Encoding Curved Tetrahedra in Face Holonomies

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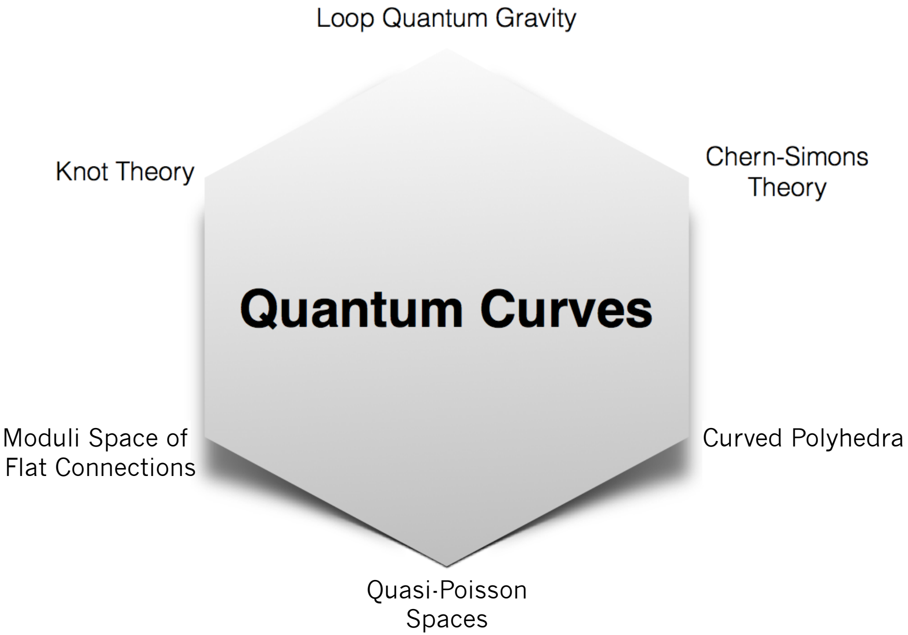
"Even a tiny cosmological constant casts a long shadow"

A. Ashtekar

$$\Lambda = 2.90 \times 10^{-122}\, \ell_P^{-2}$$

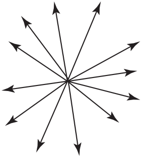
The kinematical structure of discrete geometries is strongly impacted by Λ

Symmetries have a richer structure with $\Lambda \quad \rightsquigarrow \quad$ new ideas & insights

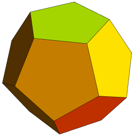

Loop Quantum Gravity
Knot Theory
Chern-Simons Theory
Quantum Curves
Moduli Space of Flat Connections
Curved Polyhedra
Quasi-Poisson Spaces

Is space discrete like a mosaic?

In 1897 H. Minkowski proved



$\Leftrightarrow$

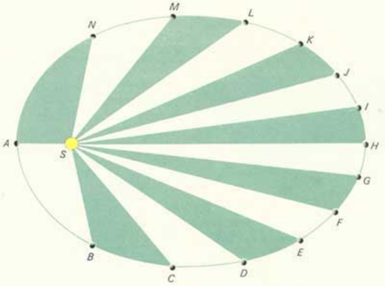


$\vec{A}_1 + \cdots + \vec{A}_N = 0,$ with $\vec{A}_i = A_i \hat{n}_i,$ $A_i =$ area face i, $\hat{n}_i = \perp$ to face i

Physical proof for one direction:

$$\vec{A}_1 + \cdots + \vec{A}_N = 0 \quad \Longleftarrow$$

Baez and Barrett combined this with Kepler's elegant realization:



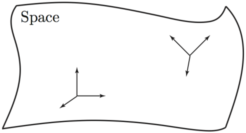
Angular momentum can be used to encode areas

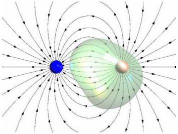
If $F(\vec{A})$ and $G(\vec{A})$ then:

$$\{F, G\} = \left(\vec{A} \times \frac{\partial F}{\partial \vec{A}}\right) \cdot \frac{\partial G}{\partial \vec{A}}$$

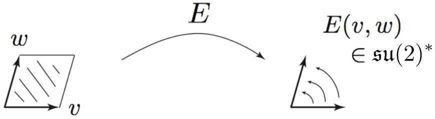
(e.g. $\{A_x, A_y\} = (\vec{A} \times \hat{x}) \cdot \hat{y} = (\hat{x} \times \hat{y}) \cdot \vec{A} = A_z$)

