# Reconstructing Spacetime from Entanglement

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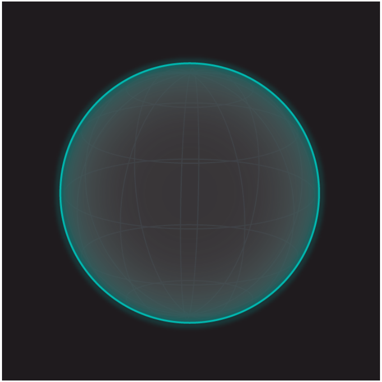
# Finite region quantum gravity

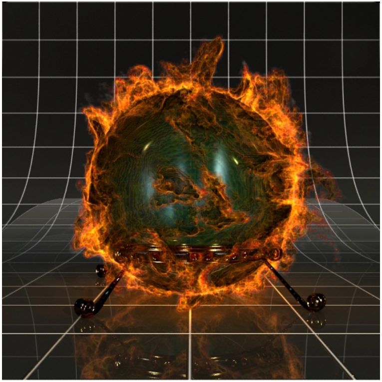
What is the physics of a finite region of a quantum spacetime?



Finite regions can incorporate diffeos of GR into a quantum context $\rightsquigarrow$ avoid notorious difficulties, e.g. [Arkani-Hamed et al].

# Entanglement thermality

Entanglement leads to even finite regions of spacetime being hot.



We illustrate this claim with a spherical entangling surface.

# Entanglement insights

Casini, Huerta & Myers have extensively studied the entanglement entropy of spherical causal domains. [Casini, Huerta & Myers '11]

Bianchi has been shedding interesting light on black holes through entanglement. [Bianchi 1-'12, 2-'12]

Disappearance of distinction between statistical and quantum fluctuations. [Bianchi, HMH, Rovelli '13]

♦ Focus on QFT while aiming for spin networks and loop gravity.

# Outline


1 Introduction to loop gravity and the discreteness of space

2 What kind of entanglement?

3 Where (and when) is the region?

4 Why the entanglement spectrum?

# Loop Quantum Gravity

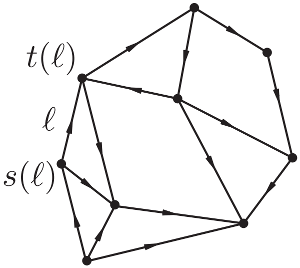
Briefly construct Hilbert space of loop gravity:

$$\mathcal{H}.$$

Similarities to Fock space of QED

and to lattice gauge theory (e.g. QCD).

Built on graphs:




Graph $\Gamma$

$L$ "links" $\ell$
$N$ "nodes" $n$

source and target:
$s : \ell \mapsto s(\ell)$ and $t : \ell \mapsto t(\ell)$

# Fock Space

Massive scalar field:

- One particle: $\mathcal{H}_1 = L^2(M)$, $M$ the Lorentz hyperboloid.
- n particles,

$$\mathcal{H}_n = L^2(M^n)/\sim$$

  with $\sim$ permutations. Factorization symmetrizes states.
- All states up to $N$ particles

$$\mathcal{H}_N = \bigoplus_{n=0}^{N} \mathcal{H}_n.$$

  Fock space

$$\mathcal{H}_{\mathsf{Fock}} = \lim_{N\to\infty} \mathcal{H}_N.$$

# Lattice Gauge Theory

Lattice $\Gamma$ with $L$ links $\ell$, $N$ nodes $n$ and gauge group $G$

$$\tilde{\mathcal{H}}_\Gamma = L^2(G^L).$$

States $\psi(h_\ell) \in \tilde{\mathcal{H}}_\Gamma$ acted on by gauge transformations

$$\psi(h_\ell) \to \psi(g_{s(\ell)}\, h_\ell\, g^{-1}_{t(\ell)}), \qquad g_n \in G.$$

Gauge invariant Hilbert space is

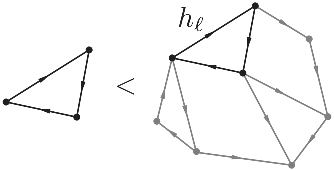
$$\mathcal{H}_\Gamma = L^2(G^L/G^N).$$

General graph $\Gamma$, called a spin network,

$$\tilde{\mathcal{H}}_\Gamma = L^2(SU(2)^L/SU(2)^N),$$

an $SU(2)$ lattice gauge theory.

