Pentahedral Volume, Chaos, and Quantum Gravity

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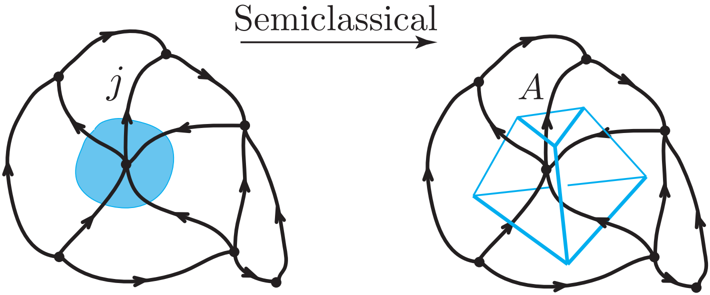
May 30, 2012

Volume

Polyhedral Volume (Bianchi, Doná and Speziale):

$$\hat{V}_{\text{Pol}} = \text{The volume of a quantum polyhedron}$$



Outline

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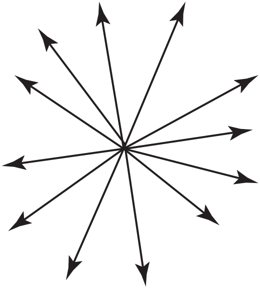
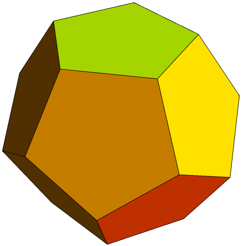
Minkowski's theorem: polyhedra

The area vectors of a convex polyhedron determine its shape:

$$\vec{A}_1 + \cdots + \vec{A}_n = 0.$$





Minkowski's theorem: polyhedra

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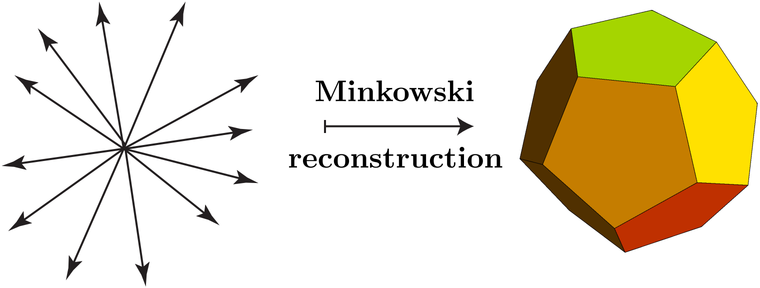
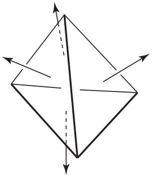


Only an existence and uniqueness theorem.

Minkowski's theorem: a tetrahdedron

Interpret the area vectors of tetrahedron as angular momenta:

$$\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 = 0 \quad \Longleftrightarrow$$



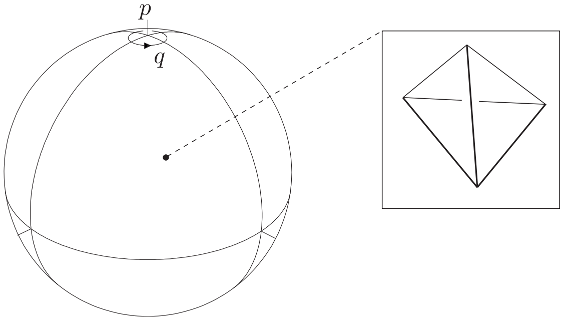
For fixed areas $A_1, \ldots, A_4$ each area vector lives in S^2.

Symplectic reduction of $(S^2)^4$ gives rise to the Poisson brackets:

$$\{f, g\} = \sum_{I=1}^{4} \vec{A}_I \cdot \left(\frac{\partial f}{\partial \vec{A}_I} \times \frac{\partial g}{\partial \vec{A}_I} \right)$$