Minkowski's theorem is a discrete version of the two roles of the gravitational Gauß law...





$$E_{ibc} = \epsilon_{ijk} e_b^j e_c^k \qquad (b, c = 1, 2, 3) \qquad (i, j, k = 1, 2, 3)$$

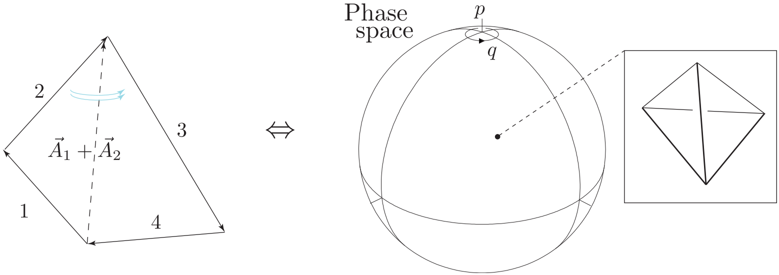


... the constraint ... | ... and the generator of gauge

$\oint E = 0$ | local choice of frame

$\vec{A}_1 + \cdots + \vec{A}_N = 0$ | diagonal rotation of all the vecs

99 years after Minkowski, in 1996, M. Kapovich and J. J. Millson found a phase space for polygons with fixed edge lengths

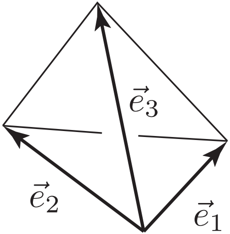


These non-planar polygons are equivalent to fixed area polyhedra

- $p = |\vec{A}_1 + \vec{A}_2|$, rotates $\vec{A}_1$ and $\vec{A}_2$ leaving others fixed
- $q =$ Angle of rotation generated by p:

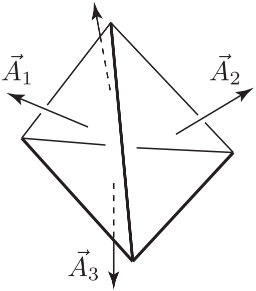
$$\{q, p\} = 1$$

Remarkably, the volume of the tetrahedron is easily expressed in terms of these variables



$$V = \frac{1}{6}\vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3)$$

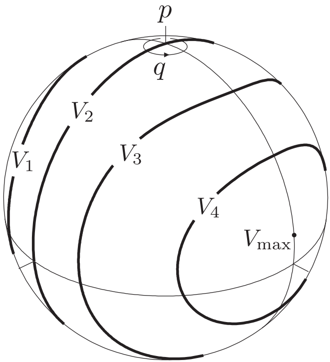
Fun to discover that, with e.g. $\vec{A}_2 = \frac{1}{2}\vec{e}_1 \times \vec{e}_3$ etc.,



$$V^2 = \tfrac{2}{9}\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)$$

With this phase space, take the volume as a Hamiltonian

$$H = V = \frac{\sqrt{2}}{3}\sqrt{|\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3|)}$$



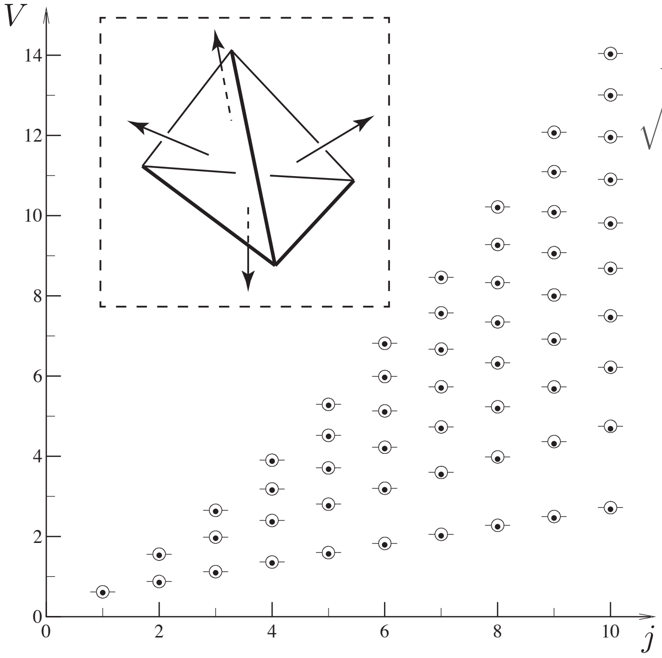
We can now quantize the system semiclassically