If $\Gamma' \leq \Gamma$ then $\tilde{\mathcal{H}}_{\Gamma'} \subset \tilde{\mathcal{H}}_\Gamma$.




$$\mathcal{H}_\Gamma = \tilde{\mathcal{H}}_\Gamma / \sim$$

$$\mathcal{H} = \lim_{\Gamma \to \infty} \mathcal{H}_\Gamma$$

# Loop Quantum Gravity

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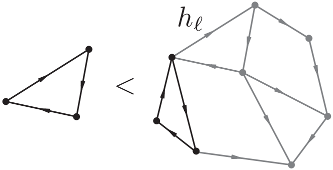


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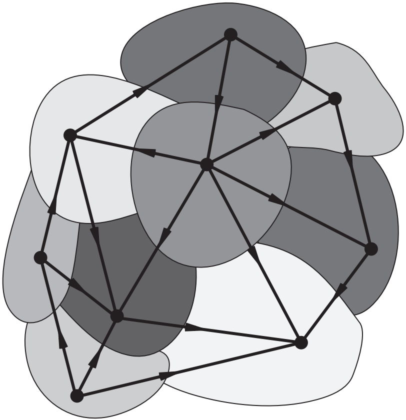
# Physical Picture

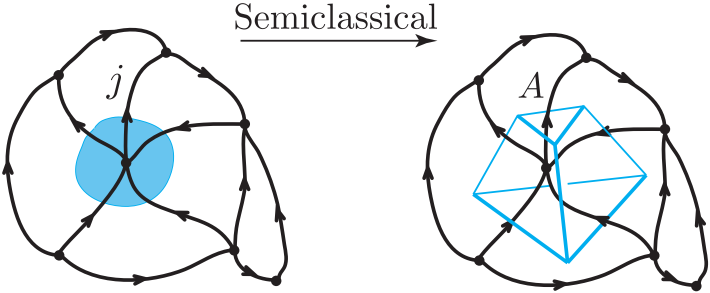


Quanta of gravity are “grains” or “chunks” of space

# Volume

Polyhedral Volume: [Bianchi, Doná and Speziale]

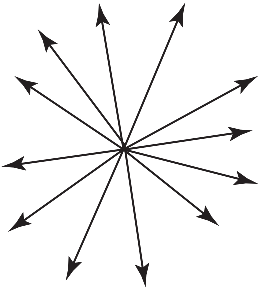
$$\hat{V}_{\text{Pol}} = \text{The volume of a quantum polyhedron}$$

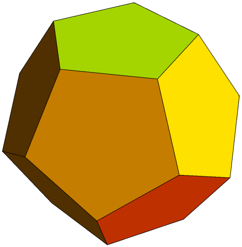


# Minkowski's theorem: polyhedra

The area vectors of a convex polyhedron determine its shape:

$$\vec{A}_1 + \cdots + \vec{A}_n = 0.$$





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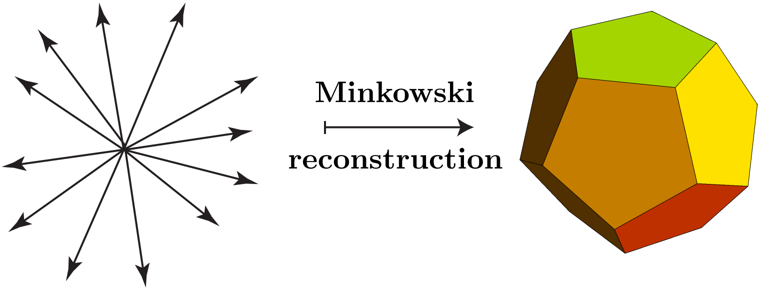
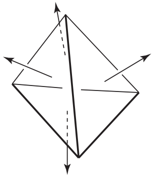



Only an existence and uniqueness theorem.

