Finite regions, spherical entanglement, and quantum gravity

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in collaboration with Eugenio Bianchi

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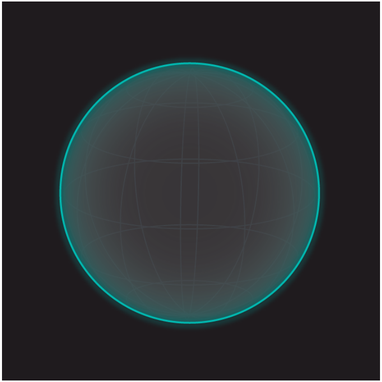
Condensed Matter Theory Seminar
University of Regensburg

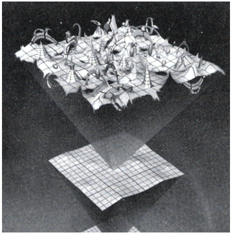
Finite region quantum gravity

What is the physics of a finite region of a quantum spacetime?

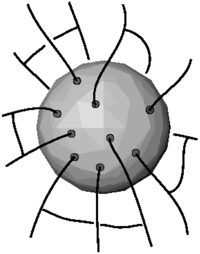


Finite regions can incorporate diffeos of GR into a quantum context $\rightsquigarrow$ avoid notorious difficulties, e.g. [Arkani-Hamed et al].

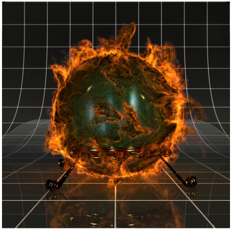
Entanglement provides a vibrant tool for quantum gravity



Entanglement correlations can distinguish vacua, ...



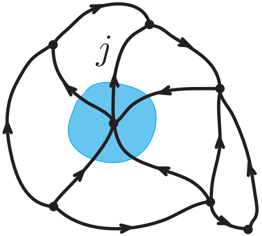
... provide a compelling calculation of black hole entropy, ...

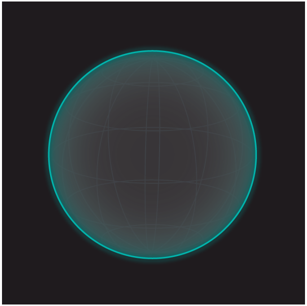


... and offer insight into spacetime thermodynamics.

What is the detailed entanglement of a finite region of spacetime?

Spin networks describe finite spatial regions $\rightsquigarrow$ their entanglement can provide consistency checks and design principles





Casini, Huerta & Myers have extensively studied the entanglement entropy of spherical causal domains.

[Casini, Huerta & Myers '11]

- New result $\rightsquigarrow$ the entanglement spectrum (or eigensystem).

Outline

Introduction to loop gravity and the discreteness of space

Entanglement and the entanglement spectrum

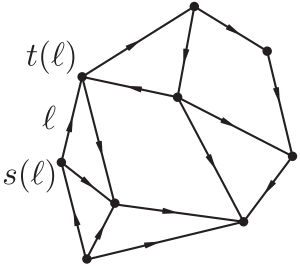
The entanglement spectrum of a sphere

Interest for quantum gravity

The Hilbert space of loop gravity is similar to Fock space of QED and to lattice gauge theory (e.g. QCD):

$$\mathcal{H}.$$

Built on graphs:



Graph Γ

L "links" ℓ
N "nodes" n

source and target:
$s : \ell \mapsto s(\ell)$ and $t : \ell \mapsto t(\ell)$

Fock Space

Massive scalar field:

- One particle: $\mathcal{H}_1 = L^2(M)$, M the Lorentz hyperboloid.
- n particles,

$$\mathcal{H}_n = L^2(M^n)/\sim$$

  with $\sim$ permutations. Factorization symmetrizes states.
- All states up to N particles

$$\mathcal{H}_N = \bigoplus_{n=0}^{N} \mathcal{H}_n.$$

  Fock space

$$\mathcal{H}_{\mathsf{Fock}} = \lim_{N\to\infty} \mathcal{H}_N.$$

Lattice Gauge Theory

Lattice Γ with L links ℓ, N nodes n and gauge group G

$$\tilde{\mathcal{H}}_\Gamma = L^2(G^L).$$

States $\psi(h_\ell) \in \tilde{\mathcal{H}}_\Gamma$ acted on by gauge transformations

$$\psi(h_\ell) \to \psi(g_{s(\ell)}\, h_\ell\, g_{t(\ell)}^{-1}), \qquad g_n \in G.$$

Gauge invariant Hilbert space is

$$\mathcal{H}_\Gamma = L^2(G^L/G^N).$$

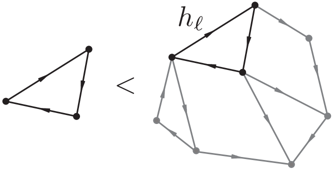
Loop Quantum Gravity

General graph Γ, called a spin network,

$$\tilde{\mathcal{H}}_\Gamma = L^2(SU(2)^L / SU(2)^N),$$

an $SU(2)$ lattice gauge theory.