Require Bohr-Sommerfeld quantization condition,

$$S = \oint_{\gamma} pdq = (n + \frac{1}{2})h.$$

Area of orbits given in terms of complete elliptic integrals,

$$S(V) = \left(aK(m) + \sum_{i=1}^{4} b_i \Pi(\alpha_i^2, m) \right) V$$



$$V_{\mathsf{Tet}} = \frac{\sqrt{2}}{3} \times \sqrt{|\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)|}$$

$$A_1 = j + 1/2$$
$$A_2 = j + 1/2$$
$$A_3 = j + 1/2$$
$$A_4 = j + 3/2$$

○ = Numerical

● = Bohr-Som

[PRL **107**, 011301]

Table

j_1 j_2 j_3 j_4	Loop gravity	Bohr-Sommerfeld	Accuracy
6 6 6 7	1.828	1.795	1.8%
	3.204	3.162	1.3%
	4.225	4.190	0.8%
	5.133	5.105	0.5%
	5.989	5.967	0.4%
	6.817	6.799	0.3%
$\frac{11}{2}$ $\frac{13}{2}$ $\frac{13}{2}$ $\frac{13}{2}$	1.828	1.795	1.8%
	3.204	3.162	1.3%
	4.225	4.190	0.8%
	5.133	5.105	0.5%
	5.989	5.967	0.4%
	6.817	6.799	0.3%

Is space discrete like a mosaic? It may be, but...

...this should be understood as a spectral discreteness.

One

Is space discrete
like a mosaic?

Two

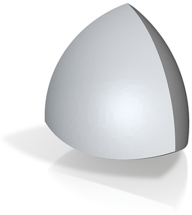
The shadow of the
cosmological constant

Three

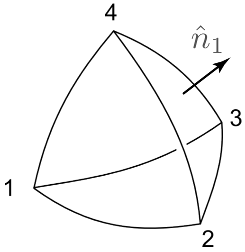
Curved spinfoams

A spherical tetrahedron is 4 points of S^3 connected by geodesics

Each face is a triangular portion of a great 2-sphere

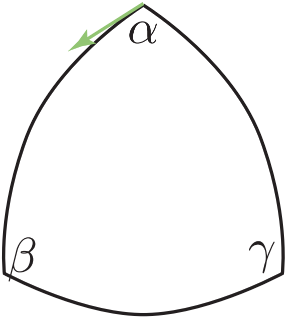


- Great spheres are flatly embedded in S^3 (i.e. $K_{ij} = 0$)

