From a curved-space reconstruction theorem to a 4d Spinfoam model with a Cosmological Constant

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$$\Lambda = 2.90 \times 10^{-122}\,\ell_P^{-2}$$

$$= 1 \times 10^{-52} m^{-2}$$

The model:

$$Z_{\mathsf{CS}}\left(S^3;\Gamma_5|\,\vec{j},\vec{\imath}\right)=\int\mathcal{D}A\mathcal{D}\bar{A}\ \mathrm{e}^{\mathrm{i}\mathsf{CS}[S^3\,|\,A,\bar{A}]}\ \Gamma_5\left(\vec{j},\vec{\imath}\,|A,\bar{A}\right),$$

with $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons action and *complex* couplings $(t,\bar{t})$

$$\mathsf{CS}[S^3\,|\,A,\bar{A}]=\frac{t}{8\pi}\int_{S^3}\mathrm{tr}\left(A\wedge\mathrm{d}A+\frac{2}{3}A\wedge A\wedge A\right)+\frac{\bar{t}}{8\pi}\int_{S^3}\mathrm{tr}\left(\bar{A}\wedge\mathrm{d}\bar{A}+\frac{2}{3}\bar{A}\wedge\bar{A}\wedge\bar{A}\right).$$

The semiclassical limit:

$$j,\,|t|\to\infty\qquad\text{while}\qquad j/|t|\sim cnst,\quad\text{and}\quad\arg(t)=cnst.$$

The result:

$$Z_{\mathsf{CS}}\left(S^3;\Gamma_5|\,\vec{j},\vec{\imath}\right)\xrightarrow{\text{d.s.l.}}\left[\mathcal{N}_+\mathrm{e}^{\mathrm{i}\left(\sum_t\mathsf{a}_t\Theta_t-\Lambda V_4^\Lambda\right)}+\mathcal{N}_-\mathrm{e}^{-\mathrm{i}\left(\sum_t\mathsf{a}_t\Theta_t-\Lambda V_4^\Lambda\right)}\right].$$

One
The result

Two
Reconstructing a curved 4-simplex

Three
Curved spinfoams

The semiclassical limit is

$$t := \frac{12\pi}{\Lambda \ell_P^2}\left(\frac{1}{\gamma} + \mathrm{i}\right)$$

$$\boxed{\ell_P \to 0, \quad j \to \infty, \quad \text{with} \quad \mathrm{a}_{\text{phys}} \equiv \gamma \ell_P^2 j = \text{cnst}}$$

$\ell_P \to 0$ means $t \to \infty$, which corresponds to CS classical flat limit,

however

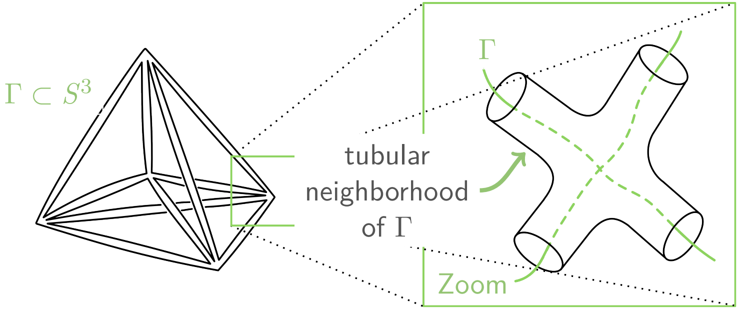
$j \to \infty$ makes the Wilson graph operator stand out and act as a distributional source for $(A, \overline{A})$,

thus avoiding flatness

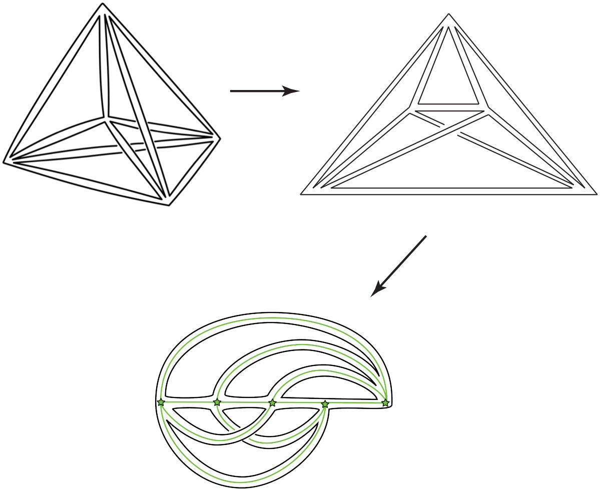
Semiclassical limit $=$
study of flat connections on the graph complement $S^3 \setminus \Gamma$