Minkowski's theorem: a tetrahdedron

For fixed areas $A_1, \ldots, A_4$



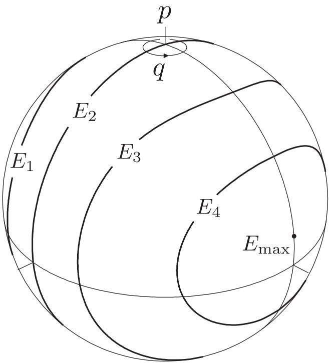
$p = |\vec{A}_1 + \vec{A}_2|$ $\quad$ $q =$ Angle of rotation generated by p:

$$\{q, p\} = 1$$

Dynamics

Take as Hamiltonian the Volume:

$$H = V^2 = \frac{2}{9}\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)$$



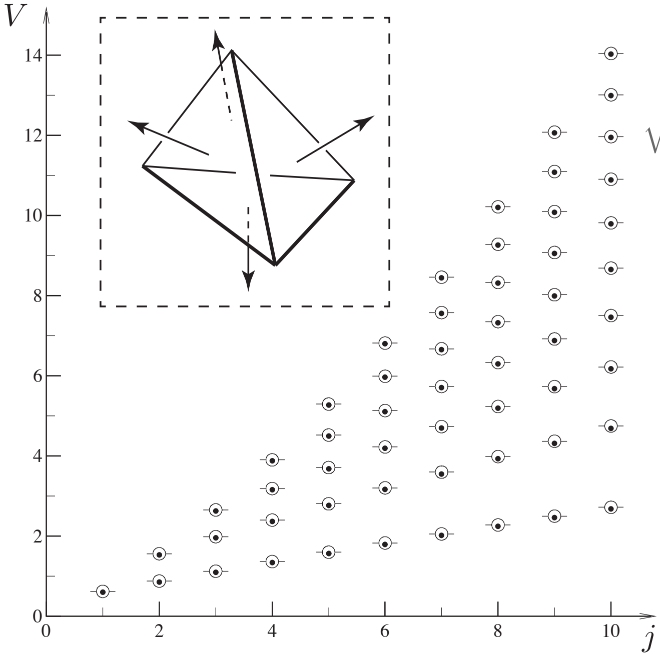
Bohr-Sommerfeld quantization

Require Bohr-Sommerfeld quantization condition,

$$S = \oint_{\gamma} pdq = (n + \frac{1}{2})2\pi\hbar.$$

Area of orbits given in terms of complete elliptic integrals,

$$S(E) = \left(\sum_{i=1}^{4} a_i K(m) + \sum_{i=1}^{4} b_i \Pi(\alpha_i^2, m) \right) E$$



$$V_{\text{Tet}} = \frac{\sqrt{2}}{3} \times \sqrt{|\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)|}$$

$$A_1 = j + 1/2$$
$$A_2 = j + 1/2$$
$$A_3 = j + 1/2$$
$$A_4 = j + 3/2$$

○ = Numerical

● = Bohr-Som

[PRL **107**, 011301]

Table

j_1 j_2 j_3 j_4	Loop gravity	Bohr-Sommerfeld	Accuracy
6 6 6 7	1.828	1.795	1.8%
	3.204	3.162	1.3%
	4.225	4.190	0.8%
	5.133	5.105	0.5%
	5.989	5.967	0.4%
	6.817	6.799	0.3%
$\frac{11}{2}$ $\frac{13}{2}$ $\frac{13}{2}$ $\frac{13}{2}$	1.828	1.795	1.8%
	3.204	3.162	1.3%
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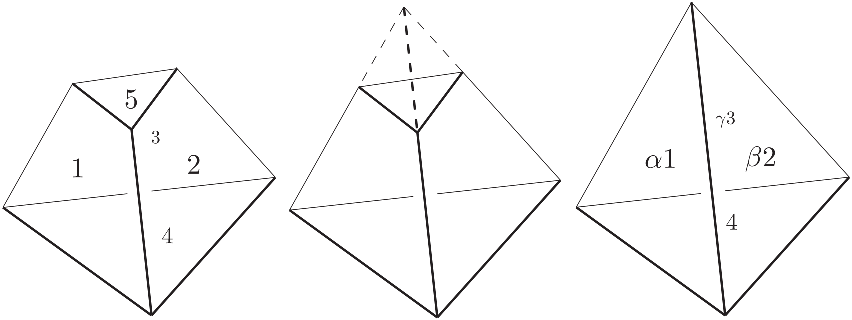
Volume of a pentahedron

A pentahedron can be completed to a tetrahedron



Volume of a pentahedron

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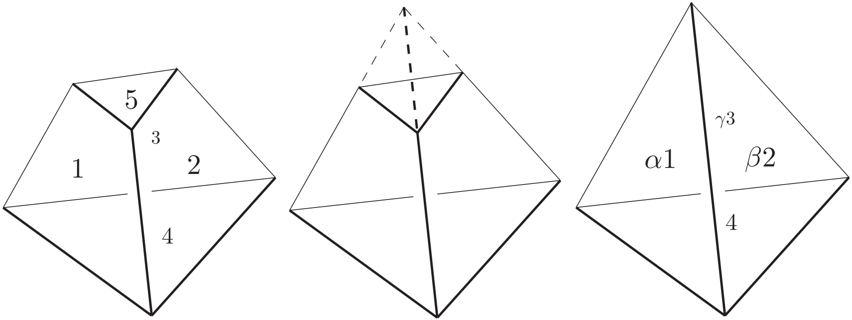

