightsquigarrow$ look closely at W_{ijk}

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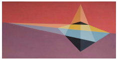

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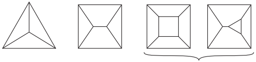

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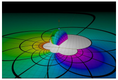


Connections with projective geometry were already highlighted in nice papers by Freidel, Krasnov, & Livine; [e.g. 1005.2090]

Using spinors they showed explicitly the isomorphism:

$$P\mathbb{C}^n \times \mathcal{S}_{\mathrm{KM}} \simeq \mathrm{Gr}_{\mathbb{C}}(2,n)$$

A projective variety: embedded in $\mathbb{C}^n \wedge \mathbb{C}^n$
via spinor bilinears $F_{ij} = z_i^0 z_j^1 - z_j^0 z_i^1$
and Plücker relations $F_{ij}F_{kl} - F_{ik}F_{jl} + F_{il}F_{jk} = 0$

Fixed total area
Fixed total phase

Spinor's norms: areas
Spinor's phases: a framing of each face (related to extrinsic geometry θ)

Questions:

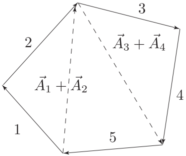
- What is the precise relation btwn the complex Grassmannian projectivity and the real projectivity of the adjacencies?
- Can we analytically reconstruct the adjacency using projective geometry, either the complex one of the F's, or the real one of the W_{ijk}'s?

Can the W_{ijk} be taken as coordinates for the space of shapes?

Consider pentahedral case for proof of principle test:

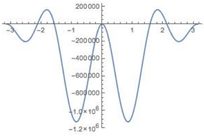
Some notation

$$\vec{A}_{ij} = \vec{A}_{ji} = \vec{A}_i + \vec{A}_j,$$
$$W_{ijk} = \vec{A}_i \cdot [\vec{A}_j \times \vec{A}_k],$$
$$\vec{B}_{ij} = \vec{A}_i \times \vec{A}_j, \quad \vec{B}_{(kl)(ij)} = \vec{A}_{kl} \times \vec{A}_{ij}$$

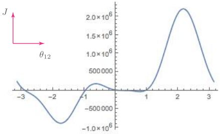


Then

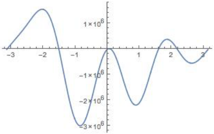
$$W_{ijk} = \vec{B}_{ij} \cdot \vec{B}_{(kl)(ij)} \frac{(\vec{A}_{kl} \cdot \vec{A}_k)}{A_{kl}^2} \frac{\sin\theta_{ij}}{A_{ij}} + \frac{\vec{B}_{ij} \cdot \vec{B}_{kl}}{A_{kl}} [\cos\theta_{ij} \sin\theta_{kl} + \frac{\vec{A}_{ij} \cdot \vec{A}_{kl}}{A_{ij} A_{kl}} \sin\theta_{ij} \cos\theta_{kl}]$$



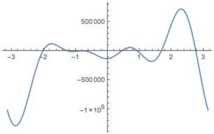
$\theta_{34} = 0$



$\theta_{34} = \frac{\pi}{1.9}$



$\theta_{34} = \frac{2\pi}{1.9}$



$\theta_{34} = \frac{3\pi}{1.9}$

Could be just good luck; already for hexahedra there are 6 KM variables, but 10 W's (after using closure)—seems hopeless...

But, in $\mathbb{R}^3$ there must be linear dependencies amongst 4 or more vectors, e.g. $\sum_{i=1}^4 a_i \vec{A}_i = 0$, solve for a_i to find

$$W_{234}\vec{A}_1 - W_{134}\vec{A}_2 + W_{124}\vec{A}_3 - W_{123}\vec{A}_4 = 0$$

and dot in $\{\vec{A}_1 \times \vec{A}_5, \vec{A}_2 \times \vec{A}_5, \vec{A}_3 \times \vec{A}_5\}$ to get

$$\begin{aligned} W_{134}W_{125} - W_{124}W_{135} + W_{123}W_{145} &= 0 \\ W_{234}W_{125} - W_{124}W_{235} + W_{123}W_{245} &= 0 \\ W_{234}W_{135} - W_{134}W_{235} + W_{123}W_{345} &= 0 \end{aligned}$$

These are quadratic Plücker relations amongst the W's and...

...now the counting works. Simpler to use $w_{ijk} = W_{ijk}/A_i A_j A_k$

A_i's n	ϕ_{ij} $(n,2)$	ϕ_{ij}'s lin.dep. $n(n-5)/2+3$	$\phi_{ij} - lin.dep.$	$\phi_{ij} - lin.dep. - clos \equiv$KM $\equiv w - lin.dep. - clos$	w's $(n,3)$	w's lin.dep. $n(n^2-3n-10)/6+3$	w's - lin.dep.
4	6	1	5	2	4	-1	4(+1 missing)
5	10	3	7	4	10	3	7
6	15	6	9	6	20	11	9
7	21	10	11	8	35	24	11

In fact, the Plückers of the W's and the $F's$ are exactly the same

$$W_{ijk} = \frac{i}{4} F_{ij} E_{ik} \bar{F}_{jk} + c.c. \qquad \begin{array}{l} F_{ij} = z_i^0 z_j^1 - z_j^0 z_i^1 = [z_i|z_j\rangle \\ E_{ij} = \bar{z}_i^0 z_j^0 + \bar{z}_i^1 z_j^1 = \langle z_i|z_j\rangle \end{array}$$

$$W_{ijk}W_{ilm} - W_{ilk}W_{ijm} + W_{ilj}W_{ikm} = 0 \leftrightarrow F_{ij}F_{kl} - F_{ik}F_{jl} + F_{il}F_{jk} = 0$$

Plücker relations amongst the complex variables also follow directly from the spinorial identity

In conclusion

The face adjacency of a pentahedron is completely determined by cross ratios that can unusually be expressed as the ratio of just two numbers, e.g. $\alpha = -W_{124}/W_{123}$

Indeed pentahedral adjacency is likely completely determined by a Desargues' configuration and projective configurations are also important to hexahedral geometry

We have provided a new set of real projective tools with rich connections to the geometry of polyhedra and exposed a computational foundation for the motto *adjacency is projective*