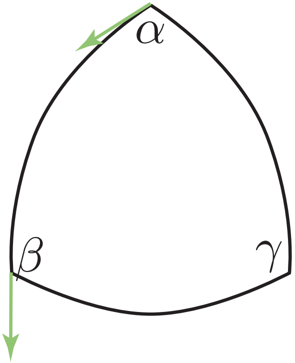


The normal to a face is well-defined and invariant under parallel transport

The holonomy around a curved triangle is the area of the triangle

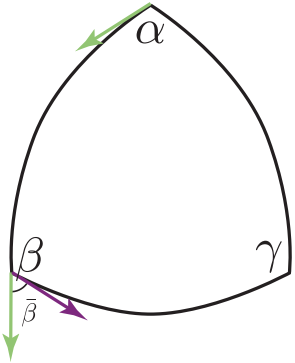


The holonomy around a curved triangle is the area of the triangle



Parallel transport of the tangent vector is easy

The holonomy around a curved triangle is the area of the triangle



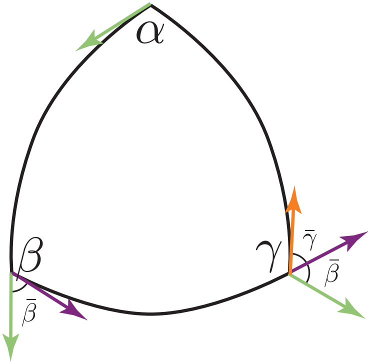
Pick up next tangent vector

and use the complement

$$\bar{\beta} = \pi - \beta$$

Transport preserves this angle

The holonomy around a curved triangle is the area of the triangle

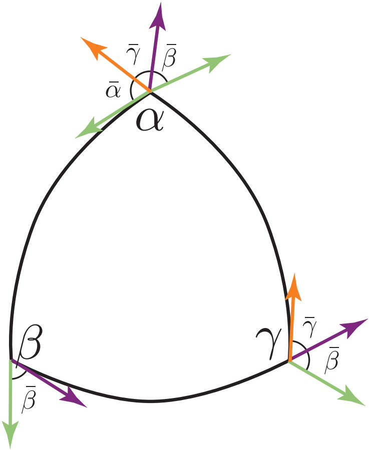


Repeat

$$\bar{\beta} = \pi - \beta$$

$$\bar{\gamma} = \pi - \gamma$$

The holonomy around a curved triangle is the area of the triangle



$$\bar{\alpha} = \pi - \alpha$$
$$\bar{\beta} = \pi - \beta$$
$$\bar{\gamma} = \pi - \gamma$$

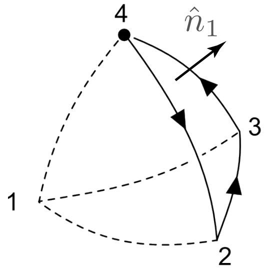
The full holonomy is a counterclockwise rotation about the normal with angle

$$\begin{aligned} a &= 2\pi - \bar{\alpha} - \bar{\beta} - \bar{\gamma} \\ &= \alpha + \beta + \gamma - \pi, \end{aligned}$$

the area of the spherical triangle!

In our work we have proved that

$$O_4\, O_3\, O_2\, O_1 = \mathbb{1} \qquad \Longleftrightarrow$$



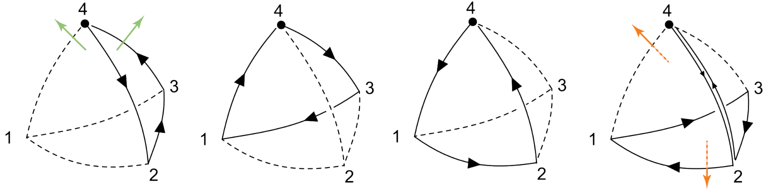
Here

$$O = \exp\left(\frac{a}{r^2}\, \hat{n} \cdot \vec{J}\right), \quad O \in SO(3)$$

The closure relation is the automatic homotopy constraint. One immediate check: for $r \to \infty$

$$O_4\, O_3\, O_2\, O_1 = \mathbb{1} + r^{-2}(a_1 \hat{n}_1 + a_2 \hat{n}_2 + a_3 \hat{n}_3 + a_4 \hat{n}_4) \cdot \vec{J} + \cdots = \mathbb{1}$$

Define simple paths to determine a geometrically meaningful curved Gram matrix.



The geometrical dot product $\hat{n}_1 \cdot \hat{n}_3$ is well defined at vertex 4, but we have to rotate $\hat{n}_4$ to give $\hat{n}_2 \cdot \hat{n}_4$ meaning at 4.

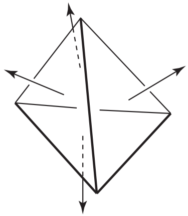
The Gram matrix is

$$\text{Gram} = \begin{pmatrix} 1 & \hat{n}_1 \cdot \hat{n}_2 & \hat{n}_1 \cdot \hat{n}_3 & \hat{n}_1 \cdot \hat{n}_4 \\ * & 1 & \hat{n}_2 \cdot \hat{n}_3 & \hat{n}_2 \cdot \mathbf{O}_1 \hat{n}_4 \\ * & * & 1 & \hat{n}_3 \cdot \hat{n}_4 \\ sym & * & * & 1 \end{pmatrix}.$$

The holonomies directly determine the sign of the curvature through Gram.

$$\begin{cases} \det \mathsf{Gram} > 0 & \text{spherical geometry} \\ \det \mathsf{Gram} < 0 & \text{hyperbolic geometry} \end{cases}$$

Consider a flat (Euclidean) tetrahedron



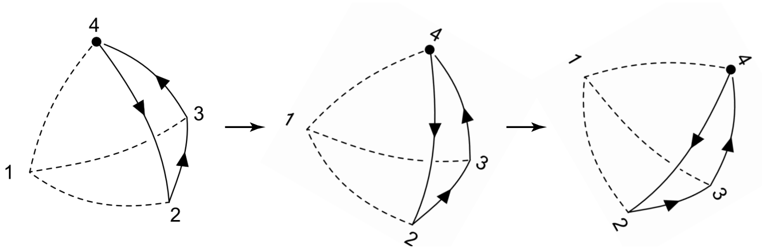
Its four vectors are linearly dependent $\rightsquigarrow$ $\det \mathsf{Gram} = 0$.

