Finite regions, spherical entanglement, and quantum gravity

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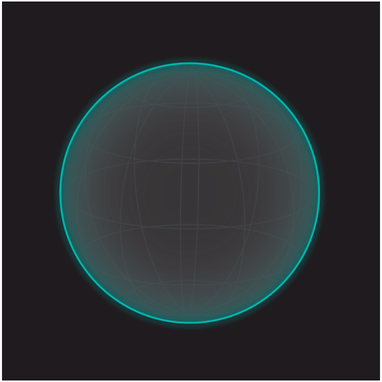
CENTRE DE PHYSIQUE THÉORIQUE MARSEILLE
NSF

Finite region quantum gravity

What is the physics of a finite region of a quantum spacetime?

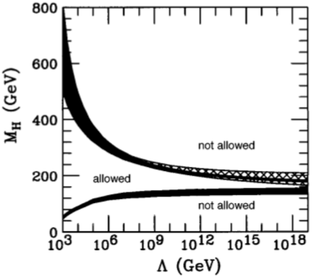


Divergences

"...no approach to quantum gravity can claim complete success that does not explain in full and convincing detail the ultimate fate of the divergences of perturbative quantum gravity." H. Nicolai

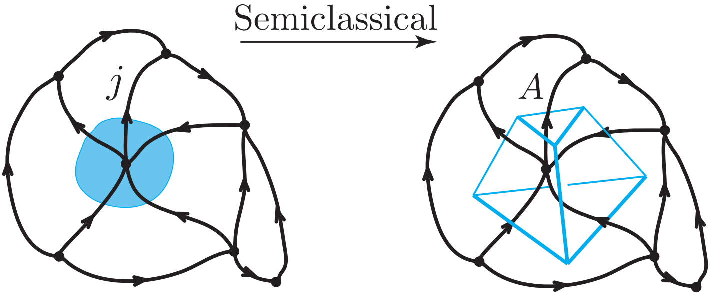
gr-qc/1301.5481

The mass of the LHC scalar boson (125 GeV) narrowly avoids vacuum instability and the Landau pole of self-couplings:



Currently understood physics may hold up to the Planck scale!

Polyhedral program



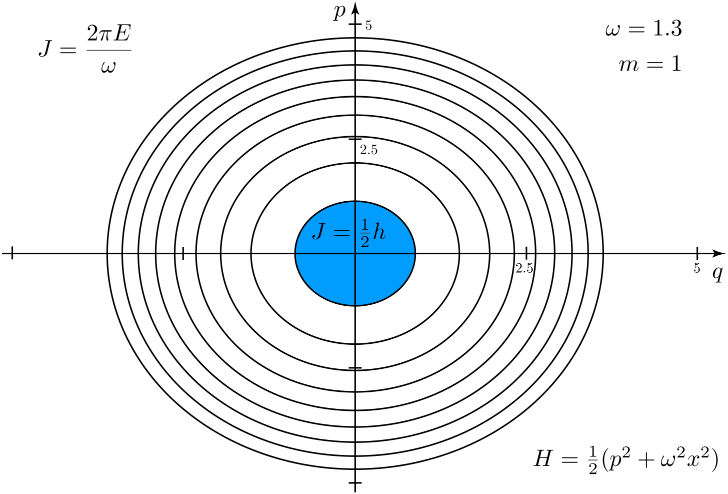
$E_i = \int \epsilon_{ijk} e^i_b e^k_c dx^b dx^c = 8\pi \ell_{\mathsf{Pl}}^2 J_i \qquad \longrightarrow \qquad \vec{A}$

- Gauss constraint $\longrightarrow$ Closure relation

$\hat{V}_{\mathsf{Pol}} =$ The volume of a quantum polyhedron

[Bianchi, Doná and Speziale]

Bohr-Sommerfeld Quantization of Harmonic Osc.