$\alpha, \beta, \gamma > 1$ found from,

$$\alpha\vec{A}_1 + \beta\vec{A}_2 + \gamma\vec{A}_3 + \vec{A}_4 = 0$$

$$\text{e.g.} \implies \alpha = -\vec{A}_4 \cdot (\vec{A}_2 \times \vec{A}_3)/\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)$$

Volume of a pentahedron

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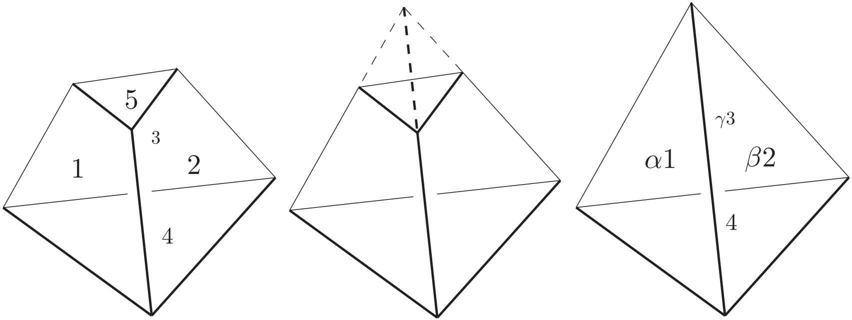
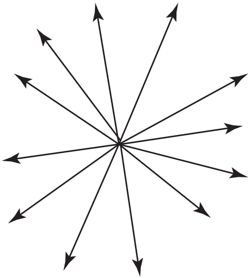


The volume of the prism is then,

$$V = \frac{\sqrt{2}}{3}\left(\sqrt{\alpha\beta\gamma} - \sqrt{(\alpha-1)(\beta-1)(\gamma-1)}\right)\sqrt{\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)}$$

Adjacency and reconstruction

What's most difficult about Minkowski reconstruction? Adjacency!



Remarkable side effect of introducing α, β and γ: they completely solve the adjacency problem!

Determining the adjacency

Let $W_{ijk} = \vec{A}_i \cdot (\vec{A}_j \times \vec{A}_k)$. Different closures imply,

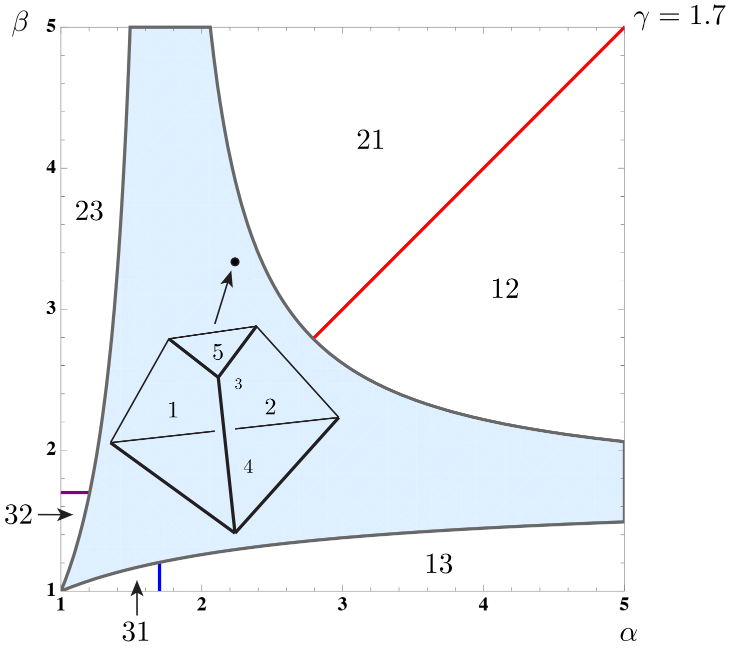
- $\alpha_1 \vec{A}_1 + \beta_1 \vec{A}_2 + \gamma_1 \vec{A}_3 + \vec{A}_4 = 0,$

$$\alpha \equiv \alpha_1 = -\frac{W_{234}}{W_{123}} \quad \beta \equiv \beta_1 = \frac{W_{134}}{W_{123}} \quad \gamma \equiv \gamma_1 = -\frac{W_{124}}{W_{123}}$$