If $\Gamma' \leq \Gamma$ then $\tilde{\mathcal{H}}_{\Gamma'} \subset \tilde{\mathcal{H}}_\Gamma$.



$$\mathcal{H}_\Gamma = \tilde{\mathcal{H}}_\Gamma / \sim$$

$$\mathcal{H} = \lim_{\Gamma \to \infty} \mathcal{H}_\Gamma$$

Loop Quantum Gravity

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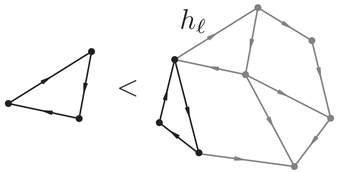

$$\mathcal{H}_\Gamma = \tilde{\mathcal{H}}_\Gamma / \sim$$

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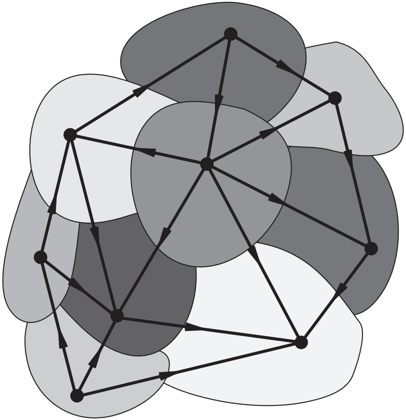
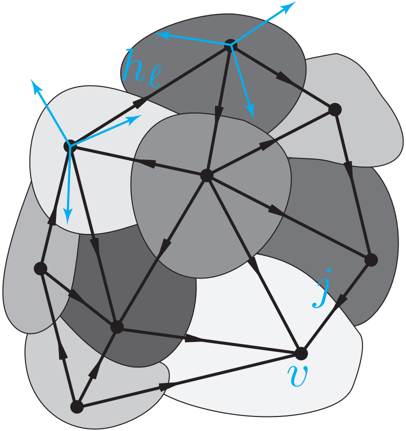
Physical Picture



Quanta of gravity are “grains” or “chunks” of space

Physical Picture

Gauge freedom is the freedom to reorient frame throughout space

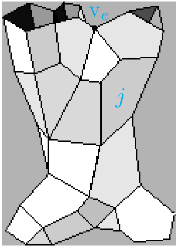


Geometry captured through volume v of nodes and area j of links

Can introduce dynamics with a discrete path integral or 'spin foam'

$$Z = \sum_{j_f, \mathsf{v}_e} \prod_f (2j_f + 1) \prod_v A_v(j_j, \mathsf{v}_e)$$

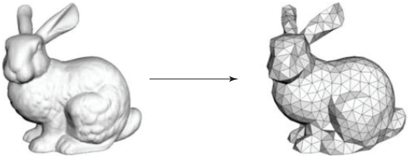
$$A_v(j_f, \mathsf{v}_e) = \mathsf{Tr}[\otimes_{e \in v} f_\gamma(\mathsf{v}_e)]$$



Remarkably,

$$\boxed{A_v \sim e^{iS_{\mathsf{Regge}}}}$$

The Regge action discretizes the Einstein-Hilbert action

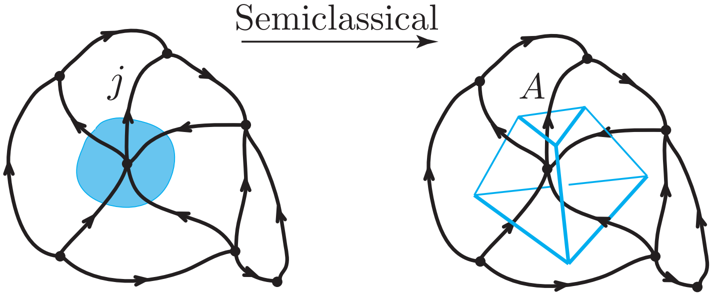


$$S_{\mathrm{EH}} \longrightarrow S_{\mathrm{Regge}}$$

Volume

Polyhedral Volume: [Bianchi, Doná and Speziale]

Gravitational Gauß law: sum of fluxes at node vanishes, $\sum_i \hat{\vec{J}}_i = 0$

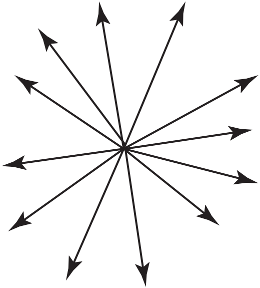


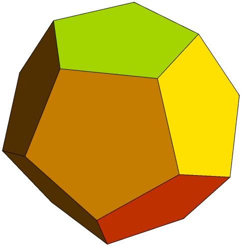
$\hat{V}_{\text{Pol}} =$ The volume of a quantum polyhedron