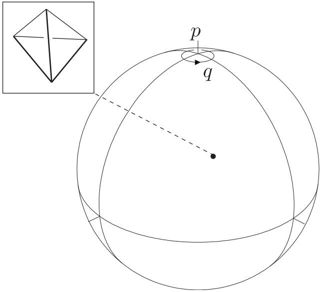


Require:

$$J = \oint_\gamma p dq = (n + \frac{1}{2})2\pi\hbar.$$

Tetrahedra

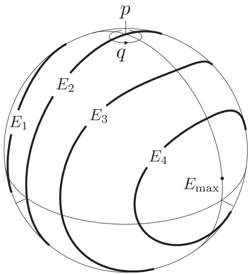
Phase space (fixed $A_1, \ldots, A_4$):



$p = |\vec{A}_1 + \vec{A}_2|, \quad q = \text{ang. gen. } p$
$\{q, p\} = 1$

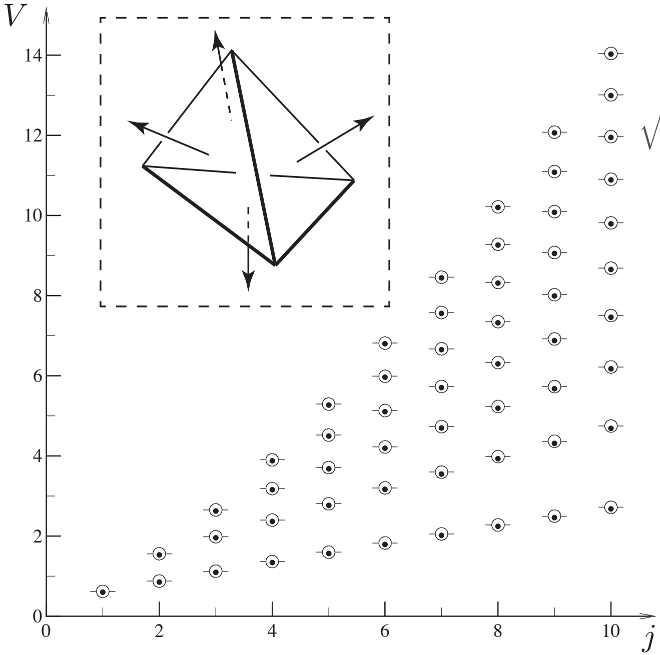
Take volume as Hamiltonian:

$$H = V^2 = \frac{2}{9} \vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)$$



Orbit areas are complete elliptic integrals,

$$J(E) = \left(\sum_{i=1}^{4} a_i K(m) + \sum_{i=1}^{4} b_i \Pi(\alpha_i^2, m) \right) E$$



$$V_{\mathsf{Tet}} = \frac{\sqrt{2}}{3} \times \sqrt{|\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)|}$$

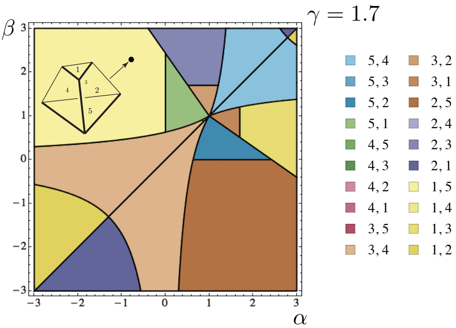
$$A_1 = j + 1/2$$
$$A_2 = j + 1/2$$
$$A_3 = j + 1/2$$
$$A_4 = j + 3/2$$

$\circ$ = Numerical

$\bullet$ = Bohr-Som

[PRL **107**, 011301]

Volume gap: pentahedra



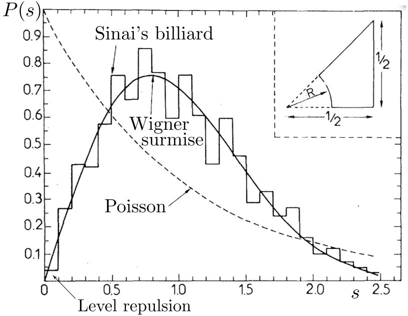
Analytic volume:

$$V_{\text{Pent}} = \frac{\sqrt{2}}{3}\left(\sqrt{\alpha\beta\gamma} - \sqrt{\bar{\alpha}\bar{\beta}\bar{\gamma}}\right) \times \sqrt{|\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)|},$$

$$\alpha \equiv \vec{A}_4 \cdot (\vec{A}_3 \times \vec{A}_2) / \vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)$$

etc.