- $\alpha_2 \vec{A}_1 + \beta_2 \vec{A}_2 + \vec{A}_3 + \gamma_2 \vec{A}_4 = 0,$

$$\alpha_2 = \frac{W_{234}}{W_{124}} = \frac{\alpha}{\gamma} \quad \beta_2 = -\frac{W_{134}}{W_{124}} = \frac{\beta}{\gamma} \quad \gamma_2 = -\frac{W_{123}}{W_{124}} = \frac{1}{\gamma}$$

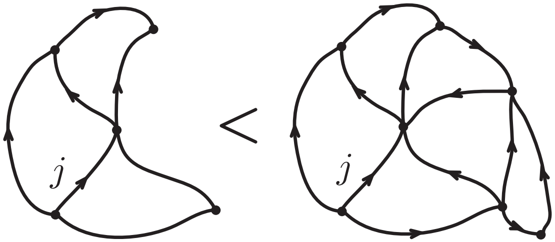
They are mutually incompatible!



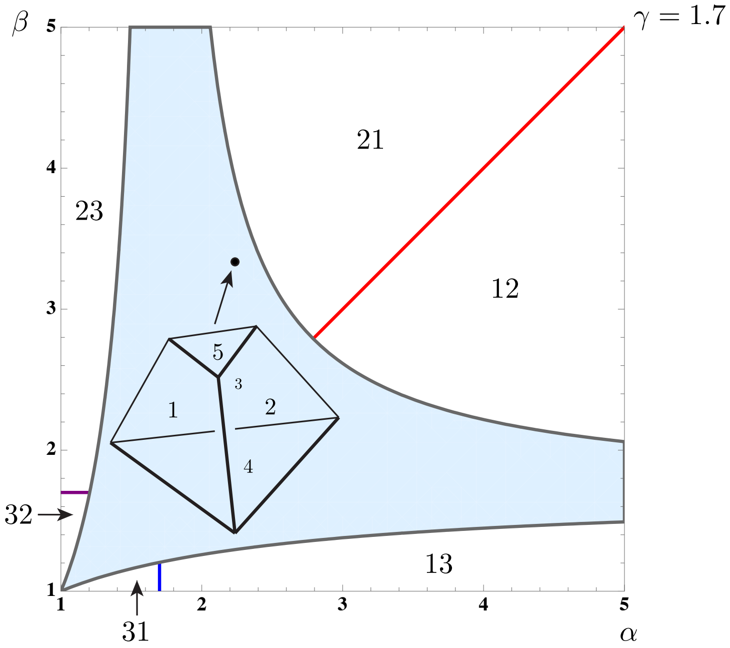
β
5
4
3
2
1
1
2
3
4
5
α
$\gamma = 1.7$
21
12
23
13
32
31
5
3
1
2
4

Cylindrical consistency

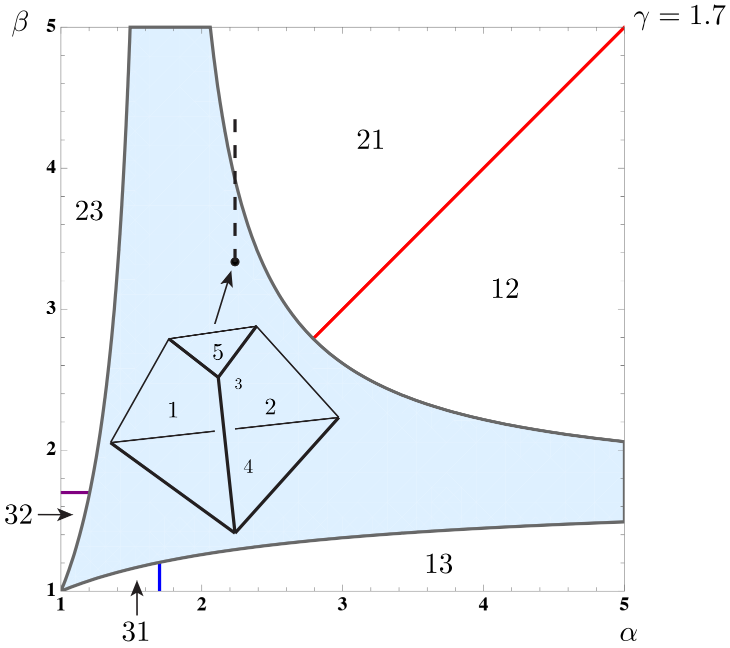
Smaller graphs and the associated observables can be consistently included into larger ones



