Edge modes, Soft Modes, Black-Hole thermodynamics and Quantum gravity

Laurent Freidel

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Abstract

On y va. Boum!

Road-Map

• Lecture 1: Canonical Formalism, Symplectic Poisson, Noether first theorem
• Lecture 2: Covariant Formalism, gauge theory, Cartan calculus, action forms, Second Noether theorem.
• Lecture 3: Gravity and Noether, Soft modes, Penrose diagrams
• Lecture 4: Gravitional and electromagnetic Edge modes and Loop Gravity.
• Lecture 4': Gravitional Thermodynamics from Noether,
• Lecture 5: gravitational Edge modes and Loop Gravit, Discretization, Non-commutation,...

References:

Iyer-Wald: gr-qc/9403028, Some properties of Noether charge and a proposal for dynamical black hole entropy
Jacobson-Mohd: arXiv:1507.01054, Black hole entropy and Lorentz-diffeomorphism Noether charge
Strominger: arXiv:1703.05448, Lectures on the Infrared Structure of Gravity and Gauge Theory
Freidel-Donnelly: arXiv:1601.04744, Local subsystems in gauge theory and gravity.
Freidel, Perez, Pranzetti: e-Print: arXiv:1611.03668, Loop gravity string.

Introduction

0.1 A brief overview: Context, Content, and Connections

This subject puts together all of the following: Noether's theorem, Covariant Formalism, Edge Modes, Soft Modes, Asymptotic symmetries, BMS symmetry, ADM mass, Boundary

symmetries, Soft theorems, Infrared anomalies, Memory effects, Black-Hole thermodynamics, Holography, Geometrical entropy formula, Black-Hole Hairs, resolution of the information paradox, Discretization of gauge theory, Discretization of gravity and extensions of loop quantum gravity.
I will not be able to cover all these subject :(. I can only skim through the surface in 5 hours with a narrow focus and make choices that I am still in the process of making.

1 Lecture 1: Preliminaries

Here I'll recall some basics that are needed for the understanding of the lecture. Would be amazing if I could skip it and there was a preliminary introductory lecture about it.

1.1 Cartan calculus and volume forms

Talk about Cartan calculus: forms, vector fields, and differentials. Talk about the Lie bracket & Jacobi identity

$$[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]] \tag{1}$$

the Lie-derivative $L_X Y = [X,Y]$ and $L_X\alpha$, Cartan's magical formula $L_x = \imath_x \mathrm{d} + \mathrm{d}\imath_x$ and present the Cartan identities: both d and ι_X are graded differential operators of degree $+1$ and -1 respectively. The graded commutator $[A,B] := AB - (-1)^{ab}BA$.

The wedge product of forms is such that

$$\mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_n} = \mathrm{sign}(\sigma)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \tag{2}$$

where $\mathrm{sign}(\sigma)$ is the signature of the permutation σ.

The Lie bracket and Lie derivative satisfy 6 Cartan identities for $(\mathrm{d}, L_X, \imath_X)$: 3 involve the differentials

$$2\mathrm{d}^2 = [\mathrm{d},\mathrm{d}] = 0 \tag{3}$$

$$\begin{aligned} [\imath_X, \imath_Y] &= 0, \\ [\mathrm{d}, \imath_X] &= L_X \\ [\mathrm{d}, L_X] &= 0 \\ [L_X, L_Y] &= L_{[X,Y]} \\ [L_X, \imath_Y] &= \imath_{[X,Y]} \end{aligned} \tag{4}$$

(5)

explain each briefly.

The first of the Cartan identities $\mathrm{d}^2 = 0$ is equivalent to the equality of mixed partials in coordinates, e.g.,

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i} \tag{6}$$

Ex. 1: Prove it.

More geometrically it can be thought of as the fact that there is no boundary to the boundary of a manifold:

$$\int_M \mathrm{d}^2\omega = \int_{\partial M} \mathrm{d}\omega = \int_{\partial\partial M} \omega = \int_{\emptyset} \omega = 0. \tag{7}$$

Volume form: Given a metric we can define volume forms, we use the Hodge star operation. It is such that

$$\epsilon := *1 = \sqrt{g}(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d), \qquad \iota_\xi * \omega = *(\omega \wedge g(\xi)) \tag{8}$$

where $g(\xi)_a := g_{ab}\xi^b$. It also satisfy

$$g(\xi) \wedge *\omega = *(\iota_\xi \omega)(-1)^{|\omega|-1} \tag{9}$$

$$\left\{ \begin{array}{rcl} \epsilon & := & *1 = \sqrt{g}(\mathrm{d}x^1 \wedge \cdots \mathrm{d}x^d), \\ \epsilon_a & := & \iota_{\partial_a}\epsilon = g_{aa'} * (\mathrm{d}x^{a'}), \\ \epsilon_{ab} & := & \iota_{\partial_b}\iota_{\partial_a}\epsilon = g_{aa'}g_{bb'} * (\mathrm{d}x^{a'} \wedge \mathrm{d}x^{b'}) \end{array} \right. \tag{10}$$