# Minkowski's theorem: a tetrahdedron

Interpret the area vectors of tetrahedron as angular momenta:

$$\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 = 0 \quad \Longleftrightarrow$$



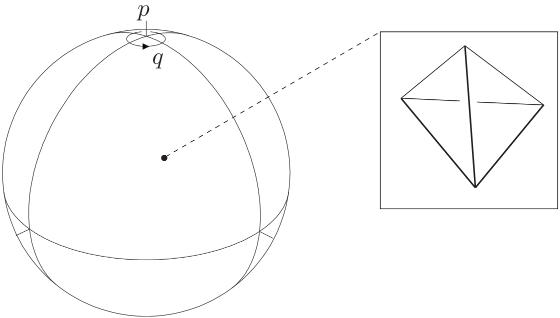
For fixed areas $A_1, \ldots, A_4$ each area vector lives in $S^2$.

Symplectic reduction of $(S^2)^4$ gives rise to the Poisson brackets:

$$\{f, g\} = \sum_{l=1}^{4} \vec{A}_l \cdot \left( \frac{\partial f}{\partial \vec{A}_l} \times \frac{\partial g}{\partial \vec{A}_l} \right)$$

# Minkowski's theorem: a tetrahdedron

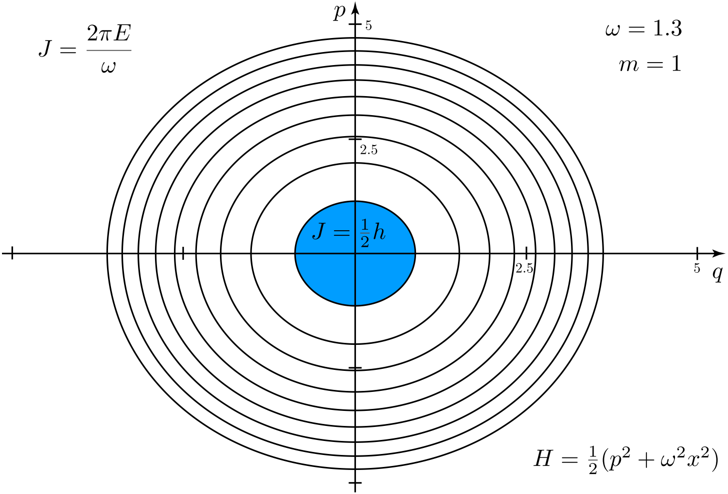
For fixed areas $A_1, \ldots, A_4$




$p = |\vec{A}_1 + \vec{A}_2|$ $\qquad$ $q =$ Angle of rotation generated by $p$:

$$\{q, p\} = 1$$

# Bohr-Sommerfeld Quantization of Harmonic Osc.



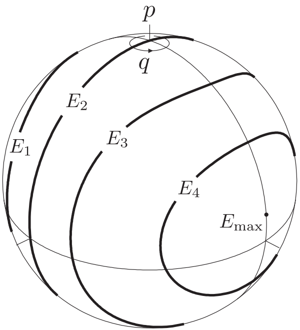

Require:

$$J = \oint_\gamma p dq = (n + \frac{1}{2})2\pi\hbar.$$

# Dynamics

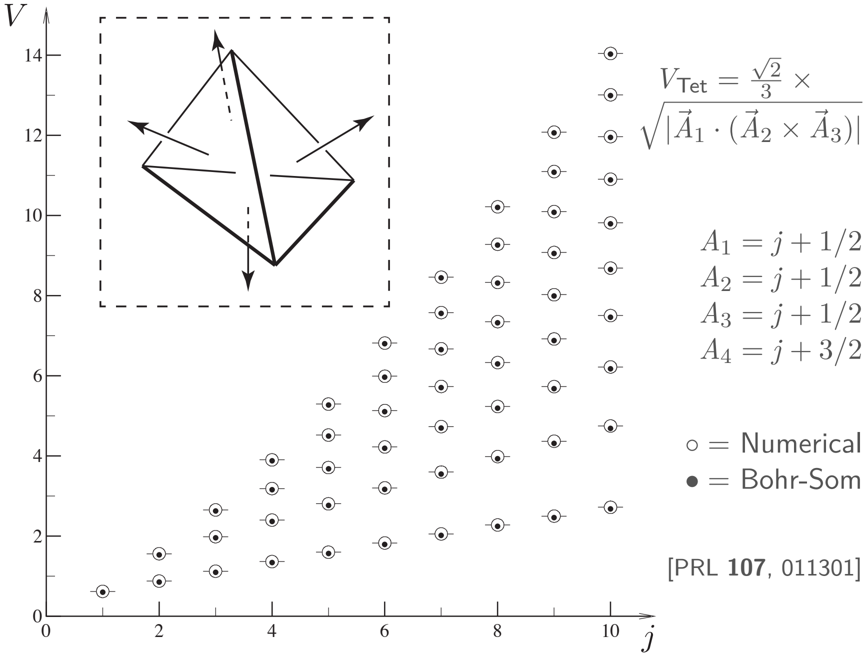
Take as Hamiltonian the Volume:

$$H = V^2 = \frac{2}{9}\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)$$




Area of orbits given in terms of complete elliptic integrals,

$$J(E) = \left( \sum_{i=1}^{4} a_i K(m) + \sum_{i=1}^{4} b_i \Pi(\alpha_i^2, m) \right) E$$



$V$
$j$
0
2
4
6
8
10
12
14
$V_{\mathsf{Tet}} = \frac{\sqrt{2}}{3} \times \sqrt{|\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)|}$
$A_1 = j + 1/2$
$A_2 = j + 1/2$
$A_3 = j + 1/2$
$A_4 = j + 3/2$
$\circ$ = Numerical
$\bullet$ = Bohr-Som
[PRL **107**, 011301]

| Table | | | |
|---|---|---|---|
| $j_1\ j_2\ j_3\ j_4$ | Loop gravity | Bohr-Sommerfeld | Accuracy |
| $6\ 6\ 6\ 7$ | 1.828 | 1.795 | 1.8% |
| | 3.204 | 3.162 | 1.3% |
| | 4.225 | 4.190 | 0.8% |
| | 5.133 | 5.105 | 0.5% |
| | 5.989 | 5.967 | 0.4% |
| | 6.817 | 6.799 | 0.3% |
| $\frac{11}{2}\ \frac{13}{2}\ \frac{13}{2}\ \frac{13}{2}$ | 1.828 | 1.795 | 1.8% |
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# Outstanding challenge

How to identify the ground state of a general relativistic theory?

*Want to coordinate individual grains of space to recover Minkowski space from this quantum theory.*

- Use entanglement

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1 Introduction to loop gravity and the discreteness of space

2 What kind of entanglement?

3 Where (and when) is the region?

4 Why the entanglement spectrum?

# Boundary conditions for the remainder

For simplicity I restrict to:

Scalar field $\varphi(x)$,

with $m = 0$

on flat $D = 3 + 1$ spacetime

unless otherwise stated. Metric signature $(-, +, +, +)$.

♦ Results apply to any conformal field theory.

# Entanglement

Pure state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

| $A$ | $B$ |
|---|---|

$\mathcal{H}_A \otimes \mathcal{H}_B$

Schmidt decomposition:

$$|\Psi\rangle = \sum_i \lambda_i |i_A\rangle \otimes |i_B\rangle$$

with $|i_A\rangle$ and $|i_B\rangle$ orthonormal bases in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

Leads to the reduced density matrix

$$\begin{aligned}\rho_B &= \mathsf{Tr}_A |\Psi\rangle\langle\Psi| \\ &= \sum_i \lambda_i^2 |i_B\rangle\langle i_B|\end{aligned}$$

# Entanglement entropy

$$S_E \equiv -\mathrm{Tr}\, \rho_B \log \rho_B = -\sum_i \lambda_i^2 \log \lambda_i^2$$

For example,

$$|\Psi\rangle = \sum_i^D \frac{1}{\sqrt{D}} |i_A\rangle \otimes |i_B\rangle$$
$$\implies S_E = \log D$$

- Interested in *bipartite entanglement of pure states*.