Cylindrical consistency is non-trivially implemented for the polyhedral volume



β
γ = 1.7
5
4
3
2
1
1
2
3
4
5
α
21
23
12
13
32
31
1
2
3
4
5



β
γ = 1.7
5
4
3
2
1
21
23
12
5
3
1
2
4
32
13
31
1
2
3
4
5
α

Outline

EBK quantization

Sommerfeld and Epstein extended Bohr's condition, $L = n\hbar$, as we have seen

$$S = \int_0^T p \frac{dq}{dt} dt = nh$$

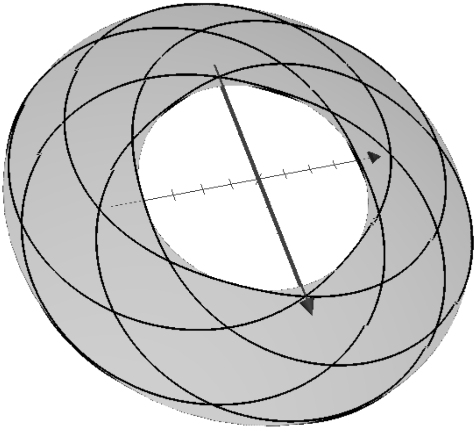
and applied it to bounded, separable systems with d degrees of freedom,

$$\int_0^{T_i} p_i \frac{dq_i}{dt} dt = n_i h, \qquad i = 1, \ldots, d$$

Here the T_i are the periods of each of the coordinates.

Einstein(!) was not satisfied. These conditions are not invariant under phase space changes of coordinates.

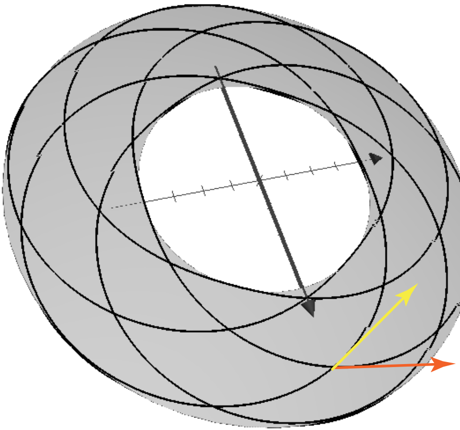
Motivating example: central force problems



In configuration space trajectories cross

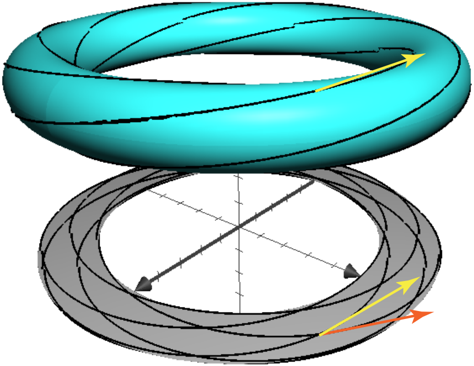
EBK quantization II

Motivating example: central force problems



Momenta are distinct at such a crossing

Motivating example: central force problems



In phase space the distinct momenta lift to the two sheets of a torus

EBK quantization III

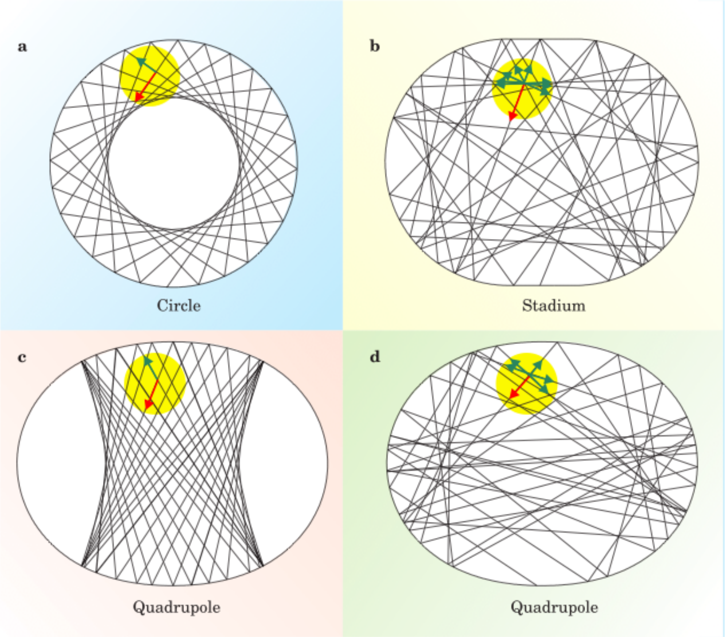
Following Poincaré, Einstein suggested that we use the invariant