Application of these definition gives. These forms can be used to integrate functions on manifold M, vectors on codimension 1 slices Σ, and charge aspects on co-dimension 2 surfaces.

$$\int_M F\epsilon = \int_M \hat{F}, \qquad \int_\Sigma \xi^a \epsilon_a = \int_\Sigma \iota_\xi \epsilon, \qquad \frac{1}{2}\int_S Q^{ab}\epsilon_{ab} = \int_S \iota_Q \epsilon. \tag{11}$$

are such that $\mathrm{d}\epsilon = \mathrm{d}\epsilon_a = \mathrm{d}\epsilon_{ab} = 0$ and we can show that

$$\mathcal{L}_\alpha \epsilon = (\partial_a \alpha^a)\epsilon, \qquad \mathcal{L}_\alpha(\beta^b \epsilon_b) = [\partial_a(\alpha^a\beta^b) - \beta^a(\partial_a\alpha^b)]\epsilon_b \tag{12}$$

And we establish that

$$\mathrm{d}(\xi^a \epsilon_a) = (\partial_a \xi^a)\epsilon, \qquad \mathrm{d}\left(\tfrac{1}{2}Q^{ab}\epsilon_{ab}\right) = (\partial_a Q^{ab})\epsilon_b. \tag{13}$$

Proof:

$$\mathrm{d}\imath_\alpha \epsilon = \partial_a \alpha^b \mathrm{d}x^b \wedge \epsilon_b = \tag{14}$$

1.2 Canonical Formalism

Talk about Canonical formalism, symplectic potential, Poisson bracket, Noether charge, action and Hamiltonians for finite dimensional systems.

A Phase space is a manifold P equipped with a two-form ω which is closed. This is the symplectic form $\omega = \omega_{ab}(\mathrm{d}x^a \wedge \mathrm{d}x^b)$,

$$\mathrm{d}\omega = 0. \tag{15}$$

When invertible, we can associate to it a Poisson structure $\{f,g\} = \Pi^{ab}\partial_a f \partial_b g$. The Poisson structure is a biderivation , it is simply given by a bivector field which is the inverse of the symplectic potential, where $\Pi^{ab}\omega_{cb} = \delta^a_c$. The central identity for the Poisson bracket is that it satisfies the Jacobi identity

$$\mathrm{Jac}(F,G,H) := \{F,\{G,H\}\} + \mathrm{cycl} = 0. \tag{16}$$

Ex. 2: Prove that it follows from $\mathrm{d}\omega = 0$ and $\Pi\omega = -1$.
The main purpose of the Poisson bracket is that it allows to map, phase space observables F onto a phase space transformation, a flow. The Flow associated to F is encoded into a vector field X_F as follows: Given F we define the Hamiltonian vector field X_F to be such that

$$\iota_{X_F}\omega + \mathrm{d}F = 0. \tag{17}$$

And we define the Poisson bracket to be given by

$$\{F,G\} := \omega(X_F, X_G). \tag{18}$$

We can establish three key properties of the Hamiltonian vector field and the Poisson bracket:

$$\mathcal{L}_{X_F}\omega = 0, \qquad \{F,G\} = \mathcal{L}_{X_F}G, \qquad [X_F, X_G] = X_{\{F,G\}}. \tag{19}$$

In other words we have that Hamiltonian vector field preserves the symplectic structure, that the bracket compute the action of a Hamiltonian vector field on a second hamiltonian and that the bracket of two Hamiltonian vector fields is an Hamiltonian vector field associated with the bracket.
Ex. 3: Prove it! Proof:

$$\begin{aligned} \mathcal{L}_{X_F}\omega &= \mathrm{d}\iota_{X_F}\omega = -\mathrm{d}^2 F = 0, \\ \{F,G\} &= \iota_{X_G}\iota_{X_F}\omega = -\iota_{X_F}\iota_{X_G}\omega = \iota_{X_F}\mathrm{d}G = \mathcal{L}_{X_F}G, \\ \iota_{[X_F,X_G]}\omega &= [\mathcal{L}_{X_F}, \iota_{X_G}]\omega = -\mathcal{L}_{X_F}\mathrm{d}G = -\mathrm{d}\mathcal{L}_{X_F}G = -\mathrm{d}\{F,G\}. \end{aligned} \tag{20}$$

In other words we have established that

$$\{F,\cdot\} = X_F \Leftrightarrow \iota_{X_F}\omega = \omega(X_F,\cdot) = -\mathrm{d}F. \tag{21}$$

We can also establish that Jacobi is satisfied

$$\{\{F,G\},H\} = \{F,\{G,H\}\} - \{G,\{F,H\}\}. \tag{22}$$

Proof:

$$\begin{aligned} \{\{F,G\},H\} &= \mathcal{L}_{\{F,G\}}H = \mathcal{L}_{[X_F,X_G]}H \\ &= [\mathcal{L}_{X_F}, \mathcal{L}_{X_G}]H = \{F,\{G,H\}\} - \{G,\{F,H\}\}. \end{aligned} \tag{23}$$