- Robust mechanism $\rightsquigarrow$ volume gap: chaotic dynamics & low density of states at low volume

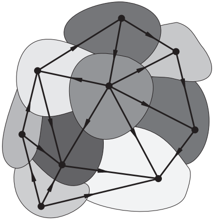


Outstanding challenge

How to identify the ground state of a general relativistic theory?

Want to coordinate individual grains of space to recover Minkowski space from this quantum theory.

- Use entanglement



Outline

Entanglement insights

Tensor networks:

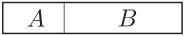
- Condensed matter and tensor networks: Matrix Product States (MPS), Projected Entangled Pair States (PEPS), Tensor networks (TNs) $\rightsquigarrow$ Area laws.
- Physical corner of the Hilbert space.
- Spin networks are tensor networks.

Casini, Huerta & Myers have extensively studied the entanglement entropy of spherical causal domains. [Casini, Huerta & Myers '11]

♦ New result $\rightsquigarrow$ the entanglement eigensystem.

Entanglement

Pure state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.



$\mathcal{H}_A \otimes \mathcal{H}_B$

Schmidt decomposition:

$$|\Psi\rangle = \sum_i \sqrt{\lambda_i} |i_A\rangle \otimes |i_B\rangle$$

with $|i_A\rangle$ and $|i_B\rangle$ orthonormal bases in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

Leads to the reduced density matrix

$$\begin{aligned} \rho_B &= \mathrm{Tr}_A |\Psi\rangle\langle\Psi| \\ &= \sum_i \lambda_i |i_B\rangle\langle i_B|, \end{aligned}$$

and the entanglement entropy

$$S_E \equiv -\mathrm{Tr}\, \rho_B \log \rho_B = -\sum_i \lambda_i \log \lambda_i.$$

Entanglement spectrum

Can always write

$$\rho_B = e^{-H_E}, \quad \text{i.e.} \quad H_E \equiv -\log \rho_B,$$

the "entanglement Hamiltonian".

Already diagonalized H_E:

$$\rho_B = \sum_i e^{-\epsilon_i} |i_B\rangle\langle i_B|$$

with ϵ_i $(i = 1, 2, \dots)$ the "entanglement spectrum". $(\lambda_i = e^{-\epsilon_i})$

[Li & Haldane '08]

Provides thorough understanding of entanglement.

- Study for spacetime fields.

Boundary conditions for the remainder

For simplicity I restrict to:

Scalar field $\varphi(x)$,

with $m = 0$

on flat $D = 3 + 1$ spacetime

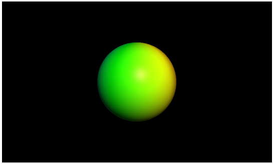
unless otherwise stated. Metric signature $(-, +, +, +)$.

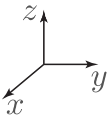
♦ Results apply to any conformal field theory.

Outline

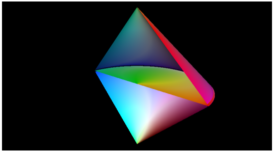
Spherical causal domain

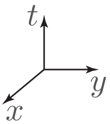
Cauchy development of 3-ball with boundary 2-sphere of radius R:





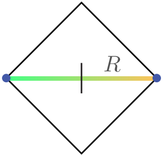
Spatial 3-ball B $\rightsquigarrow$ Cauchy development D(B).