Minkowski's theorem: polyhedra

The area vectors of a convex polyhedron determine its shape:

$$\vec{A}_1 + \cdots + \vec{A}_n = 0.$$





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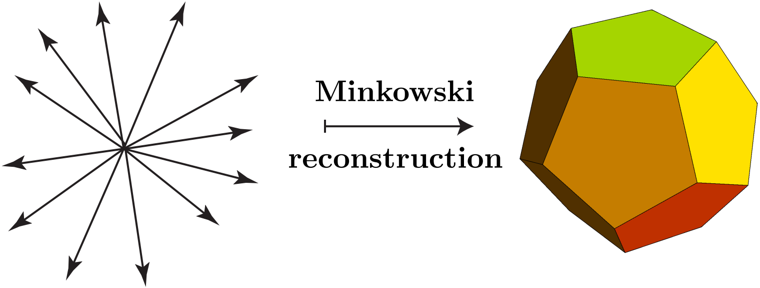
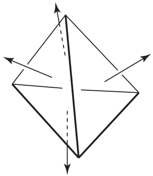


Only an existence and uniqueness theorem.

Polyhedra can be turned into dynamical systems

Interpret the area vectors as angular momenta:

$$\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 = 0 \quad \Longleftrightarrow$$



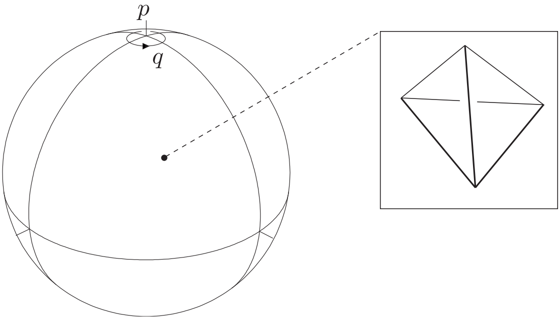
For fixed areas $A_1, \ldots, A_4$ each area vector lives in S^2.

Symplectic reduction of $(S^2)^4$ gives rise to the Poisson brackets:

$$\{f, g\} = \sum_{l=1}^{4} \vec{A}_l \cdot \left(\frac{\partial f}{\partial \vec{A}_l} \times \frac{\partial g}{\partial \vec{A}_l} \right)$$

The reduced phase space of a tetrahedron is a sphere

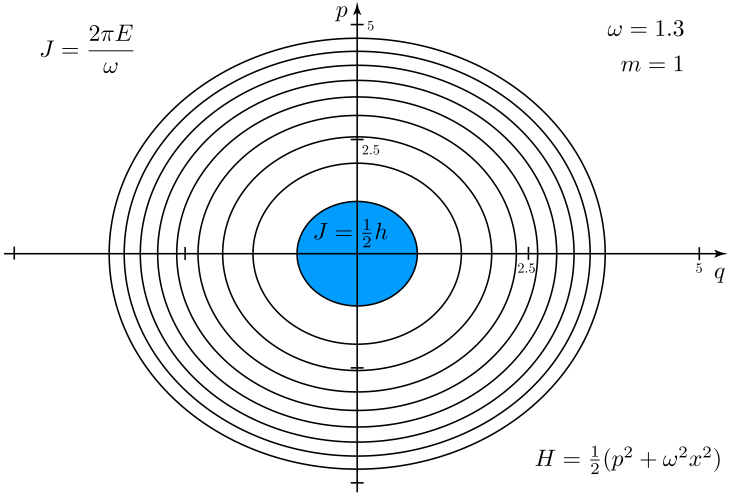
For fixed areas $A_1, \ldots, A_4$



$p = |\vec{A}_1 + \vec{A}_2|$ $q =$ Angle of rotation generated by p:

$$\{q, p\} = 1$$

Recall Bohr-Sommerfeld Quantization of Harmonic Oscillator



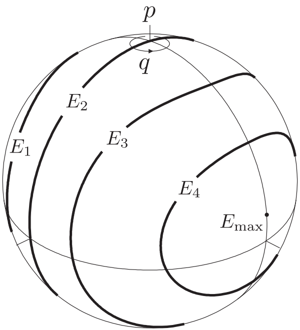
Require:

$$J = \oint_\gamma p dq = (n + \frac{1}{2}) 2\pi\hbar.$$

Dynamics

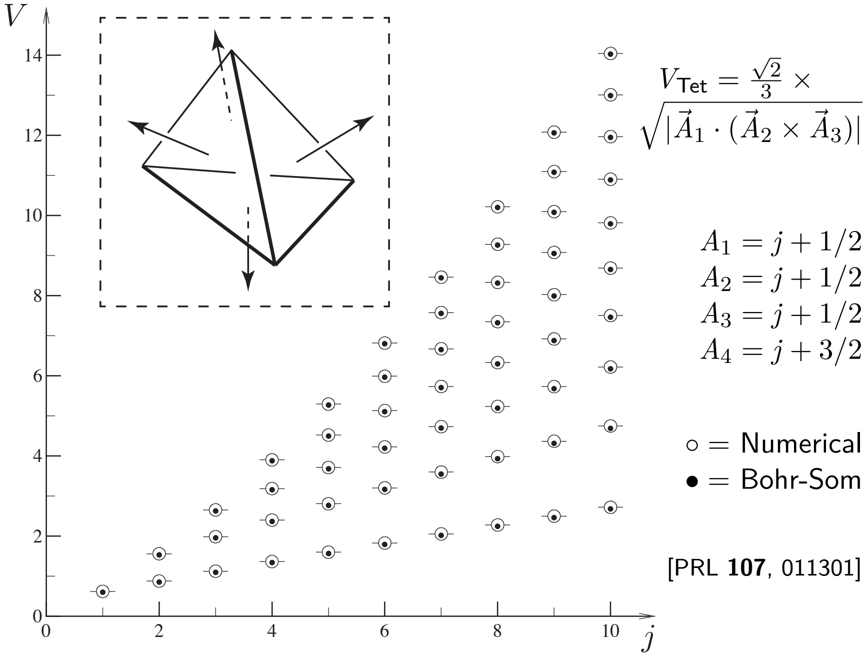
Take as Hamiltonian the Volume:

$$H = V^2 = \frac{2}{9}\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)$$



Action of orbits given in terms of complete elliptic integrals,

$$J(E) = \left(\sum_{i=1}^{4} a_i K(m) + \sum_{i=1}^{4} b_i \Pi(\alpha_i^2, m)\right) E$$



V
14
12
10
8
6
4
2
0
0
2
4
6
8
10
j
$V_{\mathsf{Tet}} = \frac{\sqrt{2}}{3} \times \sqrt{|\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)|}$
$A_1 = j + 1/2$
$A_2 = j + 1/2$
$A_3 = j + 1/2$
$A_4 = j + 3/2$
○ = Numerical
● = Bohr-Som
[PRL **107**, 011301]

Table			
$j_1\ j_2\ j_3\ j_4$	Loop gravity	Bohr-Sommerfeld	Accuracy
	1.828	1.795	1.8%
	3.204	3.162	1.3%
	4.225	4.190	0.8%
$6\ 6\ 6\ 7$	5.133	5.105	0.5%
	5.989	5.967	0.4%
	6.817	6.799	0.3%
	1.828	1.795	1.8%
	3.204	3.162	1.3%
$\frac{11}{2}\ \frac{13}{2}\ \frac{13}{2}\ \frac{13}{2}$	4.225	4.190	0.8%
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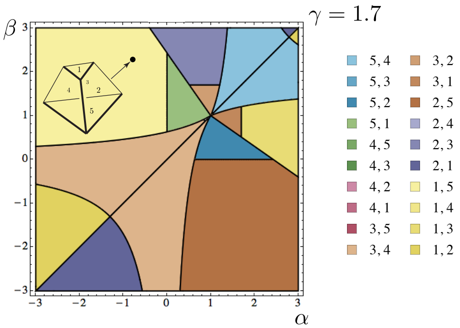
Does a volume gap persist for more complex polyhedra? (e.g # faces > 4)



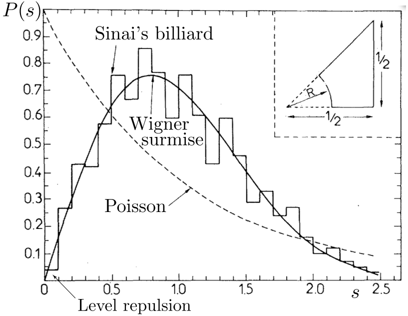
Analytic volume:

$$V_{\text{Pent}} = \frac{\sqrt{2}}{3}\left(\sqrt{\alpha\beta\gamma} - \sqrt{\bar{\alpha}\bar{\beta}\bar{\gamma}}\right)$$
$$\times\sqrt{|\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)|},$$
$$\alpha \equiv \vec{A}_4\cdot(\vec{A}_3\times\vec{A}_2)/\vec{A}_1\cdot(\vec{A}_2\times\vec{A}_3)$$

etc.

- Robust mechanism $\rightsquigarrow$ volume gap: chaotic dynamics & low density of states at low volume

[PRD **87**, 044020]



Outstanding challenge

How to identify the ground state of a general relativistic theory?

Want to coordinate individual grains of space to recover Minkowski space from this quantum theory.