The general claim follows from a special case and continuity in the curvature.

There is no need for another group.

Lift the set $\{O_\ell\}$ to a set $\{H_\ell\} \subset SU(2)$.

The new closure, $H_4 H_3 H_2 H_1 = \mathbb{1}$, is a curved Gauß constraint and should again play its role as generator of gauge transfrmtns.

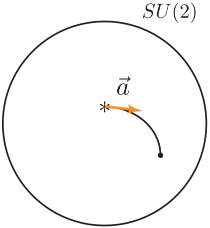


- To achieve this we must have group-valued momenta
  $\rightsquigarrow$ in this manner the theory of quasi-Poisson spaces enters

A quasi-Poisson space has a Poisson bivector, i.e. a Poisson bracket, that violates the Jacobi identity in a specific way

If we parametrize $SU(2)$ using the fundamental representation, then the holonomies around faces are

$$H = e^{-i\frac{\vec{a}\cdot\vec{\sigma}}{2r^2}}$$

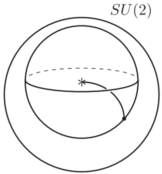


With this parametrization we can construct the quasi-Poisson brackets for the fluxes $\vec{a}$

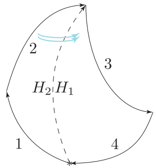
$$\{a^i, a^j\}_{qP} = \frac{a}{2r^2}\cot\frac{a}{2r^2}\epsilon^{ij}_k a^k \xrightarrow{r\to\infty} \epsilon^{ij}{}_k a^k.$$

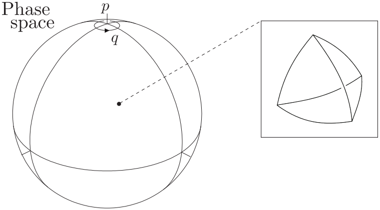
This Poisson structure naturally foliates $SU(2)$ into leaves, these are the conjugacy classes of $SU(2)$

Fixing the area of a face gives a geodesic that sweeps out a 2-sphere in $SU(2) \cong S^3$.



Forming the fusion product of 4 of these phase spaces and reducing by the Gauß constraint gives





$$\omega_H = r^2 \sin\frac{a}{r^2}\, \mathrm{d}^2\Omega \qquad \text{and} \qquad \mathcal{L}_H = 2r^2 \sin\frac{a}{2r^2}\, \mathrm{d}^2\Omega.$$

One

Is space discrete
like a mosaic?

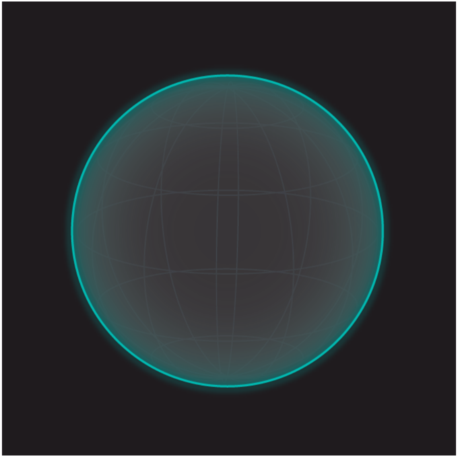
Two

The shadow of the
cosmological constant

Three

Curved spinfoams

What is the physics of a finite region of a quantum spacetime?