1.3 Lagrangian

From an action to a symplectic structure. A symplectic structure is locally associated with a symplectic potential $\omega = \mathrm{d}\theta$.

$$S = \int_0^1 \mathrm{d}t[p\dot{q} - H(p,q)]. \tag{24}$$

We have

$$\delta S = \delta p[\dot{q} - \partial_p H] - \delta q[\dot{p} + \partial_q H] + [p\delta q]_0^1 \tag{25}$$

We see here that the structure of the equation of motion is

$$\dot{q} = \{H, q\}, \qquad \dot{p} = \{H, p\}, \qquad \{p, q\} = 1. \tag{26}$$

so that $X_H = \partial_t$ generates the time flow. This Poisson structure is compatible with the symplectic structure:

$$\theta = p\delta q, \qquad \omega = \delta p \wedge \delta q. \tag{27}$$

We have that

$$X_p = \partial_q, \quad X_q = -\partial_p, \quad \{p, q\} = 1. \tag{28}$$

Here we have $X_p = \{p, \cdot\} = \partial_q$ also $X_q = \{q, \cdot\} = -\partial_p$ and therefore $\{p, q\} = 1$. Here H generates a hamiltonian flow $X_H = \partial_t$.

Thus the symplectic structure is the inverse of $\omega = \mathrm{d}\theta$ where θ is the boundary term in the action.

Difference between gauge and symmetry: A Symmetry X is a canonical transform which preserve the Hamiltonian. We denote its hamiltonian J_X. $I_X\omega + \mathrm{d}J_X = 0$. X is a symmetry if $\{J_X, H\} = 0$. Noether first theorem states that a symmetry is conserved, this follows from

$$\dot{J}_X = X_H[J_X] = \{H, J_X\} = -\{J_X, H\} = X[H] = 0. \tag{29}$$

A gauge transformation is a transformation which is in the Kernel of Ω. It's Noether charge vanishes! $J_X = 0$.

Two questions: What happens if ω is not invertible? and how to we find the symplectic form ω in Field theory?

Suppose we have a phase space (P, ω) together with a set of constraints $C = \{C_a, a = 1, \cdots, n\}$. the constraints sub space $C^{-1}(0) \equiv \{x \in P | C_a(x) = 0\} \subset P$. We denote by $i_C : C \to P$ the embedding map. $i_C^*\omega$ the pull back of ω to C, restricted to the constraint surface is a closed two form. It is a presymplectic form since it is not invertible. We denote by $N_C \equiv Ker(i_C^*\omega) \subset TC$ the set of vector field which are in the kernel of $i_C^*\omega$. Since ω is closed we have that if $X, Y \in N_C$ then $[X, Y] \in N_C$. N_C is therefore the tangent space to the space of orbits. An equivalence relation is defined by

$$x \sim y \Leftrightarrow y = e^X x, \quad X \in N_C. \tag{30}$$

and we define

$$P/\!\!/C = [C^{-1}(0)]^*/\sim \tag{31}$$

where $*$ means that we take out the fixed point of the group action. $P/\!\!/C$ is a symplectic manifold.

1.4 geometric quantisation

In the quantisation scheme the symplectic potential plays a key role. As we have seen, classically an observable F defines a vector field X_F which is such that $\imath_{X_F}\omega = -\mathrm{d}F$. At the quantum level phase space functions are promoted to sections of a line bundle over P. The additional dimension is given by the phase factor. The question we want to investigate is whether there exists a quantisation map $F \to \hat{F}$ promoting functions to operators such that

$$[\hat{F}, \hat{G}] = i\hbar\widehat{\{F, G\}} \tag{32}$$

for all functions (F, G) in Phase space? Remarkably the answer is yes! Strange because Groenewold-Van Hove theorem states that this is not possible. This is a cornerstone results of Geometric quantisation, and the symplectic potential plays a key role.

One first establish that the change in the symplectic potential along an Hamiltonina vector field is given by

$$\mathcal{L}_{X_F}\theta = \mathrm{d}\iota_{X_F}\theta + \iota_{X_F}\mathrm{d}\theta = \mathrm{d}(\iota_{X_F}\theta - F) := \mathrm{d}\ell_F. \tag{33}$$

The combination $\ell_F := \iota_{X_F}\theta - F$ is the Lagrangian associated with F.

$$\begin{aligned}
\mathcal{L}_{X_F}\ell_G - \mathcal{L}_{X_G}\ell_F &= \mathcal{L}_{X_F}\iota_{X_G}\theta - \underbrace{\mathcal{L}_{X_F}G}_{=\{F,G\}} - \underbrace{\iota_{X_G}\mathrm{d}\ell_F}_{=\iota_{X_G}\mathcal{L}_{X_F}\theta} - \mathrm{d}\underbrace{\iota_{X_G}\ell_F}_{=0} \\
&= [\mathcal{L}_{X_F}, \iota_{X_G}]\theta - \{F, G\} \\
&= \iota_{[X_F, X_G]}\theta - \{F, G\} = \ell_{\{F,G\}}.
\end{aligned} \tag{34}$$