- Use entanglement

Outline

Introduction to loop gravity and the discreteness of space

Entanglement and the entanglement spectrum

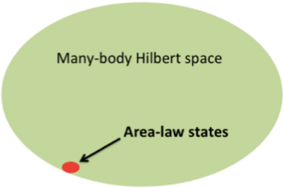
The entanglement spectrum of a sphere

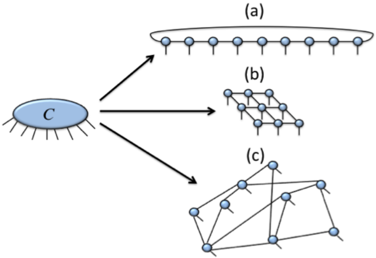
Interest for quantum gravity

Tensor networks, already valuable in condensed matter, are a natural fit for loop gravity

$$|\Psi\rangle = \sum_{i_1, i_2, \ldots, i_N} C_{i_1 i_2 \cdots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_N\rangle$$

The many body Hilbert space is too large ($\sim p^N$)





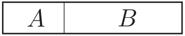
(a) Matrix Prod States (MPS)
(b) Proj Ent Pair Sts (PEPS)
(c) Tensor networks (TNs)

$\rightsquigarrow$ Area laws

- Spin networks are tensor networks

Entanglement

Pure state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.



$\mathcal{H}_A \otimes \mathcal{H}_B$

Schmidt decomposition:

$$|\Psi\rangle = \sum_i \sqrt{\lambda_i} |i_A\rangle \otimes |i_B\rangle$$

with $|i_A\rangle$ and $|i_B\rangle$ orthonormal bases in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

Leads to the reduced density matrix

$$\begin{aligned} \rho_B &= \mathsf{Tr}_A |\Psi\rangle\langle\Psi| \\ &= \sum_i \lambda_i |i_B\rangle\langle i_B|, \end{aligned}$$

and the entanglement entropy

$$S_E \equiv -\mathsf{Tr}\, \rho_B \log \rho_B = -\sum_i \lambda_i \log \lambda_i.$$

The entanglement spectrum unveils a thorough description of entanglement.

Can always write

$$\rho_B = e^{-H_E}, \quad \text{i.e.} \quad H_E \equiv -\log \rho_B,$$

the "entanglement Hamiltonian".

Already diagonalized H_E:

$$\rho_B = \sum_i e^{-\epsilon_i} |i_B\rangle\langle i_B|$$

with ϵ_i $(i = 1, 2, \dots)$ the "entanglement spectrum". $(\lambda_i = e^{-\epsilon_i})$

[Li & Haldane '08]

- Study for spacetime fields.

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Introduction to loop gravity and the discreteness of space

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The entanglement spectrum of a sphere

Interest for quantum gravity

Boundary conditions for the remainder

For simplicity I restrict to:

Scalar field $\varphi(x)$,

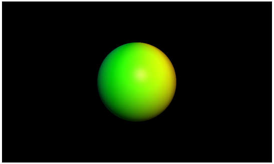
with $m = 0$

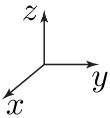
on flat $D = 3 + 1$ spacetime

unless otherwise stated. Metric signature $(-, +, +, +)$.

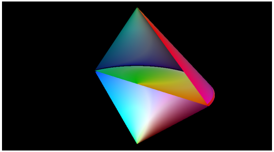
- Results apply to any conformal field theory.

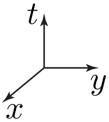
The finite region of interest is the Cauchy development of a 3-ball with boundary 2-sphere of radius R:



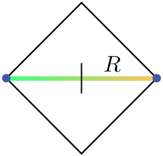


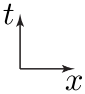
Spatial 3-ball B $\rightsquigarrow$ Cauchy development D(B).



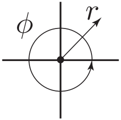


Entangling surface the boundary 2-sphere $\partial B = S^2$.





Choose adapted coordinates that preserve S^2: similar to how polar coords fix $(0,0)$...



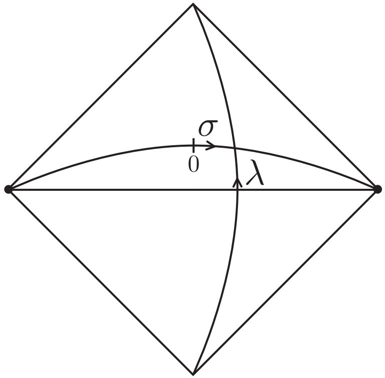
Use hyperbolas to build coordinates on the diamond

Diamond coords $(\lambda, \sigma, \theta, \phi)$:

$$t = R \frac{\mathsf{sh}\,\lambda}{\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma}$$

$$r = R \frac{\mathsf{sh}\,\sigma}{\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma}$$

with $\lambda \in (-\infty, \infty)$, $\sigma \in [0, \infty)$.



The Minkowski metric becomes

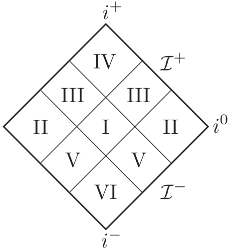
$$ds^2 = \frac{R^2}{(\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma)^2}[-d\lambda^2 + d\sigma^2 + \mathsf{sh}^2\sigma\, d\Omega^2],$$

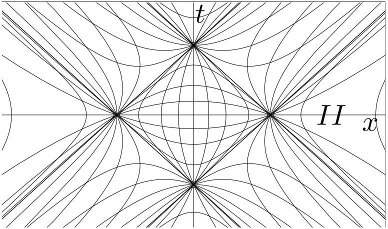
a conformal rescaling of static $\kappa = -1$ FRW.