At the level of a single building block, the EPRL amplitude of the 3d spin-network boundary state ψ_Γ is

holonomy of $\mathcal{A}$

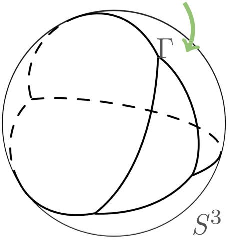
$$Z_{\mathsf{EPRL}}(\psi_\Gamma) := \int \mathcal{D}B\mathcal{D}\mathcal{A}\; \mathrm{e}^{\frac{\mathrm{i}}{2\ell_P^2}\int B\wedge\mathcal{F}[\mathcal{A}]}\; (f_\gamma\psi_\Gamma)(G[\mathcal{A}]) = (f_\gamma\psi_\Gamma)(\mathbb{1})$$

$\mathrm{SL}(2,\mathbb{C})$ spin connection

B is the 'bivector' field [$B = \star e\wedge e$ on geometric states]

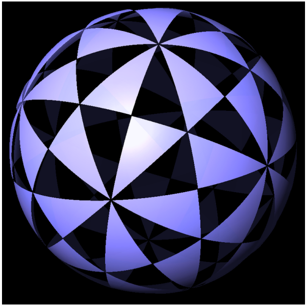
f_γ is the Dupuis-Livine map, it embeds ψ into spacetime

dual to 4-simplex boundary



*[one dimensional lower drawing]

Philosophy for ΛRegge: construct a manifold out of *homogeneously curved* building blocks & $(d-2)$-dimensional defects



At the quantum level, the homogenous curvature is implemented via $BF - \frac{\Lambda}{6} BB$ dynamics, and defects are created as in the flat case

$$\Lambda\text{-GR} = BF - \frac{\Lambda}{6} BB + \text{geometricity constraints}$$

For boundary connection functionals, ΛBF in the bulk is equivalent to CS on the boundary

$$
\begin{aligned}
Z(\psi_\Gamma) &:= \int \mathcal{D}B\mathcal{D}\mathcal{A}\, \mathrm{e}^{\frac{\mathrm{i}}{2\ell_P^2}\int B\wedge\mathcal{F}[\mathcal{A}] - \frac{\Lambda}{6} B\wedge B}\ (f_\gamma\psi_\Gamma)(G[\mathcal{A}]) \\
&= \int \mathcal{D}\mathcal{A}\, \mathrm{e}^{\frac{3\mathrm{i}}{4\Lambda\ell_P^2}\int \mathcal{F}[\mathcal{A}]\wedge\mathcal{F}[\mathcal{A}]}\ (f_\gamma\psi_\Gamma)(G[\mathcal{A}]) \\
&= \int \mathcal{D}\mathcal{A}\, \mathrm{e}^{\frac{3\pi\mathrm{i}}{\Lambda\ell_P^2}\mathsf{CS}[\mathcal{A}]}\ (f_\gamma\psi_\Gamma)(G[\mathcal{A}]),
\end{aligned}
$$

where the Chern-Simons functional is

$$
\mathsf{CS}[\mathcal{A}] := \frac{1}{4\pi}\oint_{S^3} \mathrm{d}\mathcal{A}\wedge\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}
$$

Baez

Twisting the previous construction by using the γ-Holst action gives

$$\boxed{Z_{\Lambda\mathsf{EPRL}}(\psi_\Gamma) := \int \mathcal{D}A\mathcal{D}\overline{A}\ \mathrm{e}^{\mathrm{i}\frac{h}{2}\mathsf{CS}[A]+\mathrm{i}\frac{\overline{h}}{2}\mathsf{CS}[\overline{A}]}\ (f_\gamma\psi_\Gamma)(G[A,\overline{A}])}$$

where $(A,\overline{A})$ are the self- and antiself-dual parts of $\mathcal{A}$

and $h := \frac{12\pi}{\Lambda\ell_P^2}\left(\frac{1}{\gamma}+\mathrm{i}\right)$ is the complex CS level

Note

$Z_{\Lambda\mathsf{EPRL}}$ involves only quantities living on the boundary

ΛEPRL $= \mathrm{SL}(2,\mathbb{C})$-CS evaluation of a specific Wilson graph operator

$h := \frac{12\pi}{\Lambda \ell_P^2}\left(\frac{1}{\gamma} + \mathrm{i}\right)$

Two immediate consequences:

The CS level h is complex, $\rightsquigarrow$ no (known) quantum group structure associated to the graph evaluation

Fairbairn & Meusburger, Han

Invariance of the amplitude under large gauge transformations $\mathcal{A} \mapsto \mathcal{A}^g$ implies $\Re(h) \in \mathbb{Z}$, i.e.

$$\frac{12\pi}{|\Lambda|} \equiv 4\pi R_\Lambda^2 \in \gamma \ell_P^2 \mathbb{N}$$