But, $S_E$ is a single number $\rightsquigarrow$ encodes limited information...

# Entanglement spectrum

Can always write

$$\rho_B = e^{-H_E}, \quad \text{i.e.} \quad H_E \equiv -\log \rho_B,$$

the "entanglement Hamiltonian".

Already diagonalized $H_E$:

$$\rho_B = \sum_i e^{-\epsilon_i} |i_B\rangle\langle i_B|$$

with $\epsilon_i$ $(i = 1, 2, \dots)$ $[\lambda_i^2 = e^{-\epsilon_i}]$ the "entanglement spectrum".

[Li & Haldane '08]

Provides thorough understanding of entanglement.

- Study for spacetime fields.

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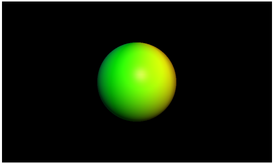
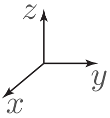
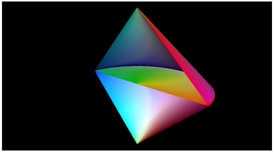
# Spherical causal domain

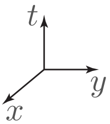
Cauchy development of 3-ball with boundary 2-sphere of radius $R$:



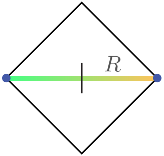


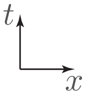

Spatial 3-ball B $\rightsquigarrow$ Cauchy development D(B).



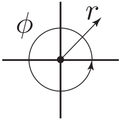



Entangling surface the boundary 2-sphere $\partial B = S^2$.




Choose adapted coordinates that preserve $S^2$: similar to how polar coords fix $(0,0)$...



# Diamond coordinates

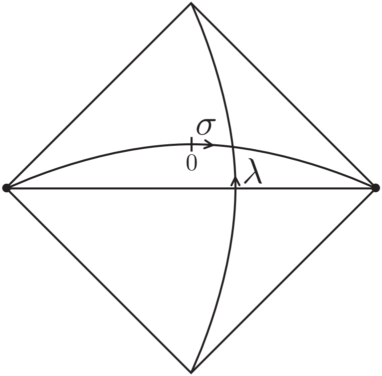
Use hyperbolas

Diamond coords $(\lambda, \sigma, \theta, \phi)$:

$$t = R\frac{\mathsf{sh}\,\lambda}{\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma}$$

$$r = R\frac{\mathsf{sh}\,\sigma}{\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma}$$

with $\lambda \in (-\infty, \infty)$, $\sigma \in [0, \infty)$.




The Minkowski metric becomes

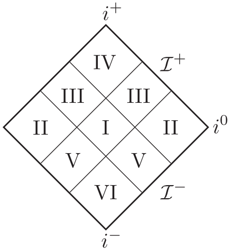
$$ds^2 = \frac{R^2}{(\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma)^2}[-d\lambda^2 + d\sigma^2 + \mathsf{sh}^2\sigma\, d\Omega^2],$$

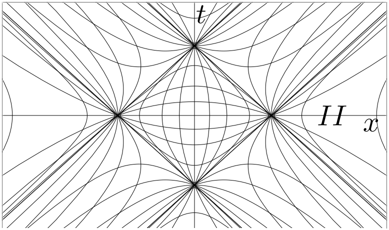
a conformal rescaling of static $\kappa = -1$ FRW.

Diamond coordinates can be extended to all of Minkowski space

E.g. region II:

$$\left.\begin{aligned} t &= R\frac{\operatorname{sh}\tilde{\lambda}}{\operatorname{ch}\tilde{\sigma}-\operatorname{ch}\tilde{\lambda}}, \\ r &= R\frac{\operatorname{sh}\tilde{\sigma}}{\operatorname{ch}\tilde{\sigma}-\operatorname{ch}\tilde{\lambda}} \end{aligned}\right\} \quad |\tilde{\lambda}| \leq \tilde{\sigma}.$$