Given a function F one defines a differential operator

$$\hat{F} := \frac{\hbar}{i}\mathcal{L}_{X_F} - \ell_F. \tag{35}$$

This operator satisfies the quantisation condition (32).

$$\begin{aligned}
\frac{i}{\hbar}[\hat{F}, \hat{G}] &= \frac{\hbar}{i}[X_F, X_G] - (\mathcal{L}_{X_F}\ell_G - \mathcal{L}_{X_G}\ell_F) \\
&= \frac{\hbar}{i}X_{\{F,G\}} - \ell_{\{F,G\}} = \widehat{\{F, G\}}.
\end{aligned} \tag{36}$$

One used that θ defines a natural hermitian connection with curvature ω:

$$\nabla_X := X - \frac{i}{\hbar}\iota_X\theta \tag{37}$$

and then define $\hat{F} = \frac{\hbar}{i}\nabla_{X_F} + F$. Applying this to (p, q) with $(X_p, X_q) = (\partial_q, -\partial_p)$ and $(\Lambda_p, \Lambda_q) = (0, -q)$ one gets

$$\hat{p} = \frac{\hbar}{i}\partial_q, \qquad \hat{q} = -\frac{\hbar}{i}\partial_p + q. \tag{38}$$

In order to get the usual quantisation we have to restrict to a polarisation where $\partial_p\phi = 0$.

1.5 Connections and curvature

Present in an elementary manner the concept of connection and its curvature in gauge and gravity. We will use Yang-Mills connections A one-form valued into a Lie algebra $\mathfrak{g}$

$$\nabla_a\phi = \partial_a + A_a, \qquad F(A) = [\nabla_a, \nabla_b] = \partial_a A_b - \partial_b A_a + [A_a, A_b]. \tag{39}$$

We will also use Levi-Civita connection Which are connection in the tangent bundle. And Levi-civita connection $\nabla_X Y - \nabla_Y X = [X, Y]$, $\nabla_a g_{ab} = 0$, the coefficient of the conection are $\nabla_a\partial_b = \Gamma_{ab}{}^c$ are given by

$$\Gamma_{ab}{}^c = \frac{1}{2}\left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\right) g^{dc}. \tag{40}$$

and the curvature tensor is

$$[\nabla_a, \nabla_b]\partial_c = R^d{}_{cab}\partial_d. \tag{41}$$

While the Ricci tensor is $R_{ab} = R^c{}_{acb}$. Taking the variation of these relations we get

$$\delta R^d{}_{cab} = \nabla_a\delta\Gamma_{bc}{}^d - \nabla_b\delta\Gamma_{ac}{}^d, \quad \delta R_{ab} = \nabla_c\delta\Gamma_{ba}{}^c - \nabla_b\delta\Gamma_{ca}{}^c. \tag{42}$$

1.6 Variational calculus

Introduce the variational Cartan calculus (δ, I_X, L_X). The two differentials anticommute $\mathrm{d}\delta + \delta\mathrm{d} = 0$, so that $\mathrm{d} + \delta$ is itself a differential. Talk about the concept of Field space vector field. And introduce as first example QCD.

2 Lecture 2: Covariant phase space and Noether's theorem

We are going to see that in the covariant formalism a Lagrangian determines both the equation of motion and a presymplectic structure on the system's phase space. We will also see that we can analyze symmetries and Hamiltonian structure without having to specify a global time foliation.

We start with the QCD Lagrangian

$$L = \frac{1}{4g^2}\mathrm{Tr}(F_{ab}F^{ab} + j_m^a A_a), \tag{43}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]$ is the curvature.

Its variation gives the equation of motion up to a boundary term:

$$\delta L = \partial_a \theta^a - \underbrace{\mathrm{Tr}(E^a \delta A_a)}_{:=E}. \tag{44}$$

where θ^a is the symplectic current, and E^a are the equations of motion

$$\theta^a := \frac{1}{g^2}\mathrm{Tr}(F^{ab}\delta A_b), \qquad \text{and} \qquad E^a := \frac{1}{g^2}\nabla_b F^{ba} - j^a_m. \tag{45}$$

The covariant derivative is such that $\nabla_a X := \partial_a X + [A_a, X]$.