The graph complement $S^3 \setminus \Gamma$ is obtained by removing a tubular neighborhood of Γ from S^3

Here and below Γ is the graph dual to the 4-simplex boundary



Planar projection of Γ



The boundary of $S^3 \setminus \Gamma$ is a genus 6 surface

There are two types of holonomies in $S^3 \setminus \Gamma$:

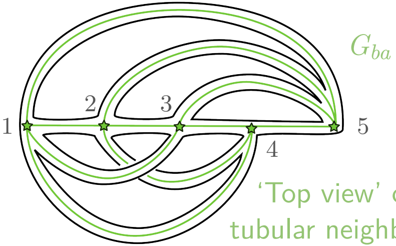
- transverse $H_b(a)$
- longitudinal G_{ba}

where $a, b, \ldots$ label the graph vertices

[connects with Bahr, Dittrich, Geiller]

We need to specify the exact paths, called a choice of framing for Γ

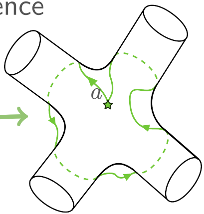
longitudinal paths run on the 'top' of the tubes



'Top view' of the tubular neighborhood

Equations of motion

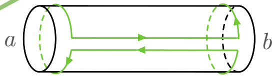
The connection on the graph complement is flat, hence holonomies along contractible paths are trivial:



closures $\overleftarrow{\prod}_b H_b(a) = \mathbb{1}$

parallel transports

$$G_{ba} H_b(a) G_{ab} = H_a(b)^{-1}$$

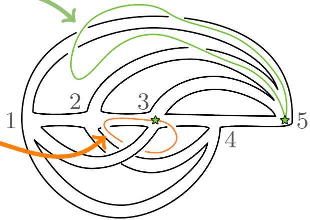


around 5 out of the 6 independent 'faces'

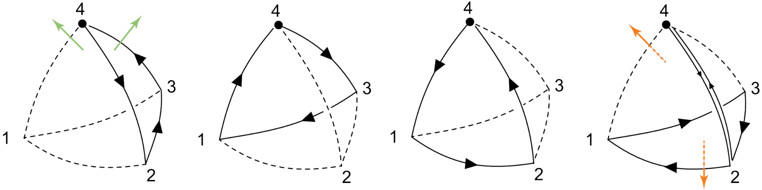
$$G_{ac} G_{cb} G_{ba} = \mathbb{1}$$

while, around the last independent 'face':

$$G_{34} G_{42} G_{23} = H_1(3)$$



In 3D, we defined simple paths to determine a geometrically meaningful curved Gram matrix.



The geometrical dot product $\hat{n}_1 \cdot \hat{n}_3$ is well defined at vertex 4,
but we have to rotate $\hat{n}_4$ to give $\hat{n}_2 \cdot \hat{n}_4$ meaning at 4.

The Gram matrix is

$$\text{Gram} = \begin{pmatrix} 1 & \hat{n}_1 \cdot \hat{n}_2 & \hat{n}_1 \cdot \hat{n}_3 & \hat{n}_1 \cdot \hat{n}_4 \\ * & 1 & \hat{n}_2 \cdot \hat{n}_3 & \hat{n}_2 \cdot \mathbf{O}_1 \hat{n}_4 \\ * & * & 1 & \hat{n}_3 \cdot \hat{n}_4 \\ sym & * & * & 1 \end{pmatrix}.$$

Trivializing the G's,

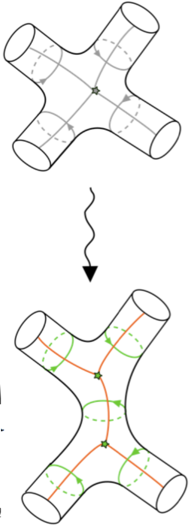
$$G_{ab} = g_a^{-1} g_b, \quad \text{except} \quad G_{42} = g_4^{-1} \left[g_3 H_{31} g_3^{-1} \right] g_2,$$

we have, in terms of $\tilde{H}_{ab} := g_a H_{ab} g_a^{-1}$,

$$\tilde{H}_{ba} = \tilde{H}_{ab}^{-1} \quad \text{except} \quad \tilde{H}_{42} = \tilde{H}_{13}^{-1} \tilde{H}_{24}^{-1} \tilde{H}_{13} \, .$$

Gives a complete understanding of the 4D Gram matrix and a 4-simplex reconstruction:

The CS phase space is $\mathcal{P} = \mathcal{M}_{\text{flat}}(\Sigma, SL(2, C))$



Natural complex coordinates are obtained via a trivalent decomposition of the graph and considering:

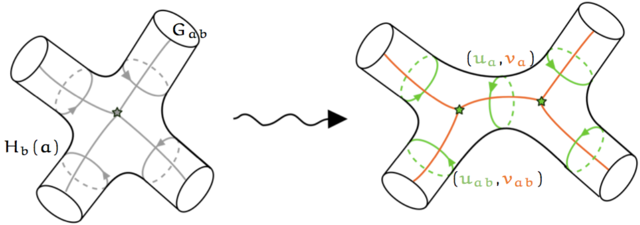
$$H_m \sim \begin{pmatrix} x_m & \\ & x_m^{-1} \end{pmatrix} \quad \text{and} \quad G_m \sim \begin{pmatrix} y_m & \\ & y_m^{-1} \end{pmatrix}$$

hence $u_m := \log x_m$ and $v_m := -2\pi \log y_m$

The Atiyah-Bott symplectic structure $\frac{h}{4\pi} \int_{\partial M} \text{Tr}\delta A \wedge \delta A$ induces the canonical Poisson brackets

$$\{u_m, v_n\} = \delta_{m,n} \qquad \text{and} \qquad \{\bar{u}_m, \bar{v}_n\} = \delta_{m,n}$$

To do WKB, we relate (u, v) to simplicial geometries



The 4-simplex reconstruction theorem shows that

$$u_{ab} = -i\frac{\Lambda}{6}a_{ab} + 2\pi i n_{ab}$$
$$v_{ab} = \frac{h}{4\pi}\Theta_{ab} + i\frac{h}{4\pi}\phi_{ab} + i\frac{h}{2}m_{ab}$$

where $n_{ab}, m_{ab} \in \mathbb{Z}$ are lifting ambiguities.

At 4-vertex a, (u_a, v_a) encodes shape of tet a with areas $\{a_{ab}\}_b$
$\rightsquigarrow$ $\exists$ parity related solution with: $(\tilde{v}_a, \tilde{v}_{ab}) = (v_a, -v_{ab})$

The WKB approximation for simplicial geometries is

$$Z(u,\bar{u}|M) \sim Z^{\alpha}\, e^{\frac{i}{\hbar}\Re(\frac{\Lambda h}{12\pi i})(\Sigma a_{ab}\Theta_{ab} - \Lambda V_4^{\Lambda}) + \frac{i}{\hbar}\Re(\frac{\Lambda h}{6})\Sigma m'_{ab} a} + Z^{\tilde{\alpha}}\, e^{-\frac{i}{\hbar}\cdots}$$

♦ the Regge action of simplicial General Relativity with a cosmological constant [Regge 1961; Barrett, Foxon 1994; Bahr, Dittrich 2010]

■ the two branches of opposite parity ($\sim$ 3d QG, mini-superspace QC)

• arbitrary term depending on the choice of lift $v := \log y + 2\pi i m$

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The result

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At the level of a single building block, the EPRL amplitude of the 3d spin-network boundary state ψ_Γ is

holonomy of $\mathcal{A}$

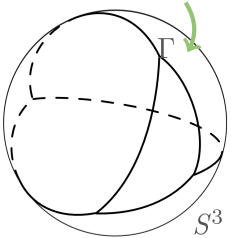
$$Z_{\mathsf{EPRL}}(\psi_\Gamma) := \int \mathcal{D}B\,\mathcal{D}\mathcal{A}\; \mathrm{e}^{\frac{\mathrm{i}}{2\ell_P^2}\int B\wedge\mathcal{F}[\mathcal{A}]}\; (f_\gamma\psi_\Gamma)(G[\mathcal{A}]) = (f_\gamma\psi_\Gamma)(\mathbb{1})$$

$\mathrm{SL}(2,\mathbb{C})$ spin connection

B is the 'bivector' field
[$B = \star e \wedge e$ on geometric states]

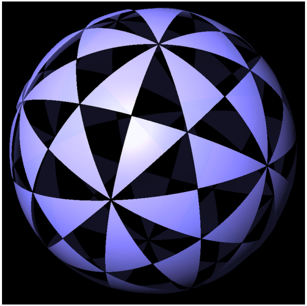
f_γ is the Dupuis-Livine map,
it embeds ψ into spacetime

dual to 4-simplex boundary



*[one dimensional lower drawing]

Philosophy for ΛRegge: construct a manifold out of *homogeneously curved* building blocks & $(d-2)$-dimensional defects



At the quantum level, the homogenous curvature is implemented via $BF - \frac{\Lambda}{6}BB$ dynamics, and defects are created as in the flat case

$$\Lambda\text{-GR} = BF - \frac{\Lambda}{6}BB + \text{geometricity constraints}$$