$$\sum_{i=1}^{d} p_i dq_i$$

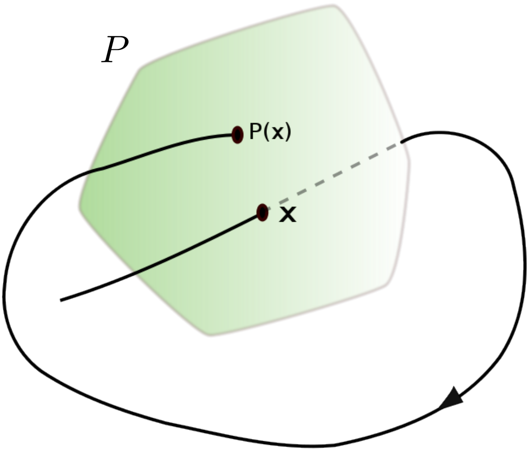
to perform the quantization.

The topology of the torus remains under coordinate changes, and so the quantization condition should be,

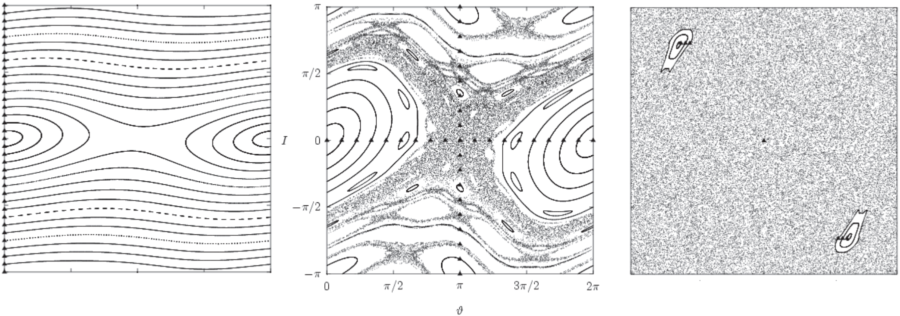
$$S_i = \oint_{C_i} \vec{p} \cdot d\vec{q} = n_i h.$$


a
Circle
b
Stadium
c
Quadrupole
d
Quadrupole

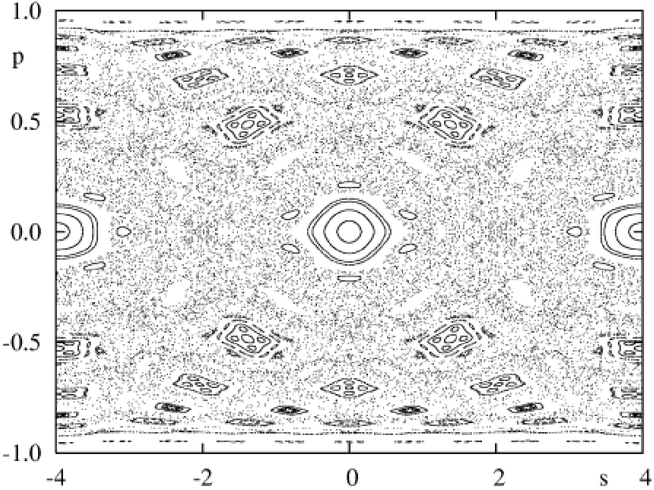
Visualizing dynamics with a surface of section



KAM: Weak perturbation of an integrable system $\to$ Break up of those tori foliated by trajectories with rational frequency ratios



KAM: Weak perturbation of an integrable system → Break up of those tori foliated by trajectories with rational frequency ratios