Diamond coordinates can be extended to all of Minkowski space

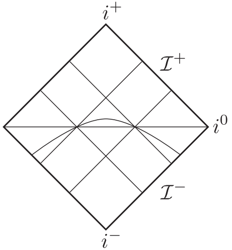
E.g. region II:

$$\left.\begin{aligned} t &= R\frac{\mathsf{sh}\,\tilde{\lambda}}{\mathsf{ch}\,\tilde{\sigma} - \mathsf{ch}\,\tilde{\lambda}}, \\ r &= R\frac{\mathsf{sh}\,\tilde{\sigma}}{\mathsf{ch}\,\tilde{\sigma} - \mathsf{ch}\,\tilde{\lambda}} \end{aligned}\right\} \quad |\tilde{\lambda}| \leq \tilde{\sigma}.$$





All hyperbolas asymp. null:

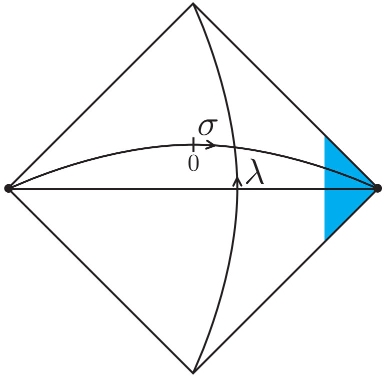


Near the L and R corners the diamond is approximately Rindler.

Large σ limit:

$$t \approx 2Re^{-\sigma}\,\mathsf{sh}\,\lambda = \ell\,\mathsf{sh}\,\lambda$$
$$R - r \approx 2Re^{-\sigma}\,\mathsf{ch}\,\lambda = \ell\,\mathsf{ch}\,\lambda$$

coord transformation to (left) Rindler wedge.



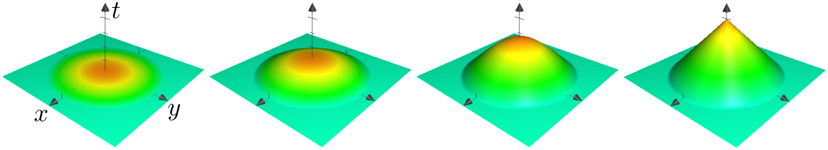
The proper distance from right corner is $\ell = 2Re^{-\sigma}$.

Entanglement (or foliation) Hamiltonian

$\xi^\mu = \left(\frac{\partial}{\partial\lambda}\right)^\mu \rightsquigarrow$ current $J^\mu = T^{\mu\nu}\xi_\nu$ with $T_{\mu\nu}$ the stress-tensor. If $T^\mu{}_\mu = 0$ the charge is conserved

$$C = \int T_{\mu\nu}\xi^\mu d\Sigma^\nu$$

and generates the spatial foliation discussed above:

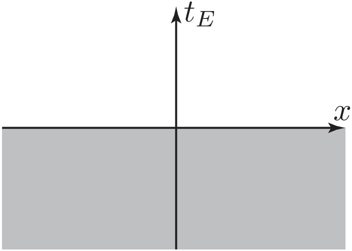


Explicitly this charge is,

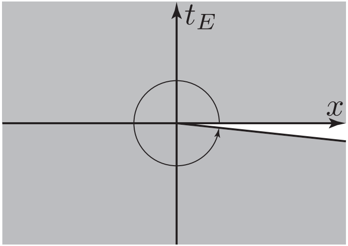
$$C_{in} = \frac{1}{2R}\int_B r^2 dr\, d\tilde{\Omega}\,\left(\frac{1}{2}(R^2 - r^2)(\dot{\varphi}^2 + \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi) + \varphi^2\right)$$

The density matrix for B is given by a Euclidean path integral

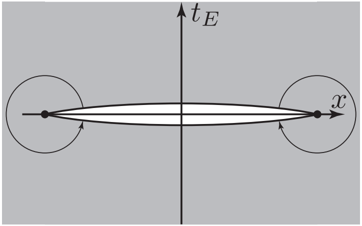
Minkowski vacuum:



Rindler density matrix:



Spherical density matrix:



$$\left.\begin{aligned} t_E &= R\frac{\sin\lambda_E}{\cos\lambda_E + \mathsf{ch}\,\sigma} \\ r &= R\frac{\mathsf{sh}\,\sigma}{\cos\lambda_E + \mathsf{ch}\,\sigma} \end{aligned}\right\}$$ Bipolar coords

$$\boxed{\rho_B = \int \mathcal{D}\varphi\, e^{-S_E} = e^{-2\pi C_{in}}}$$

Inverted strategy: study finite region of flat spacetime using QFT on a curved background