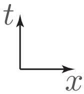


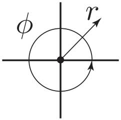


Entangling surface the boundary 2-sphere $\partial B = S^2$.

Choose adapted coordinates that preserve S^2: similar to how polar coords fix $(0, 0)$...

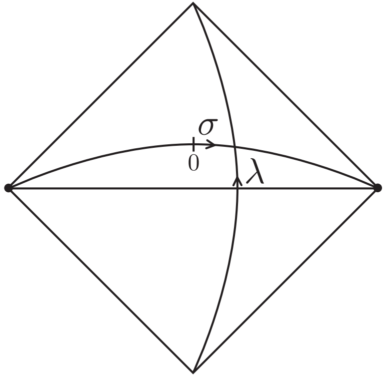






Diamond coordinates

Use hyperbolas



Diamond coords $(\lambda, \sigma, \theta, \phi)$:

$$t = R \frac{\mathsf{sh}\,\lambda}{\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma}$$

$$r = R \frac{\mathsf{sh}\,\sigma}{\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma}$$

with $\lambda \in (-\infty, \infty)$, $\sigma \in [0, \infty)$.

The Minkowski metric becomes

$$ds^2 = \frac{R^2}{(\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma)^2}[-d\lambda^2 + d\sigma^2 + \mathsf{sh}^2\sigma\, d\Omega^2],$$

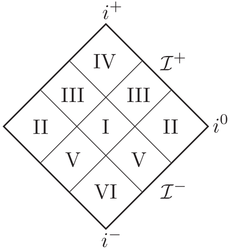
a conformal rescaling of static $\kappa = -1$ FRW.

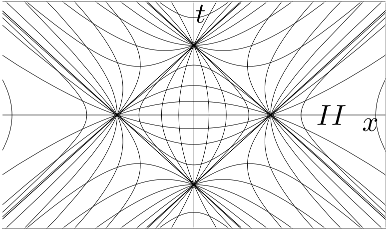
Conformal completion

Diamond coordinates can be extended to all of Minkowski space

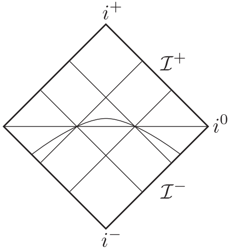
E.g. region II:

$$\left.\begin{aligned} t &= R\frac{\operatorname{sh}\tilde\lambda}{\operatorname{ch}\tilde\sigma - \operatorname{ch}\tilde\lambda}, \\ r &= R\frac{\operatorname{sh}\tilde\sigma}{\operatorname{ch}\tilde\sigma - \operatorname{ch}\tilde\lambda} \end{aligned}\right\} \qquad |\tilde\lambda| \le \tilde\sigma.$$





All hyperbolas asymp. null:

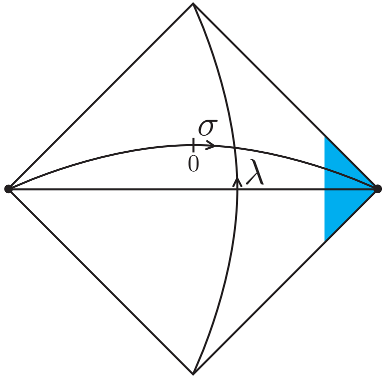


Rindler Limit

Large σ limit:

$$t = 2Re^{-\sigma}\,\mathrm{sh}\,\lambda = \ell\,\mathrm{sh}\,\lambda$$
$$R - r = 2Re^{-\sigma}\,\mathrm{ch}\,\lambda = \ell\,\mathrm{ch}\,\lambda$$

coord transformation to (left) Rindler wedge.



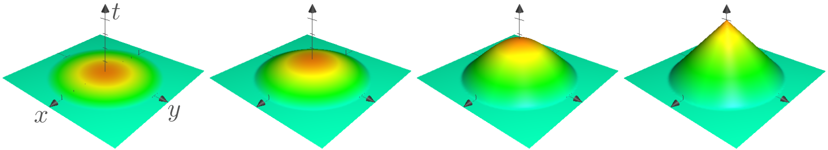
The proper distance from right corner is $\ell = 2Re^{-\sigma}$.