Kodama, Randono, Smolin, Wieland

$h := \frac{12\pi}{\Lambda \ell_P^2}\left(\frac{1}{\gamma} + \mathrm{i}\right)$

Three interesting limits:

Vanishing cosmological constant $\Lambda \to 0$:
$h \to \infty$, & CS is projected onto its classical solutions $\rightsquigarrow$ flat EPRL

q-deformed Lorentzian Barrett-Crane amplitude:
when $\gamma \to \infty$, the EPRL graph operator $\to$ Barrett & Crane's,
while h becomes $\in \mathrm{i}\mathbb{R}$, giving $q = \exp\left(-\ell_P^2/R_\Lambda^2\right)$

Noui & Roche

Semiclassical ΛRegge limit: more about this on next slide

The semiclassical ΛRegge limit is $h := \frac{12\pi}{\Lambda \ell_P^2}\left(\frac{1}{\gamma} + \mathrm{i}\right)$

$$\boxed{\ell_P \to 0, \quad j \to \infty, \quad \text{with} \quad \mathrm{a}_{\text{phys}} \equiv \gamma \ell_P^2 j = \text{cnst}}$$

$\ell_P \to 0$ means $h \to \infty$, which corresponds to CS classical flat limit,

however

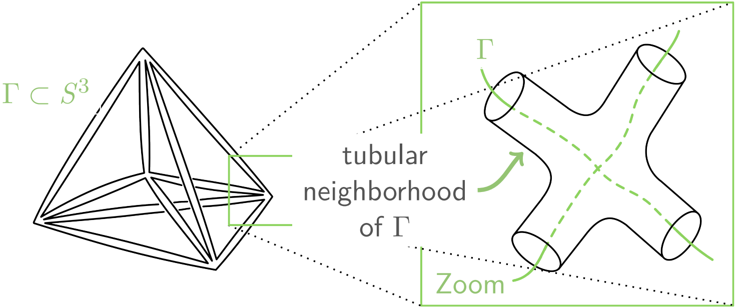
$j \to \infty$ makes the Wilson graph operator stand out and act as a distributional source for $(A, \overline{A})$,

thus avoiding flatness

Semiclassical limit =
study of flat connections on the graph complement $S^3 \setminus \Gamma$

The graph complement $S^3 \setminus \Gamma$ is obtained by removing a tubular neighborhood of Γ from S^3

Here and below Γ is the graph dual to the 4-simplex boundary

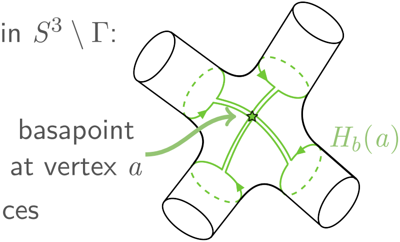


The boundary of $S^3 \setminus \Gamma$ is a genus 6 surface

There are two types of holonomies in $S^3 \setminus \Gamma$:

- transverse $H_b(a)$
- longitudinal G_{ba}

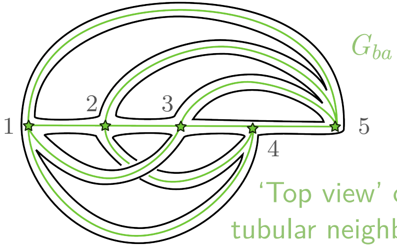
where $a, b, \ldots$ label the graph vertices



Zoom on vertex a

We need to specify the exact paths, called a choice of framing for Γ

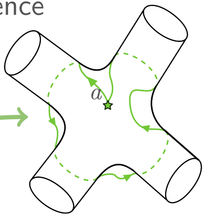
longitudinal paths run on the 'top' of the tubes