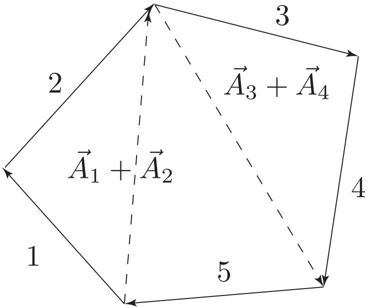


Toroidal Islands and island chains are left within a sea of chaos

Outline

Phase space of the pentahedron I

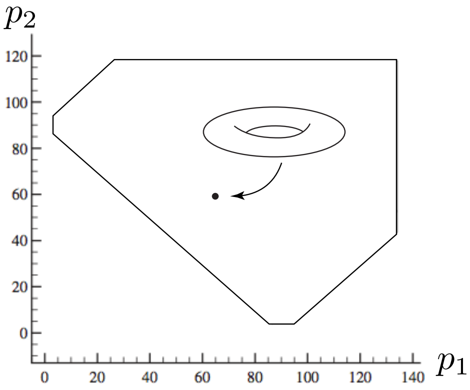
The pentahedron has two fundamental degrees of freedom,



The angles generated by $p_1 = |\vec{A}_1 + \vec{A}_2|$ and $p_2 = |\vec{A}_3 + \vec{A}_4|$.

Phase space of the pentahedron II

For fixed p_1 and p_2 these angles sweep out a torus.



The phase space consists of tori over a convex region of the $p_1 p_2$-plane.

Volume is nonlinear

The volume is a very nonlinear function of any of the variables we have considered:

$$V = \frac{\sqrt{2}}{3}\left(\sqrt{\alpha\beta\gamma} - \sqrt{(\alpha-1)(\beta-1)(\gamma-1)}\right)\sqrt{|\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)|}$$

Recall,

$$\alpha = -\frac{\vec{A}_4 \cdot (\vec{A}_2 \times \vec{A}_3)}{\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)}, \qquad \text{similarly for } \beta,\ \gamma$$

Forced to integrate it numerically.

Numerical integration

Fortunately, the angular momenta can be lifted into the phase space of a collection of harmonic oscillators. This allows the use of a geometric (i.e. symplectic) integrator.

Explicit Euler: $u_{n+1} = u_n + h \cdot a(u_n)$

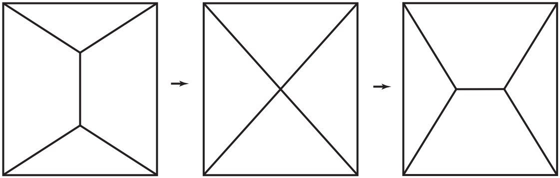
Implicit Euler: $u_{n+1} = u_n + h \cdot a(u_{n+1})$

Symplectic Euler:
$$\begin{aligned} u_{n+1} &= u_n + h \cdot a(u_n, v_{n+1}) \\ v_{n+1} &= v_n + h \cdot b(u_n, v_{n+1}) \end{aligned}$$

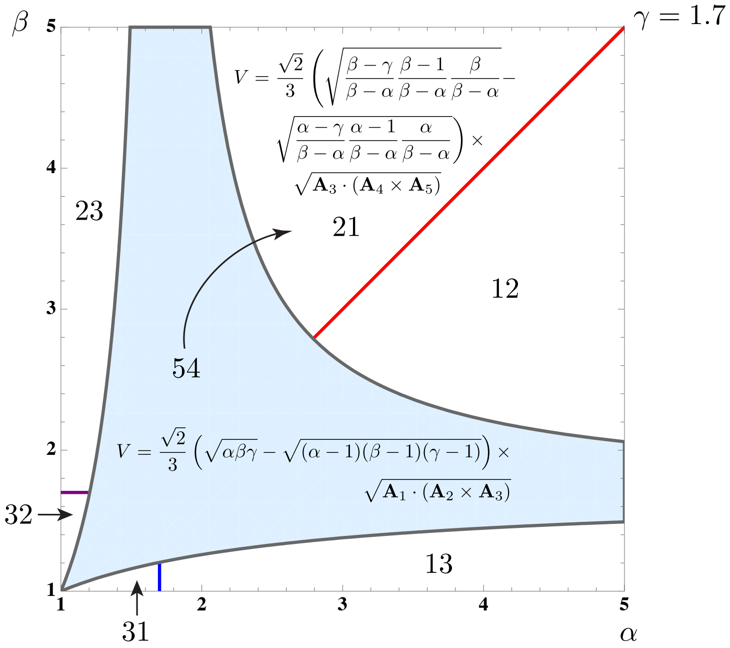
Implementation: Symplectic integrator preserves face areas to machine precision and volume varies in 14th digit