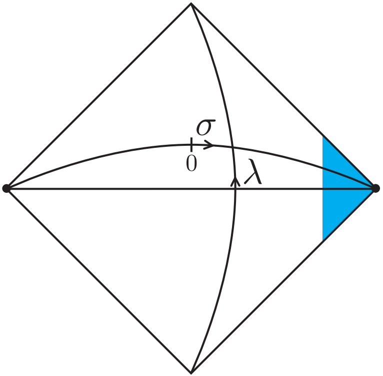



# Rindler Limit

Large $\sigma$ limit:

$$t = 2Re^{-\sigma}\text{sh}\,\lambda = \ell\,\text{sh}\,\lambda$$
$$R - r = 2Re^{-\sigma}\text{ch}\,\lambda = \ell\,\text{ch}\,\lambda$$

coord transformation to (left) Rindler wedge.




The proper distance from right corner is $\ell = 2Re^{-\sigma}$.

# Current

Congruence $\xi^\mu = \left(\frac{\partial}{\partial\lambda}\right)^\mu$ with current $J^\mu = T^{\mu\nu}\xi_\nu$,

$$\nabla_\mu J^\mu = \cancelto{0}{(\nabla_\mu T^{\mu\nu})}\xi_\nu + T^{\mu\nu}\nabla_\mu\xi_\nu = T^{\mu\nu}\nabla_{(\mu}\xi_{\nu)}.$$

▲ $\xi^\mu$ is a conformal Killing $\implies \nabla_\mu J^\mu = \frac{1}{2}\theta\, T^\mu{}_\mu \quad (\theta = \nabla_\rho \xi^\rho)$.

For dilatation invariant field theory $T^\mu{}_\mu = 0$ (on shell) so

$$\boxed{\nabla_\mu J^\mu = 0,}$$

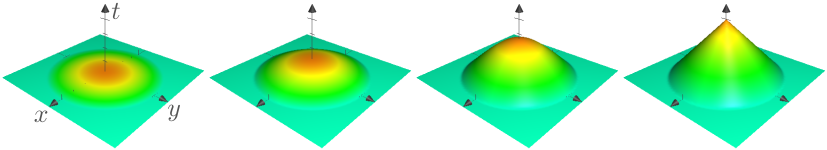
and $J^\mu$ is a Nöther current.

# Entanglement (or foliation) Hamiltonian

The conserved charge is

$$C = \int T_{\mu\nu} \xi^\mu d\Sigma^\nu$$

with $T_{\mu\nu}$ the stress-tensor. It generates the spatial foliation discussed above:




Explicitly this charge is,

$$C_{in} = \frac{1}{2R} \int_B r^2 dr\, d\tilde{\Omega} \left( \frac{1}{2}(R^2 - r^2)(\dot{\varphi}^2 + \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi) + \varphi^2 \right)$$

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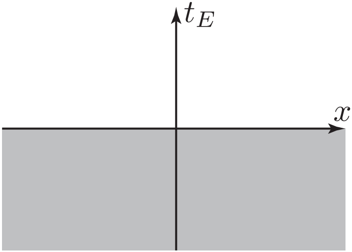
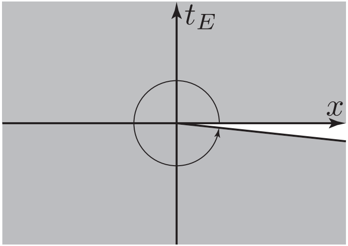
# Density matrix from Euclidean path integral

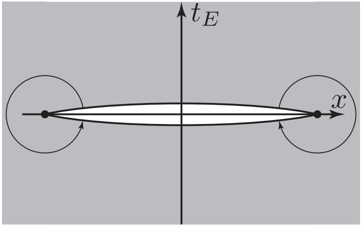
Minkowski vacuum:




Rindler density matrix:




Spherical density matrix:




$$\left.\begin{aligned} t_E &= R\frac{\sin\lambda_E}{\cos\lambda_E + \mathrm{ch}\,\sigma} \\ r &= R\frac{\mathrm{sh}\,\sigma}{\cos\lambda_E + \mathrm{ch}\,\sigma} \end{aligned}\right\} \text{Bipolar coords}$$

$$\boxed{\rho_B = \int \mathcal{D}\varphi e^{-S_E} = e^{-2\pi C_{in}}}$$

# Temperatures

$\rho_B = e^{-2\pi C_{in}}/Z$: thermal density matrix satisfying KMS condition.