Entanglement (or foliation) Hamiltonian

$\xi^\mu = \left(\frac{\partial}{\partial\lambda}\right)^\mu \rightsquigarrow$ current $J^\mu = T^{\mu\nu}\xi_\nu$ with $T_{\mu\nu}$ the stress-tensor. If $T^\mu{}_\mu = 0$ the charge is conserved

$$C = \int T_{\mu\nu}\xi^\mu d\Sigma^\nu$$

and generates the spatial foliation discussed above:

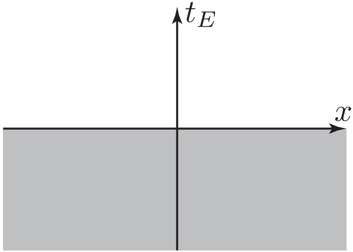


Explicitly this charge is,

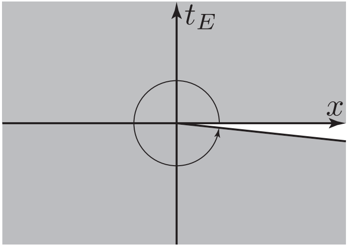
$$C_{in} = \frac{1}{2R}\int_B r^2 dr\, d\tilde{\Omega}\left(\frac{1}{2}(R^2 - r^2)(\dot{\varphi}^2 + \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi) + \varphi^2\right)$$

Density matrix from Euclidean path integral

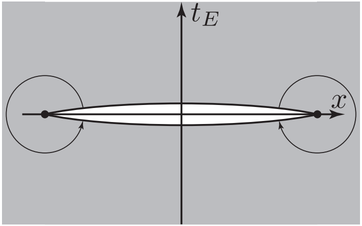
Minkowski vacuum:



Rindler density matrix:



Spherical density matrix:



$$\left.\begin{aligned} t_E &= R\frac{\sin\lambda_E}{\cos\lambda_E + \mathrm{ch}\,\sigma} \\ r &= R\frac{\mathrm{sh}\,\sigma}{\cos\lambda_E + \mathrm{ch}\,\sigma} \end{aligned}\right\}\ \text{Bipolar coords}$$

$$\boxed{\rho_B = \int \mathcal{D}\varphi\, e^{-S_E} = e^{-2\pi C_{in}}}$$

Curved background QFT

Inverted strategy: study finite region of flat spacetime using QFT on a curved background

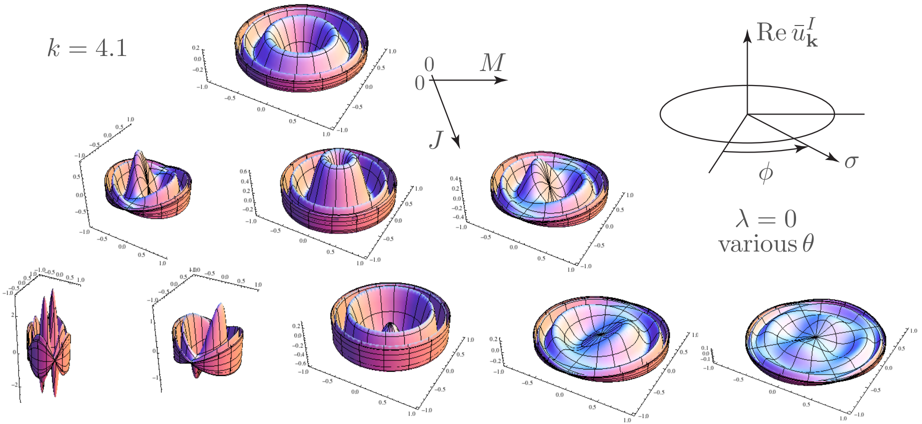
$$\mathcal{L} = \frac{1}{2}\sqrt{-g}[g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + (\cancelto{0}{m^2} + \frac{1}{6}\mathcal{R}(x))\varphi^2].$$