'Top view' of the tubular neighborhood

Equations of motion

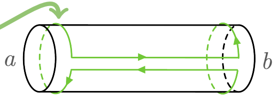
The connection on the graph complement is flat, hence holonomies along contractible paths are trivial:



closures $\overleftarrow{\prod}_b H_b(a) = \mathbb{1}$

parallel transports

$$G_{ba} H_b(a) G_{ab} = H_a(b)^{-1}$$

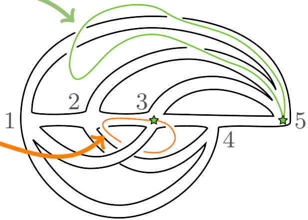


around 5 out of the 6 independent ‘faces’

$$G_{ac} G_{cb} G_{ba} = \mathbb{1}$$

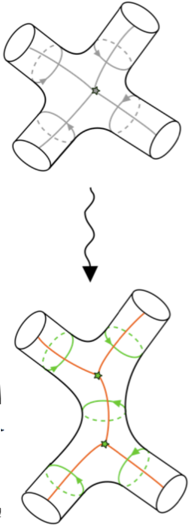
while, around the last independent ‘face’:

$$G_{34} G_{42} G_{23} = H_1(3)$$



The CS phase space is $\mathcal{P} = \mathcal{M}_{\text{flat}}(\Sigma, SL(2,C))$

Natural complex coordinates are obtained via a trivalent decomposition of the graph and considering:



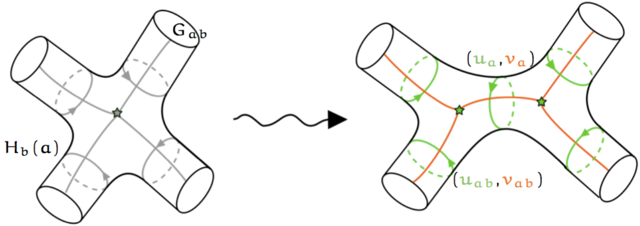
$$H_m \sim \begin{pmatrix} x_m & \\ & x_m^{-1} \end{pmatrix} \quad \text{and} \quad G_m \sim \begin{pmatrix} y_m & \\ & y_m^{-1} \end{pmatrix}$$

hence $u_m := \log x_m$ and $v_m := -2\pi \log y_m$

The Atiyah-Bott symplectic structure $\frac{h}{4\pi} \int_{\partial M} \text{Tr}\delta A \wedge \delta A$ induces the canonical Poisson brackets

$$\{u_m, v_n\} = \delta_{m,n} \qquad \text{and} \qquad \{\bar{u}_m, \bar{v}_n\} = \delta_{m,n}$$

To implement the WKB approximation for simplicial geometries, we relate (u, v) to geometrical quantities



The 4-simplex reconstruction theorem shows that

$$u_{ab} = -i\frac{\Lambda}{6}a_{ab} + 2\pi i n_{ab}$$

$$v_{ab} = \frac{h}{4\pi}\Theta_{ab} + i\frac{h}{4\pi}\phi_{ab} + i\frac{h}{2}m_{ab}$$

where $n_{ab}, m_{ab} \in \mathbb{Z}$ are lifting ambiguities.

Also, at each 4-vertex, (u_a, v_a) encodes shape of tet a with face areas $\{a_{ab}\}_b$; $\exists$ parity related solution with: $(\tilde{v}_a, \tilde{v}_{ab}) = (v_a, -v_{ab})$

The WKB approximation for simplicial geometries is

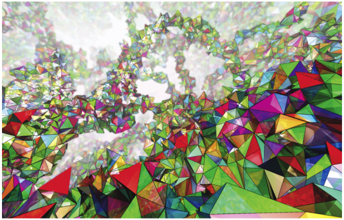
$$Z(u,\bar{u}|M) \sim Z^{\alpha}\, e^{\frac{i}{\hbar}\Re(\frac{\Lambda h}{12\pi i})(\Sigma a_{ab}\Theta_{ab} - \Lambda\, V_4^{\Lambda}) + \frac{i}{\hbar}\Re(\frac{\Lambda h}{6})\Sigma m'_{ab} a}$$
$$+ Z^{\tilde{\alpha}}\, e^{-\frac{i}{\hbar}\cdots}$$

♦ the Regge action of simplicial General Relativity with a cosmological constant [Regge 1961; Barrett, Foxon 1994; Bahr, Dittrich 2010]

■ the two branches of opposite parity (∼ 3d QG, mini-superspace QC)

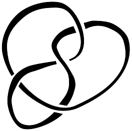
• arbitrary term depending on the choice of lift $v := \log y + 2\pi i m$

■ Established a classical foundation on which to build the quantization of spacetimes with a cosmological constant.



♦ Conjecture: the curved Minkowski theorem holds in general ⇝ study of flat connections on Riemann surfaces closely related to study of discrete, curved polyhedra.

♣ Provide an enriched context for understanding the role of quantum groups in cosmological spacetimes.



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