Ex. 4: Prove it.

$$\begin{aligned} \delta L &= \frac{1}{g^2}\mathrm{Tr}(*F \wedge \mathrm{d}_A \delta A) + \mathrm{Tr}(*j\delta A). \\ &= \mathrm{d}\left(\frac{1}{g^2}\mathrm{Tr}(*F \wedge \delta A)\right) - \frac{1}{g^2}\mathrm{Tr}((\mathrm{d}_A * F - *j) \wedge \delta A) \end{aligned} \tag{46}$$

A Lagrangian symmetry is a transformation of the field that leaves the Lagrangian invariant up to a boundary term. A gauge symmetry is a Lagrangian symmetry whose parameter is a local functional. Look to the gauge transformation $L_X A_a := \nabla_a X$. The action of this vector field on the local functional L is given on the one hand by

$$L_X L = \partial_a \underbrace{(X j^a_m)}_{:=\ell^a_X}, \tag{47}$$

where j_m is the charge matter current and we have denoted the boundary term ℓ_X.

On the other hand we have, since $L_X L = \delta I_X L$, that

$$L_X L = \partial_a(I_X \theta^a) - \underbrace{\mathrm{Tr}(E^a \nabla_a X)}_{I_X E}, \qquad \text{and} \qquad I_X \theta^a = \frac{1}{g^2}\mathrm{Tr}(F^{ab}\nabla_b X). \tag{48}$$

We can conclude two important equations from this. First taking the difference we obtain the conservation law for the Noether current:

$$\partial_a \underbrace{(I_X \theta^a - \ell^a_X)}_{:=J_X} = I_X E \mathrel{\hat{=}} 0. \tag{49}$$

This is Noether's first theorem. The Noether current $J_X := (I_X \theta^a - \ell^a_X)$ is conserved on-shell which is represented by the hatted equality.

In the case of a gauge symmetry we have more: we can decompose I_X into a total derivative plus a term that does not depend on derivatives of X,

$$I_X E = \mathrm{Tr}(E^a \nabla_a X) = \partial_a \underbrace{\mathrm{Tr}(E^a X)}_{:=C^a_X} - \mathrm{Tr}(X \nabla_a E^a), \tag{50}$$

and we can write the current conservation as

$$\partial_a (J^a_X - C^a_X) = \mathrm{Tr}(X \nabla_a E^a). \tag{51}$$

Here J_X is the Noether current while C_X is the *c*onstraints that follows from gauge symmetry.

If X is a local variation, this equality can be true only if both sides vanish, this gives us the Bianchi identity:

$$\nabla_a E^a = 0. \tag{52}$$

This is indeed an identity in the example of QCD

$$\nabla_a E^a = \frac{1}{g^2}[\nabla_a, [\nabla_b, F^{ba}]] - \nabla_a j^a_m = \frac{1}{2g^2}[F_{ab}, F^{ba}] - \nabla_a j^a_m = -\nabla_a j^a_m. \tag{53}$$

The Noether Bianchi identity means that the matter Current needs to be covariantly conserved. It also means that the Noether conservation Law reads

$$\partial_a (I_X \Theta^a - \ell^a_X - C^a_X) = 0, \qquad C^a_X = \frac{1}{g^2}\mathrm{Tr}(E^a X). \tag{54}$$

The fact that the divergence of $J_X - C_X$ vanishes independently of the equations of motion means that it is *trivially conserved*. In other words, there exists a bivector Q^{ab}_X called the *charge aspect* such that

$$J^a_X = C^a_X + \partial_b (Q^{ab}_X). \tag{55}$$

The fact that the Noether Current is a pure boundary term on-shell is the hallmark of gauge invariant theories.

Ex. 5: Check the Bianchi identity and the trivial conservation.

We can check the trivial conservation of the current directly by evaluating the charge aspect for QCD: one finds that

$$J^a_X = \frac{1}{g^2}\mathrm{Tr}\underbrace{(F^{ab}\nabla_b X)}_{\text{soft-current}} - \underbrace{\mathrm{Tr}(j^a_m X)}_{\text{hard-current}}, \qquad \text{and} \qquad Q^{ab}_X = \frac{1}{g^2}\underbrace{\mathrm{Tr}\left(F^{ab} X\right)}_{\text{charge-aspect}}. \tag{56}$$

Ex. 6: Prove it!

The solution is

$$\begin{aligned} J_X^a - C_X^a &= I_X \Theta^a - \mathrm{Tr}(j_m^a X) - \mathrm{Tr}(E^a X) \\ &= \frac{1}{g^2}\mathrm{Tr}(F^{ab}\nabla_b X) - \mathrm{Tr}(j_m^a X) - \mathrm{Tr}\left(X\left(\frac{1}{g^2}\nabla_b F^{ba} - j_m^a\right)\right) \\ &= \partial_b \underbrace{\left(\frac{1}{g^2}\mathrm{Tr}\left(F^{ab}X\right)\right)}_{Q_X^{ab}}. \end{aligned} \tag{57}$$

This means that in QCD we have an infinite number of $l$ocal charges of symmetries. These are boundary charges

$$Q_S[X] = \int_S QX. \tag{58}$$

And they form a non-abelian group of symmetry.! In QED this group is called the Spy group.