Action is conformally invariant under $\begin{cases} \bar{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu} \\ \bar{\varphi} = \Omega(x)^{-1}\varphi \end{cases}$

With $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ the EOM transform as ($\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$)

$$\bar{\Box}\bar{\varphi} = \Omega^{-3}[\Box - \frac{1}{6}\mathcal{R}]\varphi.$$

Find $\bar{\varphi}$ by finding φ.



Sphere modes:

$$\bar{u}^I_{\mathbf{k}}(x) = \frac{(2k)^{-\frac{1}{2}}}{R}(\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma)\Pi^-_{kJ}(\sigma)\,Y^M_J(\theta,\phi)\,e^{-ik\lambda}$$

Sphere modes analytically

Metric $g = -d\lambda^2 + d\sigma^2 + \mathsf{sh}^2\sigma\, d\tilde{\Omega}^2$ is static $\kappa = -1$ FRW.

Separate: $u_{\mathbf{k}}(x) = \chi_k(\lambda)\Pi^-_{kJ}(\sigma)\, Y^M_J(\theta,\phi)$ with

$$\chi_k(\lambda) = (2k)^{-\frac{1}{2}}\, e^{-ik\lambda}$$

$$\Pi^-_{kJ}(\sigma) = N(k,J)\, \mathsf{sh}^J\sigma \left(\frac{d}{d\,\mathsf{ch}\,\sigma}\right)^{1+J} \cos(k\sigma)$$

$$Y^M_J(\theta,\phi) \qquad \text{spherical harmonics}$$

$$M = -J, -J+1, \ldots, J; \qquad J = 0, 1, \ldots; \qquad 0 < k < \infty.$$

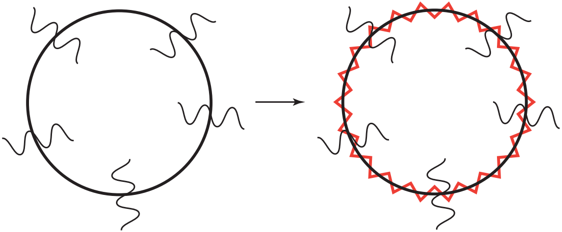
Sphere modes: $\bar{\varphi} = \Omega(x)^{-1}\varphi$, (recall $\Omega = R/(\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma)$)

$$\bar{u}^I_{\mathbf{k}}(x) = \frac{(2k)^{-\frac{1}{2}}}{R}(\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma)\Pi^-_{kJ}(\sigma)\, Y^M_J(\theta,\phi)\, e^{-ik\lambda}.$$

Sphere vacuum and spherons

Minkowski case: $\varphi(x) = \int d^{D-1}k[a_{\mathbf{k}}u_{\mathbf{k}}(x) + a_{\mathbf{k}}^{\dagger}u_{\mathbf{k}}^{*}(x)]$
The vacuum satisfies $a_{\mathbf{k}}|0\rangle_M = 0$.

Cut out all in-out entanglement to get sphere vacuum



Sphere case: $\varphi(x) = \int dk \sum_{J,M}[s_{\mathbf{k}}^{I}u_{\mathbf{k}}^{I} + s_{\mathbf{k}}^{I\dagger}u_{\mathbf{k}}^{I*} + s_{\mathbf{k}}^{II}u_{\mathbf{k}}^{II} + s_{\mathbf{k}}^{II\dagger}u_{\mathbf{k}}^{II*}]$
The sphere vacuum satisfies $s_{\mathbf{k}}^{I}|0\rangle_S = s_{\mathbf{k}}^{II}|0\rangle_S = 0$.

- $s_{\mathbf{k}}^{I\dagger}$ creates *spherons*, excitations localized within the spherical entangling surface.