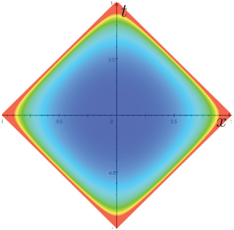
1) Geometric temperature

$$T_G = \frac{1}{2\pi}$$

2) Martinetti-Rovelli temperature

$$T_{MR} = \frac{1}{2\pi}\frac{d\lambda}{d\tau} = \frac{1}{2\pi}\frac{(\text{ch}\,\lambda + \text{ch}\,\sigma)}{R}$$

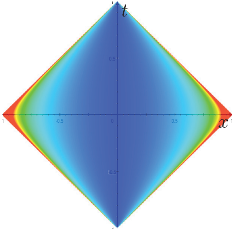
$$ds^2 = \frac{R^2}{(\text{ch}\,\lambda + \text{ch}\,\sigma)^2}[-d\lambda^2 + d\sigma^2 + \text{sh}^2\sigma\, d\Omega^2]$$




[M & R '02, Wong et al '13]

3) Thermometer (Unruh) temp.

$$T_U = \frac{a}{2\pi} = \frac{1}{2\pi}\frac{\text{sh}\,\sigma}{R}$$




Thermometer measures $T_{\mu\nu}\xi^\mu u^\nu$.

- The region is not clearly in equilibrium.

[Chirco, HMH & Rovelli '13]

# Conformal symmetry

Curved spacetime Lagrangian density (D=4 spacetime dimensions)

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}[g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + (m^2 + \frac{1}{6}\mathcal{R}(x))\varphi^2].$$

For $\bar{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}$ and $\bar{\varphi} = \Omega(x)^{-1}\varphi$ the $m = 0$ action is invariant.

With $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ the EOM transform as

$$\bar{\Box}\bar{\varphi} = \Omega^{-3}[\Box - \frac{1}{6}\mathcal{R}]\varphi,$$

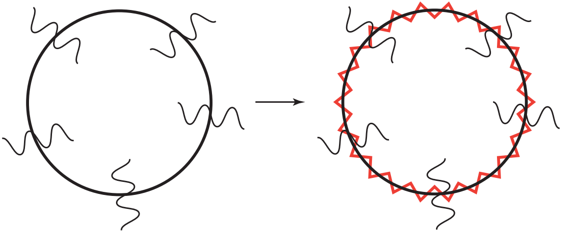
with $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. Using this, sphere modes are

$$\bar{u}^I_{\mathbf{k}}(x) = \frac{(2k)^{-\frac{1}{2}}}{R}(\mathrm{ch}\,\lambda + \mathrm{ch}\,\sigma)\Pi^-_{kJ}(\sigma)\,Y^M_J(\theta,\phi)\,e^{-ik\lambda}.$$

# Sphere vacuum and spherons

Minkowski case: $\varphi(x) = \int d^{D-1}k[a_{\mathbf{k}} u_{\mathbf{k}}(x) + a^\dagger_{\mathbf{k}} u^*_{\mathbf{k}}(x)]$
The vacuum satisfies $a_{\mathbf{k}}|0\rangle_M = 0$.

Cut out all in-out entanglement to get sphere vacuum



Sphere case: $\varphi(x) = \widetilde{\sum}_{\mathbf{k}}[s^I_{\mathbf{k}} u^I_{\mathbf{k}} + s^{I\dagger}_{\mathbf{k}} u^{I*}_{\mathbf{k}} + s^{II}_{\mathbf{k}} u^{II}_{\mathbf{k}} + s^{II\dagger}_{\mathbf{k}} u^{II*}_{\mathbf{k}}]$
The sphere vacuum satisfies $s^I_{\mathbf{k}}|0\rangle_S = s^{II}_{\mathbf{k}}|0\rangle_S = 0$.

- $s^{I\dagger}_{\mathbf{k}}$ creates *spherons*, excitations localized within the spherical entangling surface.