Now that we understand the Conservation equation and Bianchi identities lets investigate the canonical property of the charges. Given the symplectic current, we define the symplectic potential:

$$\omega = \delta\theta. \tag{59}$$

For QCD this is $\omega = \omega^a \epsilon_a$ given by

$$\omega^a = \frac{1}{g^2}\mathrm{Tr}(\delta F^{ab} \wedge \delta A_b) = \frac{1}{g^2}\mathrm{Tr}\left((\nabla^a \delta A^b - \nabla^b \delta A^a) \curlywedge \delta A_b\right). \tag{60}$$

The Symplectic potential is a 2-form in field space and a codimension one form. It can therefore be integrated over codimension one manifold Σ embedded in space-time to define the symplectic structure

$$\Omega_\Sigma := \int_\Sigma \omega. \tag{61}$$

It is customary to define Σ at a constant time slice $T = t$ of a global foliation. Here as the figure shows (**??**) we do not have to restrict to a given foliation or a particular time-slice. The question arises whether the symplectic structure depends on the codimension one surface that one choses to evaluate it? The fact that it doesn't for on-shell variations, that is variations that preserves the

Taking the differential of the defining equation (67) and using that $\delta^2 = 0$ we get

$$\boxed{\delta E = \mathrm{d}\omega.} \tag{62}$$

This equation is the classical version of the unitarity condition. WE call it Noether 0-th Law. Given two cohomologous hypersurface $\partial\Sigma = \partial\Sigma'$ enclosing a region R such that $\partial R = \Sigma \cup (-\Sigma')$ (see fig. 2) , Noether zeroth law means that

$$\Omega_\Sigma - \Omega_{\Sigma'} = \int_R \delta E \hat{=} 0. \tag{63}$$

This means that the symplectic potential is conserved. If the two regions intersect the boundary at different time then we need to impose boundary conditions that insure that the boundary symplectic potential vanish. This is the reason behind Dirichlet or Neuman boundary conditions.

Given the symplectic potential we can define the bracket of two Noether current to be given by

$$\{J_X, J_Y\} := \omega(X, Y) = -I_X I_Y \omega. \tag{64}$$

This bracket satisfy Jacobi-identity, since $\delta\omega = 0$. The bracket of charges is therefore

$$\{Q_S(X), Q_S(Y)\} = Q_S([X, Y]), \qquad \{Q^a(x), Q^b(y)\} = F^{ab}{}_c Q^c(x)\delta^{(2)}(x, y). \tag{65}$$

3 Lecture 3: Edge modes and Gravity

Summary: Let us summarise this lecture. We denote ϵ the volume form[1] and we consider the Lagrangian density $\hat{L}(\phi) = L(\phi)\epsilon$ for a set of fields ϕ_a. Its variation defines the symplectic potential and the equations of motion.

$$\delta\hat{L}(\phi) = \mathrm{d}\theta(\phi, \delta\phi) - E(\phi, \delta\phi), \qquad E = \delta\phi_a E^a. \tag{67}$$

The Lagrangian variation is a sum of a boundary term plus the equations of motion.

$$E = \delta\phi^a E_a, \qquad \theta = \theta^a \epsilon_a \tag{68}$$

where we denote the basis of codimension-one forms by $\epsilon_a = \imath_{\partial_a}\epsilon$ and $\theta = \theta^a\epsilon_a$ is a $d-1$-form on M, determined by the symplectic current θ^a, it is also a 1 form in field space. E is also a one-form on field space and a volume form.

We have seen the property of unitarity, Noether zeroth-law:

$$\boxed{\delta E = \mathrm{d}\omega, \qquad \omega = \delta\theta.} \tag{69}$$

By contracting the defining equation with a symmetry transformation and using that $L_X\hat{L} = I_X\delta\hat{L} = \mathrm{d}\ell_X$ we get that

$$\boxed{I_X E = \mathrm{d}J_X, \qquad J_X := I_X\theta - \ell_X.} \tag{70}$$

This gives us Noether first property: The Noether current is conserved on-shell.

Moreover of gauge transformation $\delta_X\phi_a = D_a X$, we have that the Bianchi identity is satisfied:

$$I_X E = \mathrm{d}C_X - X\cdot(D_a^\dagger E^a) = \mathrm{d}C_X. \tag{71}$$

[1] which is given by

$$\epsilon := \sqrt{g}(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d). \tag{66}$$

Here $C_X := X{\cdot}E^a\epsilon_a$ is a "constraint" a quantity that vanish when the equations of motions are imposed.