Spherical entanglement spectrum

Curved space, traceless (improved) stress tensor

$$T_{\mu\nu} = \frac{2}{3}\nabla_\mu\bar{\varphi}\nabla_\nu\bar{\varphi} - \frac{1}{6}g_{\mu\nu}(\nabla^\rho\bar{\varphi}\nabla_\rho\bar{\varphi}) - \frac{1}{3}\bar{\varphi}\nabla_\mu\nabla_\nu\bar{\varphi} + \frac{1}{3}g_{\mu\nu}\bar{\varphi}\Box\bar{\varphi} + \frac{1}{6}[\mathcal{R}_{ab} - \frac{1}{2}g_{ab}\mathcal{R}]\bar{\varphi}^2.$$

Entanglement Hamiltonian

$$C_{in} = \int_B T_{\mu\nu}\xi^\mu d\Sigma^\nu.$$

Entanglement spectrum

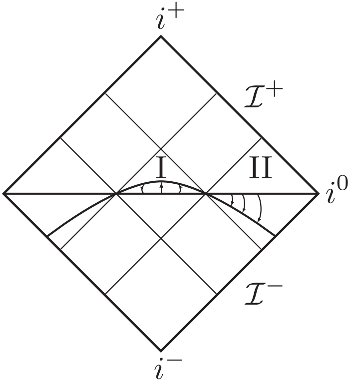
$$C_{in} = \int dk \sum_{J,M} k(s^{I\dagger}_{\mathbf{k}} s^I_{\mathbf{k}} - \cancelto{\text{zero pt. ener.}}{\frac{1}{2}[s^{I\dagger}_{\mathbf{k}}, s^I_{\mathbf{k}}]}).$$

Schmidt decomposition of Minkowski vacuum

Total Hamiltonian

$$C = C_{in} - C_{out}$$

$$C = \int dk \sum_{J,M} k(s_{\mathbf{k}}^{I\dagger} s_{\mathbf{k}}^{I} + s_{\mathbf{k}}^{II\dagger} s_{\mathbf{k}}^{II})$$



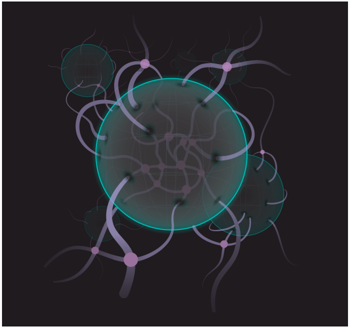
$$\boxed{|0\rangle_M = \int dk \sum_{J,M} e^{-\epsilon_k} u_{\mathbf{k}}^{I} \otimes u_{\mathbf{k}}^{II}.}$$

Outline

Spin network entanglement

This suggests an intriguing perspective on spin networks.

Individual nodes are like the interior of the sphere with no entanglement between a given node and its neighbors.



Can we choreograph entanglement to yield the Minkowski vacuum?

♦ A wealth of condensed matter research on entanglement and tensor networks to draw from.

Enticing direction

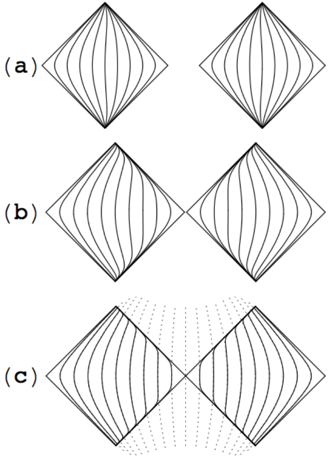
Clearly we want more nodes. The entanglement Hamiltonian for multiple diamonds in Minkowski space studied by [Casini & Huerta].

Results for fermion in $D = 1 + 1$ spacetime:

$$H = H_{\text{loc}} + H_{\text{nonloc}}.$$

Two regions of radius R separated by distances (a) $\frac{1}{8}R$, (b) $\frac{1}{140}R$, and (c) $\frac{1}{2000}R$.