# Two-point functions

Leveraging conformal symmetry we can achieve a more substantial characterization of the vacuum, finding its two-point functions:

$$D^+(x,x') = \Omega^{\frac{D-2}{2}}(x)\tilde{D}^+(x,x')\Omega^{\frac{D-2}{2}}(x')$$

- 1+1 spacetime:

$$D^+(x,x') = -\frac{1}{4\pi}\log|-\Delta\lambda^2 + \Delta\sigma^2|$$

- 3+1 spacetime:

$$D^+(x,x') = \frac{(\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma)\Delta\sigma(\mathsf{ch}\,\lambda' + \mathsf{ch}\,\sigma')}{4\pi^2 R^2\,\mathsf{sh}\,(\Delta\sigma)\,[-\Delta\lambda^2 + \Delta\sigma^2]}$$

# Sorkin-Johnston vacuum

Recently researchers working on causal sets have been investigating a remarkable state, the Sorkin-Johnston vacuum:

[Johnston '09, Sorkin '11]

$\rightsquigarrow$ proposed vacuum for a causal set within a causal diamond. It

- is defined only through referencing the diamond's interior
- has no entanglement with outside
- and so does not satisfy the Reeh-Schlieder theorem.

Afshordi et al have just investigated the Sorkin-Johnston vacuum in 2D analytically and numerically... [Afshordi et al '12]

# Comparison

... and amongst many other things they found a surprise: near the L and R diamond corners the Sorkin-Johnston two-point function is that of a static mirror in Minkowski spacetime.

- How does the sphere vacuum compare?

In the limit $\lambda = \lambda' = 0$, $\sigma, \sigma' \to \infty$:

$$D^+_{2D} \sim -\frac{1}{4\pi} \log\left(\log \frac{\ell}{\ell'}\right) \qquad\qquad D^+_{4D} \sim \frac{1}{2\pi(\ell^2 - \ell'^2)\log\frac{\ell}{\ell'}}$$

The Rindler two-point functions. [Troost & Van Dam '79]
(Recall $\ell = 2Re^{-\sigma}$ is the proper distance from the right corner.)

$\implies$ the sphere and Sorkin-Johnston vacuums differ in 2D.
Is this still true in 4D?

# Entanglement spectrum

Let us return to considering the Minkowski vacuum $|0\rangle_M$.

Explicitly constructed the entanglement Hamiltonian $\rightsquigarrow$ diagonalize it to recover the entanglement spectrum $\lambda_i = e^{-\epsilon_i/2}$ and

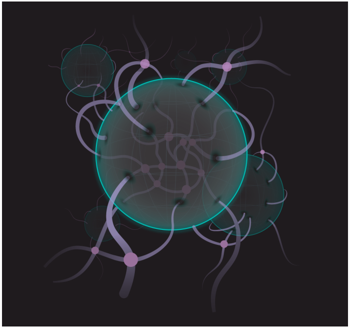
$$|0\rangle_M = \sum_i \lambda_i |i_{in}\rangle_S \otimes |i_{out}\rangle_S.$$

This shows that despite the absence of in-out entanglement you can recover the Minkowski vacuum through a well-tuned superposition of sphere states.

# Spin network entanglement

This suggests an intriguing perspective on spin networks.

Individual nodes are like the interior of the sphere with no entanglement between a given node and its neighbors.



Can we choreograph entanglement to yield the Minkowski vacuum?

♦ A wealth of condensed matter research on entanglement to draw from (MPS, PEPs, etc).

# Conclusions

- Interesting connections to causal sets. What is the nature of the Sorkin-Johnston vacuum in 4D?
- Exhibited an example outside the hypotheses of the Reeh-Schlieder theorem.
- Looking to engineer the Minkowski vacuum and its entanglement from spin network superposition.

Numerous possibilities

- More general entangling surfaces
- Deeper insights from condensed matter
- Anomalies

# Credits

Burning orb image: http://www.beautifullife.info/graphic-design/the-sphere-is-not-enough/

Spherical spin network: Z. Merali, "The origins of space and time," Nature News, Aug. 28, 2013

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