For boundary connection functionals, ΛBF in the bulk is equivalent to CS on the boundary

$$\begin{aligned} Z(\psi_\Gamma) &:= \int \mathcal{D}B \mathcal{D}\mathcal{A}\, \mathrm{e}^{\frac{\mathrm{i}}{2\ell_P^2}\int B\wedge\mathcal{F}[\mathcal{A}]-\frac{\Lambda}{6}B\wedge B}\, (f_\gamma\psi_\Gamma)(G[\mathcal{A}]) \\ &= \int \mathcal{D}\mathcal{A}\, \mathrm{e}^{\frac{3\mathrm{i}}{4\Lambda\ell_P^2}\int \mathcal{F}[\mathcal{A}]\wedge\mathcal{F}[\mathcal{A}]}\, (f_\gamma\psi_\Gamma)(G[\mathcal{A}]) \\ &= \int \mathcal{D}\mathcal{A}\, \mathrm{e}^{\frac{3\pi\mathrm{i}}{\Lambda\ell_P^2}\mathsf{CS}[\mathcal{A}]}\, (f_\gamma\psi_\Gamma)(G[\mathcal{A}]), \end{aligned}$$

where the Chern-Simons functional is

$$\mathsf{CS}[\mathcal{A}] := \frac{1}{4\pi}\oint_{S^3} \mathrm{d}\mathcal{A}\wedge\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}$$

Baez

Twisting the previous construction by using the γ-Holst action gives

$$\boxed{Z_{\Lambda\mathsf{EPRL}}(\psi_\Gamma) := \int \mathcal{D}A\mathcal{D}\overline{A}\; \mathrm{e}^{\mathrm{i}\frac{t}{2}\mathsf{CS}[A]+\mathrm{i}\frac{\bar{t}}{2}\mathsf{CS}[\overline{A}]}\;(f_\gamma\psi_\Gamma)(G[A,\overline{A}])}$$

where $(A,\overline{A})$ are the self- and antiself-dual parts of $\mathcal{A}$

and $t := \frac{12\pi}{\Lambda\ell_P^2}\left(\frac{1}{\gamma}+\mathrm{i}\right)$ is the complex CS level

Note

$Z_{\Lambda\mathsf{EPRL}}$ involves only quantities living on the boundary

ΛEPRL $= \mathrm{SL}(2,\mathbb{C})$-CS evaluation of a specific Wilson graph operator

$t := \frac{12\pi}{\Lambda \ell_P^2}\left(\frac{1}{\gamma} + \mathrm{i}\right)$

Two immediate consequences:

The CS level t is complex, $\rightsquigarrow$ no (known) quantum group structure associated to the graph evaluation

Fairbairn & Meusburger, Han

Invariance of the amplitude under large gauge transformations $\mathcal{A} \mapsto \mathcal{A}^g$ implies $\Re(t) \in \mathbb{Z}$, i.e.

$$\frac{12\pi}{|\Lambda|} \equiv 4\pi R_\Lambda^2 \in \gamma \ell_P^2 \mathbb{N}$$

Kodama, Randono, Smolin, Wieland

$t := \frac{12\pi}{\Lambda \ell_P^2} \left(\frac{1}{\gamma} + \mathrm{i}\right)$

Three interesting limits:

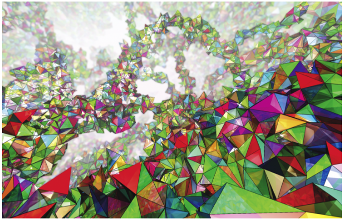
Semiclassical ΛRegge limit:
$j,\ |t| \to \infty$ while $j/|t| \sim cnst,$ and $\arg(t) = cnst.$

Vanishing cosmological constant $\Lambda \to 0$:
$t \to \infty$, & CS is projected onto its classical solutions $\rightsquigarrow$ flat EPRL

q-deformed Lorentzian Barrett-Crane amplitude:
when $\gamma \to \infty$, the EPRL graph operator $\to$ Barrett & Crane's,
while t becomes $\in \mathrm{i}\mathbb{R}$, giving $q = \exp\left(-\ell_P^2 / R_\Lambda^2\right)$

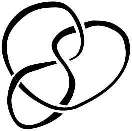
Noui & Roche

■ $SL(2,\mathbb{C})$ Chern-Simons theory implements BF-$\frac{\Lambda}{6}$BB, leads to a quantized cosmological constant, and has a rich semiclassical limit



♦ Conjecture: the curved Minkowski theorem holds in general $\rightsquigarrow$ study of flat connections on Riemann surfaces closely related to study of discrete, curved polyhedra.

♣ Provide an enriched context for understanding the role of quantum groups in cosmological spacetimes.



Λ