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}[g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + (\cancelto{0}{m^2} + \frac{1}{6}\mathcal{R}(x))\varphi^2].$$

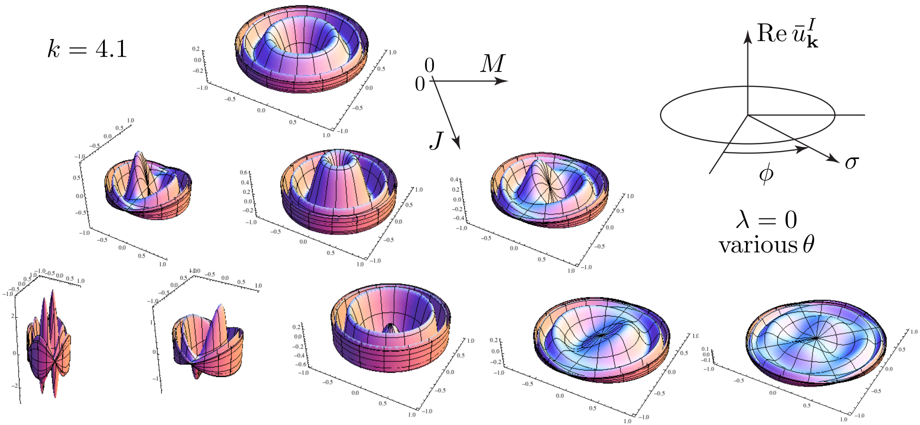
Action is conformally invariant under $\begin{cases} \bar{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu} \\ \bar{\varphi} = \Omega(x)^{-1}\varphi \end{cases}$

With $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ the EOM transform as ($\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$)

$$\bar{\Box}\bar{\varphi} = \Omega^{-3}[\Box - \frac{1}{6}\mathcal{R}]\varphi.$$

Find $\bar{\varphi}$ by finding φ.

Sphere modes



Sphere modes:

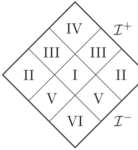
$$\bar{u}^I_{\mathbf{k}}(x) = \frac{(2k)^{-\frac{1}{2}}}{R}(\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma)\Pi^-_{kJ}(\sigma)\, Y^M_J(\theta,\phi)\, e^{-ik\lambda}$$

Mode decomposition defines vacuum $|0\rangle_S$ and excitations above it

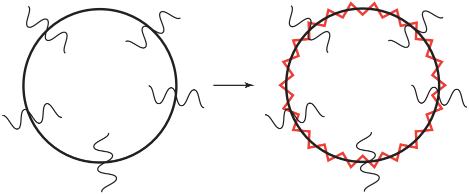
Minkowski space:
$\varphi(x) = \int d^{D-1}k[a_{\mathbf{k}}u_{\mathbf{k}}(x) + a^\dagger_{\mathbf{k}}u^*_{\mathbf{k}}(x)]$ with vacuum $a_{\mathbf{k}}|0\rangle_M = 0$.

Sphere: $\varphi(x) = \int dk \sum_{J,M}[s^I_{\mathbf{k}}u^I_{\mathbf{k}} + s^{I\dagger}_{\mathbf{k}}u^{I*}_{\mathbf{k}} + s^{II}_{\mathbf{k}}u^{II}_{\mathbf{k}} + s^{II\dagger}_{\mathbf{k}}u^{II*}_{\mathbf{k}}]$

$s^{I\dagger}_{\mathbf{k}}$ creates *spherons*,
excitations localized within
the S^2 entangling surface



The sphere vacuum satisfies $s^I_{\mathbf{k}}|0\rangle_S = s^{II}_{\mathbf{k}}|0\rangle_S = 0$.



This vacuum has no in-out entanglement due to mode localization

Spherical entanglement spectrum

Curved space, traceless (improved) stress tensor

$$T_{\mu\nu} = \frac{2}{3}\nabla_\mu\bar{\varphi}\nabla_\nu\bar{\varphi} - \frac{1}{6}g_{\mu\nu}(\nabla^\rho\bar{\varphi}\nabla_\rho\bar{\varphi}) - \frac{1}{3}\bar{\varphi}\nabla_\mu\nabla_\nu\bar{\varphi} + \frac{1}{3}g_{\mu\nu}\bar{\varphi}\Box\bar{\varphi} + \frac{1}{6}[\mathcal{R}_{ab} - \frac{1}{2}g_{ab}\mathcal{R}]\bar{\varphi}^2.$$

Entanglement Hamiltonian

$$C_{in} = \int_B T_{\mu\nu}\xi^\mu d\Sigma^\nu.$$

Entanglement spectrum

$$C_{in} = \int dk \sum_{J,M} k(s_{\mathbf{k}}^{I\dagger} s_{\mathbf{k}}^{I} + \frac{1}{2}[s_{\mathbf{k}}^{I}, s_{\mathbf{k}}^{I\dagger}]).$$

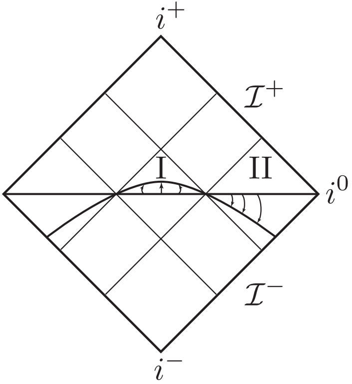
zero pt. ener.