From this we conclude that we have the Bianchi identity and the trivial conservation law We have seen that

$$\mathrm{d}(J_X - C_X) = 0, \qquad D_a^\dagger E^a = 0. \tag{72}$$

This means that

$$J_X = J_X^{\text{soft}} + J_X^{\text{hard}} = C_X + \mathrm{d}Q_X. \tag{73}$$

That is the current which is a sum of a soft component due to the gauge fields and a hard component due to the sources is a pure boundary term on-shell. This means that we have a new form of charge conservation in any gauge theory

$$\mathrm{d}Q_X \hat{=} J_X^{\text{soft}} + J_X^{\text{hard}}. \tag{74}$$

Holography: The previous conservation law is *equivalent* to the validity of the equations of motion. If One defines the boundary charges

$$Q_X = \frac{1}{g^2}\mathrm{Tr}(*FX), \tag{75}$$

we have the conservation equation:

$$Q_{S'}(X) - Q_S(X) = \int_\Sigma \mathrm{Tr}(\mathrm{d}_A X \wedge *F) + \int_\Sigma \mathrm{Tr}(X j_m), \qquad Q_S(X) := \int_S Q_X \tag{76}$$

where $\partial\Sigma = S' \cup \bar{S}$. Here we need to empphasize that we have an *infinite* number of charges. This are *local charge of symmetry* not gauge. One for each spherical Harmonics $X = Y_{\ell m}\tau_\alpha$ with τ_α a basis of $\mathfrak{g}$.

$$Q_{\ell m \alpha} = \int_S Y_{\ell m} Q_\alpha^{ab}\epsilon_{ab}. \tag{77}$$

where α labels a basis of the Lie algebra $\mathfrak{g}$.

3.1 Symmetry and equations of motion

Now that we have the symplectic potential we want to understand whether we have Noether third identity: That is is it true that not only the Noether current is also the canonical generator of gauge transformations? That is we are want to know if it is true that

$$I_X\omega + \delta J_X \hat{=} 0? \tag{78}$$

It turns out that there exist an anomaly in this equation. We are going to show that in fact we have

$$I_X\omega + \delta J_X \hat{=} \mathrm{d}\vartheta_X. \tag{79}$$

where ϑ_X is the symplectic anomaly. The presence of this anomaly term violates the on-shell conservation of the symplectic structure when $\delta\vartheta_X \neq 0$ since $L_X\omega = \mathrm{d}\delta\vartheta_X$. The fact that the symplectic structure is not preserved means that the numbers of degrees of freedom of the system is not preserved by the evolution. This is the Hallmark of open systems and what this means is that we are not tracking all the relevant degrees of freedom necessary to understand the symmetry of the system.

The resolution of this fundamental puzzle rely in accepting that we are missing certain degrees of freedom and that we therefore have to add them back to our description in order to have a complete description. The main point is that the missing degrees of freedom are related to boundary terms. Indeed if we integrate (**??**) on a region Σ with boundary$S$ then we get

$$L_X\Omega_\Sigma = \int_S \delta\vartheta_X. \tag{80}$$

And of course if $\partial\Sigma = \emptyset$ then the theory is unitary. The missing degrees of freedom are therefor associated with the edges of our spacelike regions, they are *e*dge modes.

The way we reveal these edge mode is by identifying a boundary symplectic structure denoted ϑ which is a one-form in Field space and a codimension 2 form in spacetime such that the anomaly can by written

$$\delta(\vartheta_X + L_X\vartheta) \hat{=} 0. \tag{81}$$

As we will see doing so requires adding boundary goldstone modes, new degrees of freedom that renders the theory unitary. When this is the case we can define an extended symplectic potential and an extended charge

$$\theta^{\mathrm{ext}} := \theta + \mathrm{d}\vartheta, \qquad J_X^{\mathrm{ext}} = J_X + \mathrm{d}I_X\vartheta. \tag{82}$$

This potential is such that

$$\mathrm{d}J_X^{\mathrm{ext}} + I_X\omega^{\mathrm{ext}} \hat{=} 0, \tag{83}$$

which insures that X is Hamiltonian and that J_X^{ext} is its Hamiltonian generator.
Ex. 7: Prove it!.

This follows from

$$I_X\omega^{\mathrm{ext}} = I_X\omega + \mathrm{d}I_X\delta\vartheta = -\delta J_X + \mathrm{d}\vartheta_X + \mathrm{d}(L_X\theta^{\mathrm{ext}} - \delta I_X\theta^{\mathrm{ext}}) = -\delta(J_X + I_X\theta^{\mathrm{ext}}). \tag{84}$$

We are now going to study the general structure of the symplectic anomaly, that determines the nature of the edge modes. What we are going to show in the case of QCD and in the case of gravity, but which holds in general is the following. First given a symmetry transformation X acting on fields space, there exist a space-time vector field $X^\sharp$ acting on spacetime which is such that

$$L_X E \quad = \quad I_{\delta X}E + \mathrm{d}\imath_{X^\sharp}E. \tag{85}$$

The symplectic anomaly can be written explicitely in terms of this vector field and the charge aspect as

$$\vartheta_X = \iota_{X^\sharp}\theta + Q_{\delta X}. \tag{86}$$

Given this we can now prove that[2]

$$\{J^{\rm ext}_X, J^{\rm ext}_Y\} = J^{\rm ext}_{[X,Y]}. \qquad I_{[X,Y]} := [L_X, I_Y] \tag{87}$$

