Symmetry Reduction and Polyhedral Dynamics

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[PRL **107**, 011301]

Motivation

Physical motivation:

- Study the quantization of gravity
- Investigate the proposal that space is quantized
- Discover the volume spectrum of tetrahedral grains of space

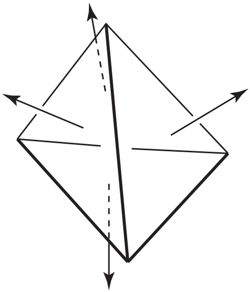
Mathematical Motivation:

- Provides convex polyhedra with a dynamical structure
- Pose and answer questions about polyhedra
- Leads to interesting integrable systems

Minkowski's theorem: a tetrahdedron

The area vectors of tetrahedron determine its shape:

$$\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 = 0.$$



Kinematics: Penrose

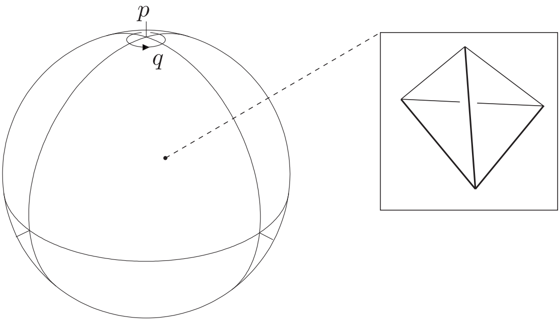
- Physical input: impose $\vec{A}_1, \ldots, \vec{A}_4$ are angular momenta

Angular momenta have Poisson brackets,

$$\{f, g\} = \sum_{\ell=1}^{4} \vec{A}_\ell \cdot \left(\frac{\partial f}{\partial \vec{A}_\ell} \times \frac{\partial g}{\partial \vec{A}_\ell} \right)$$

Kinematics II: Kapovich & Millson

$\vec{A}_1, \ldots, \vec{A}_4$ angular momenta



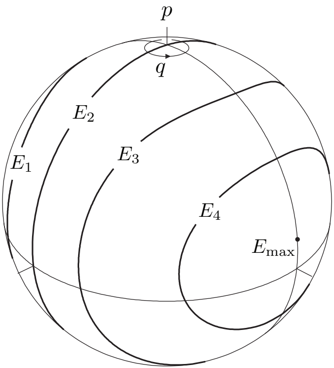
$$p = |\vec{A}_1 + \vec{A}_2|, \qquad q = \text{angle of rotation generated by } p:$$

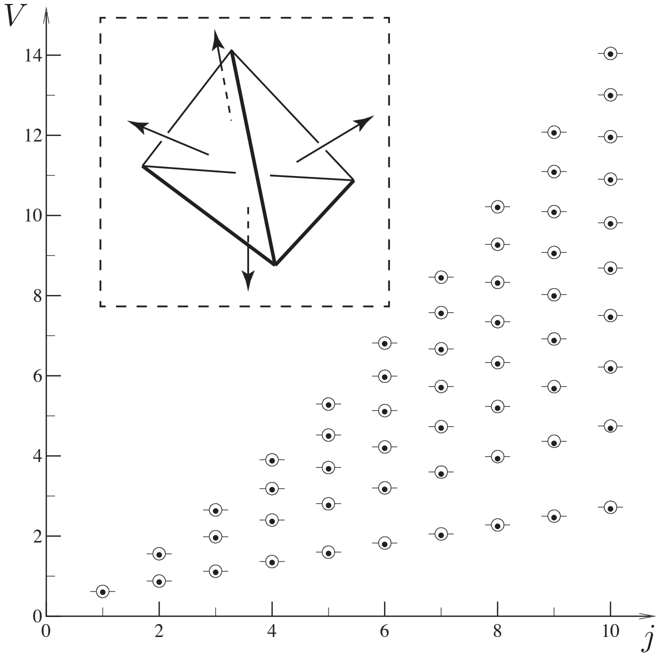
$$\{q, p\} = 1$$

Dynamics

Take as Hamiltonian the volume squared:

$$H = V^2 = \frac{2}{9} \vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3).$$





$$V_{\text{Tet}} = \frac{\sqrt{2}}{3} \times \sqrt{\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)}$$

$$A_1 = j + 1/2$$
$$A_2 = j + 1/2$$
$$A_3 = j + 1/2$$
$$A_4 = j + 3/2$$

○ = Numerical

● = Bohr-Som

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Poisson reduction

Instead of a symplectic reduction, consider Poisson reduction by overall rotation (diagonal action), $\vec{A}_1 + \cdots + \vec{A}_4$ $(= 0)$.

Independent invariants: $\vec{A}_1 \cdot \vec{A}_2, \vec{A}_2 \cdot \vec{A}_3$, and $\vec{A}_1 \cdot \vec{A}_3$

Convenient coordinates: trade $\vec{A}_1 \cdot \vec{A}_2$ for $A \equiv |\vec{A}_1 + \vec{A}_2|^2$ and similarly, $B \equiv |\vec{A}_2 + \vec{A}_3|^2$ and $C \equiv |\vec{A}_1 + \vec{A}_3|^2$

Equations of motion

In these coordinates

$$V^2 = ABC + aA + bB + cC + d,$$

where a, b, c, and d are constants (functions of $|\vec{A}_1|, \ldots, |\vec{A}_4|$).

Computation with the Lie-Poisson brackets gives,

$$\dot{A} = \{A, V^2\} = A(B - C) + c - b,$$
$$\dot{B} = \{B, V^2\} = B(C - A) + a - c,$$
$$\dot{C} = \{C, V^2\} = C(A - B) + b - a,$$

a non-trivial deformation of the Lotka-Volterra system!

Nambu Bracket

Evidently, $K \equiv A + B + C = \text{const.}$

An easy check demonstrates,

$$\dot{A} = \{A, K, V^2\} \equiv \left|\frac{\partial(A, K, V^2)}{\partial(A, B, C)}\right|, \cdots$$

This leads to two compatible Poisson structures,

$$\{f, g\}_K = -\vec{\nabla} K \cdot (\vec{\nabla} f \times \vec{\nabla} g) \quad \text{and} \quad \{f, g\}_{V^2} = \vec{\nabla} V^2 \cdot (\vec{\nabla} f \times \vec{\nabla} g)$$

i.e. this is a bi-Hamiltonian system.

Questions

This system does not fall into the usual classification of the Lotka-Volterra systems.

- Where does it belong in the bi-Hamiltonian classification?
- Is there a Lax pair for this system?
- What about an R-matrix and q-deformation?