Schmidt decomposition of Minkowski vacuum

Total Hamiltonian

$$C = C_{in} - C_{out}$$

$$C = \int dk \sum_{J,M} k(s_{\mathbf{k}}^{I\dagger} s_{\mathbf{k}}^{I} + s_{\mathbf{k}}^{II\dagger} s_{\mathbf{k}}^{II})$$

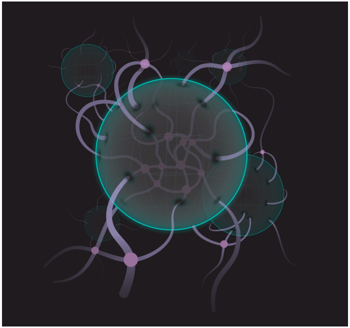


$$\boxed{|0\rangle_M = \int dk \sum_{J,M} e^{-\epsilon_k} u_{\mathbf{k}}^{I} \otimes u_{\mathbf{k}}^{II}.}$$

Outline

Investigation of a spherical region suggests an intriguing perspective on spin networks.

Individual nodes are entangled to neighbors through links. How shall we supperpose spin network states to recover Minkowski geometry?



Can we choreograph entanglement to yield the Minkowski vacuum?

♦ A wealth of condensed matter research on entanglement and tensor networks to learn from.

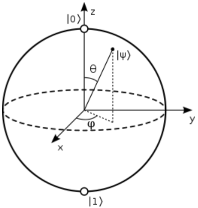
Reeh-Schlieder theorem

If quantum gravity cuts off the continuum what becomes of the Reeh-Schlieder theorem?

A special, initial pure state of two q-bits,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)$$

can be used to steer the q-bits onto whole state space.



More generally:

A	B

$$\mathcal{H}_A \otimes \mathcal{H}_B$$

Schmidt rank of the decomp.

$$|\Psi\rangle = \sum_i \sqrt{\lambda_i}|i_A\rangle \otimes |i_B\rangle$$

is at most dim$\mathcal{H}_A$. There is no way to steer onto all of $\mathcal{H}_B$.

In what sense, if any, does the Reeh-Schlieder theorem hold for a large but finite #d.o.f.?

Conclusions

I presented a new independent road to the granularity of space and the computation of the spectrum of the volume.

There are many surprises lurking in the continuum-discrete transition (e.g. Reeh-Schlieder).

Looking to evaluate vacuum proposals and provide design criteria to recover the Minkowski vacuum in quantum gravity. Your suggestions?

Credits

Special thanks to John Schliemann for the invitation.

Spherical spin network: Z. Merali, "The origins of space and time," Nature News, Aug. 28, 2013

Thanks also to the Cracow School of Theoretical Physics, LIII Course, 2013 for a lovely school and setting while this work was begun. And to the Perimeter Institute for their gracious hosting of visitors and support during the continuation of this work.

Sphere modes analytically

Metric $g = -d\lambda^2 + d\sigma^2 + \mathsf{sh}^2\sigma\, d\tilde{\Omega}^2$ is static $\kappa = -1$ FRW.

Separate: $u_{\mathbf{k}}(x) = \chi_k(\lambda)\Pi^-_{kJ}(\sigma)\, Y^M_J(\theta, \phi)$ with

$$\chi_k(\lambda) = (2k)^{-\frac{1}{2}}\, e^{-ik\lambda}$$

$$\Pi^-_{kJ}(\sigma) = N(k, J)\, \mathsf{sh}^J\sigma \left(\frac{d}{d\,\mathsf{ch}\,\sigma}\right)^{1+J} \cos(k\sigma)$$

$$Y^M_J(\theta, \phi) \qquad \text{spherical harmonics}$$

$$M = -J, -J+1, \ldots, J; \qquad J = 0, 1, \ldots; \qquad 0 < k < \infty.$$

Sphere modes: $\bar{\varphi} = \Omega(x)^{-1}\varphi$, (recall $\Omega = R/(\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma)$)

$$\bar{u}^I_{\mathbf{k}}(x) = \frac{(2k)^{-\frac{1}{2}}}{R}(\mathsf{ch}\,\lambda + \mathsf{ch}\,\sigma)\Pi^-_{kJ}(\sigma)\, Y^M_J(\theta, \phi)\, e^{-ik\lambda}.$$