3.2 QCD Edge mode

We now show that for QCD the symplectic anomaly takes the form (86) with $X^\sharp = 0$. We have seen that $\delta E = \mathrm{d}\omega$ and that $I_X = \mathrm{d}J_X$ therefore we get that

$$L_X E = I_X\delta E + \delta I_X E = I_X \mathrm{d}\omega + \delta \mathrm{d} J_X = \mathrm{d}(I_X\omega + \delta J_X). \tag{88}$$

This shows that

$$\delta J_X + I_X\omega \,\hat{=}\, \mathrm{d}\vartheta_X. \tag{89}$$

where ϑ_X is a one-form on field space and a codimension 2 form on space-time. As we will see it can be interpreted as the edge mode symplectic potential. We can evaluate this directly for QCD. We find

$$I_X\omega + \delta J_X = L_X\theta - \delta\ell_X = J_{\delta X}. \tag{90}$$

This follows from this in QCD:

$$\begin{aligned} L_X\theta^a - \delta\ell^a_X &= \frac{1}{g^2}\mathrm{Tr}(F^{ab}\nabla_b\delta X) - \delta X j^a_m \\ &= \frac{1}{g^2}\partial_b\mathrm{Tr}(F^{ab}\delta X) + \frac{1}{g^2}\mathrm{Tr}(\nabla_b F^{ba}\delta X) - \delta X j^a_m \\ &= \partial_b Q^{ab}_{\delta X} + \mathrm{Tr}(E^a\delta X) = J^a_{\delta X}. \end{aligned} \tag{91}$$

One introduce

$$\vartheta^{ab} := -\frac{1}{g^2}\mathrm{Tr}(F^{ba}\varphi^{-1}\delta\varphi), \qquad \vartheta = -Q_{\varphi^{-1}\delta\varphi} \tag{92}$$

where $\varphi \in G$ is a group valued field, and define

$$L_X A = \nabla X, \qquad L_X\varphi = \varphi X. \tag{93}$$

This satisfy the identity $L_X\vartheta = -\mathrm{d}Q_{\delta X}$.
Ex. 8: Prove it!.

This follows form $L_X F = [F, X]$ and $L_X(\varphi^{-1}\delta\varphi) = [\varphi^{-1}\delta\varphi, X] + \delta X$. In this case we have that $I_X\vartheta = -Q_X$. Therefore we have that the extended symplectic potential is

$$\theta^{\rm ext} = \frac{1}{g^2}\mathrm{Tr}(*F\wedge\delta A) - \mathrm{d}\left(\frac{1}{g^2}\mathrm{Tr}(*F\varphi^{-1}\delta\varphi)\right). \tag{94}$$

[2] Central extension

The extended Noether charge is then

$$J_X^{\text{ext}} = J_X + \mathrm{d}I_X\vartheta = C_X \hat{=} 0. \tag{95}$$

Therefore we see that the edge modes not only restore the unitarity but also the gauge invariance. What is remarkable is that this induces in terms a new form of symmetry: One can look at the edge mode transformation

$$L_{\tilde{X}}A = 0, \qquad L_{\tilde{X}}\varphi = -X\varphi. \tag{96}$$

This transformation is Hamiltonian and the generator of the symmetry transformation is $\widetilde{Q}_X = I_{\tilde{X}}\theta^{ext}$ given by

$$\widetilde{Q}_X = \frac{1}{g^2}\mathrm{Tr}(*\varphi F\varphi^{-1}X). \tag{97}$$

This charge is gauge invariant: it commutes with the Constraint

$$\{J_X, \widetilde{Q}_Y\} = 0. \tag{98}$$

3.3 Edge mode or no edge modes

WE could also have decided that the Hamiltonian parameter is not part of phase space $\delta X = 0$. In this case J_X is no longer a gauge generator, the variables conjugated to boundary gauge transformations becomes physical and we have new physical degrees of freedom anyway.

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4 Lecture 4*: Gravitational Edge modes and BH entropy

This lecture starts with two exercises:
Ex. 9: Ex 3: Compute $(\theta, \ell_\xi J_\xi, C_\xi, Q_\xi)$ for gravity. One start with the Lagrangian

$$L = \frac{\epsilon}{8\pi G}\left(\frac{1}{2}R(g) - \Lambda\right) \tag{99}$$

with $R = g^{ab}R_{ab}$ and $[\nabla_a, \nabla_b]v^a = R_{ab}v^a$.
Ex. 10: First establish that

$$\theta^a[g, \delta g] = \frac{1}{16\pi G}(\delta\Gamma^a_{bc}g^{bc} - \delta\Gamma^c_{cb}g^{ab}) = \frac{1}{16\pi G}\nabla_b[\delta g^{ab} - g^{ab}\delta g]\epsilon_a \tag{100}$$

Here we used the abbreviated notation $\delta g = g^{ab}\delta g_{ab}$, and $\delta g^{ab} = g^{ac}g^{bd}\delta g_{cd}$ (i.e. δg^{ab} is the variation of the metric with the indices raised, *not* the variation of the inverse metric, which is given by $\delta(g^{ab}) = -\delta g^{ab})\Box$.