What is the physical significance of the non-local mixing?



Two-point functions

Leveraging conformal symmetry we can achieve a more substantial characterization of the sphere vacuum, finding its two-point functions:

$$D^{+}(x,x') = \Omega^{\frac{D-2}{2}}(x)\tilde{D}^{+}(x,x')\Omega^{\frac{D-2}{2}}(x')$$

- 1+1 spacetime:

$$D^{+}(x,x') = -\frac{1}{4\pi}\log|-\Delta\lambda^{2}+\Delta\sigma^{2}|$$

- 3+1 spacetime:

$$D^{+}(x,x') = \frac{(\mathrm{ch}\,\lambda+\mathrm{ch}\,\sigma)\Delta\sigma(\mathrm{ch}\,\lambda'+\mathrm{ch}\,\sigma')}{4\pi^{2}R^{2}\,\mathrm{sh}\,(\Delta\sigma)\left[-\Delta\lambda^{2}+\Delta\sigma^{2}\right]}$$

Sorkin-Johnston vacuum

Causal sets provide the only fully controlled Lorentz invariant discretization of spacetime.

A remarkable state, the Sorkin-Johnston vacuum:

[Johnston '09, Sorkin '11]

$\rightsquigarrow$ proposed vacuum for a causal set within a causal diamond. It

- is defined only through referencing the diamond's interior
- has no entanglement with outside
- and so does not satisfy the Reeh-Schlieder theorem.

Afshordi et al have just investigated the Sorkin-Johnston vacuum in 2D analytically and numerically... [Afshordi et al '12]

Comparison

... and amongst many other things they found a surprise: near the L and R diamond corners the Sorkin-Johnston two-point function is that of a static mirror in Minkowski spacetime.

- How does the sphere vacuum compare?

In the limit $\lambda = \lambda' = 0$, $\sigma, \sigma' \to \infty$:

$$D^+_{2D} \sim -\frac{1}{4\pi} \log\left(\log \frac{\ell}{\ell'}\right) \qquad D^+_{4D} \sim \frac{1}{2\pi(\ell^2 - \ell'^2) \log \frac{\ell}{\ell'}}$$

The Rindler two-point functions. [Troost & Van Dam '79]
(Recall $\ell = 2Re^{-\sigma}$ is the proper distance from the right corner.)

$\implies$ the sphere and Sorkin-Johnston vacuums differ in 2D.
Is this still true in 4D?

♦ Have the tools to completely characterize these differences.

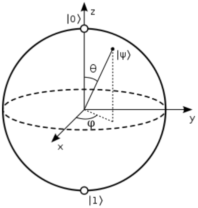
Reeh-Schlieder theorem

If quantum gravity cuts off the continuum what becomes of the Reeh-Schlieder theorem?

A special, initial pure state of two q-bits,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)$$

can be used to steer one of the q-bits onto whole bloch sphere.



More generally:

A	B

$$\mathcal{H}_A \otimes \mathcal{H}_B$$

Schmidt rank of the decomp.

$$|\Psi\rangle = \sum_i \sqrt{\lambda_i} |i_A\rangle \otimes |i_B\rangle$$

is at most dim$\mathcal{H}_A$. There is no way to steer onto all of $\mathcal{H}_B$.

In what sense, if any, does the Reeh-Schlieder theorem hold for a large but finite #d.o.f.?

Conclusions

- Looking to engineer the Minkowski vacuum and its entanglement from spin network superposition.
- Interesting connections to causal sets. What is the nature of the Sorkin-Johnston vacuum in 4D?
- The fate of the Reeh-Schlieder theorem for a large but finite number of degrees of freedom needs to be understood.

Numerous possibilities

- More general entangling surfaces
- Continue to draw from condensed matter
- Anomalies

Credits

Spherical spin network: Z. Merali, “The origins of space and time,” Nature News, Aug. 28, 2013

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