Volume dynamics: first results

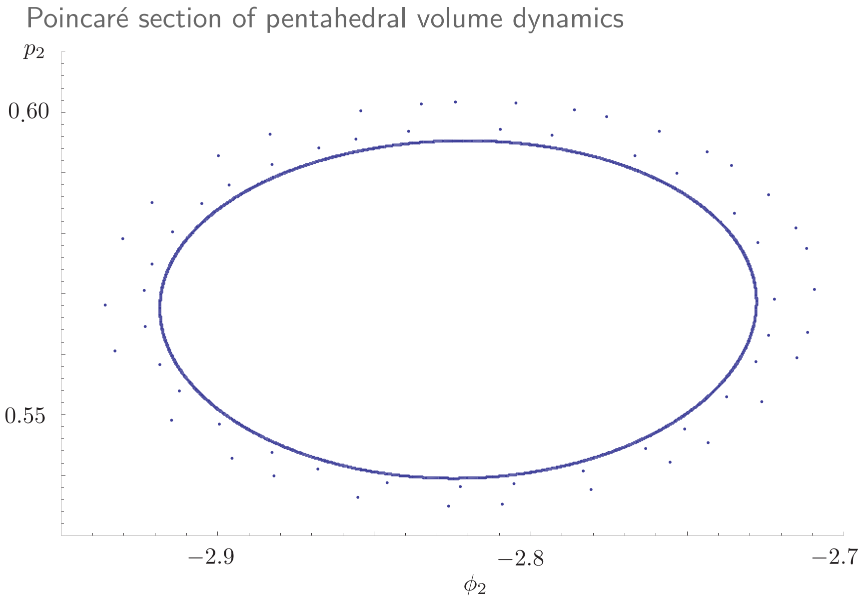
A Schlegel diagram projects a 3D polyhedron into one of its faces (left panel):



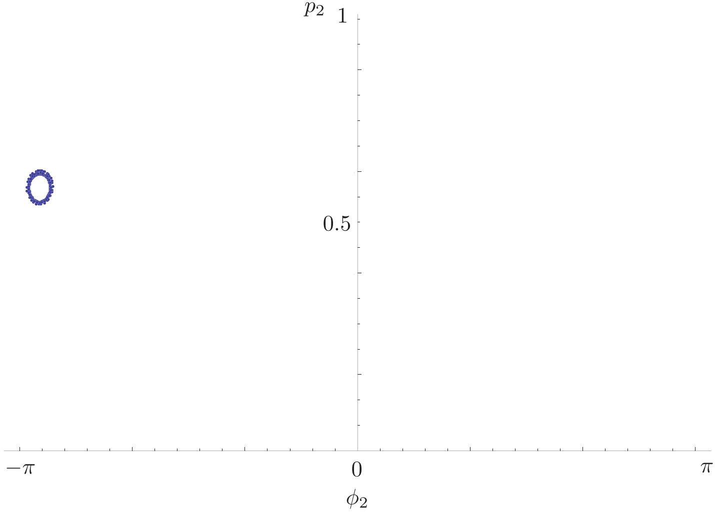
A Schlegel move merges two vertices of the diagram and and splits them apart in a different manner. This is precisely how the volume dynamics changes adjacency.



β
5
4
3
2
1
1
2
3
4
5
α
$\gamma = 1.7$
$V = \frac{\sqrt{2}}{3}\left(\sqrt{\frac{\beta-\gamma}{\beta-\alpha}\frac{\beta-1}{\beta-\alpha}\frac{\beta}{\beta-\alpha}} - \sqrt{\frac{\alpha-\gamma}{\beta-\alpha}\frac{\alpha-1}{\beta-\alpha}\frac{\alpha}{\beta-\alpha}}\right) \times \sqrt{\mathbf{A}_3 \cdot (\mathbf{A}_4 \times \mathbf{A}_5)}$
23
21
12
54
$V = \frac{\sqrt{2}}{3}\left(\sqrt{\alpha\beta\gamma} - \sqrt{(\alpha-1)(\beta-1)(\gamma-1)}\right) \times \sqrt{\mathbf{A}_1 \cdot (\mathbf{A}_2 \times \mathbf{A}_3)}$
32
13
31


Poincaré section of pentahedral volume dynamics
p_2
0.60
0.55
−2.9
−2.8
−2.7
ϕ_2

Guess: Analogy with billiards systems suggests that the dynamics will be mixed, containing chaos



Conclusions

- Minkowski reconstruction for 5 vectors solved
- There is cylindrical consistency in the polyhedral picture and it is non-trivial
- The classical polyhedral volume is only twice continuously differentiable
- Can explore the classical dynamics of the volume operator in the case of a polyhedron with 5 faces
- Does this dynamics exhibit chaos?