I. Introduction

Almost twenty-five years ago the W-coefficient appeared for the first time in a paper by Racah\(^1\) as an auxiliary tool for the computation of matrix elements in the theory of complex spectra. Today there is hardly any branch of physics involving angular momenta\(^2\) where the use of W-coefficients is not needed in order to carry out the simplest computation. Yet we feel that the W-coefficient is something more than an extremely successful computational tool and a beautiful toy for theoretical physicists to play with. In fact, a complete understanding of the properties of this remarkable function may very well yield to a new insight into the theory of angular momenta.

We think, and we know that our view is shared by others\(^2\), that a complete investigation of the semiclassical limit of W-coefficients and related functions is a prerequisite for a deeper understanding of their properties.

The present paper contains a heuristic derivation of an asymptotic formula, or better, of a set of asymptotic formulae with separate ranges of validity, for the W-coefficient. These formulae are certainly a useful complement to the existing tables of Racah coefficients, since they are remarkably accurate for surprisingly low values of the angular momenta involved.

A similar point of view could be adopted for the Clebsch–Gordan coefficients. However, since their definition depends on the particular labelling

\(^1\) Work supported in part by EOAR under grant 66–29.

\(^2\) On leave of absence from Istituto di Fisica, Torino, Italy. N.A.T.O. visiting fellow.
adopted for the vector basis of the representations, and since, moreover, they can be deduced as a particular limit carried on the W-coefficients, we are definitely tempted to regard them as subsidiary quantities in this paper.

Coming back to the W-coefficient or, rather, to the symmetric version of it, i.e., the 6j-symbol defined by Wigner, it has been for years a normal practice to associate to it a diagram or graph which exhibits the symmetry properties in a most obvious way. A further advantage of these graphs is that they can be generalized to the higher order 3nj-symbols defined by Wigner and others. There are at least three different versions of these graphical algorithms, all having approximately the same content, the translation of one into the others being achieved through some principle of plane or space duality. The reason for choosing any one of them is rather sentimental and largely related to individual habits.

We shall prefer here a three-dimensional representation in which angular momenta appear as vectors satisfying “bona fide” graphical composition rules. In this particular calculation the 6j-symbol \[ \begin{bmatrix} a & c & d \\ b & e & f \end{bmatrix} \] is associated to the tetrahedron shown in fig. 1. So far the tetrahedron is just a mnemonical device. However, we may think about a real solid T whose edges are just \( a + \frac{1}{2}, b + \frac{1}{2}, \) etc. With reference to fig. 1, we shall use also the following notation for the edges: \( j_{12} = a + \frac{1}{2}, j_{13} = b + \frac{1}{2}, j_{14} = c + \frac{1}{2}, j_{24} = d + \frac{1}{2}, j_{23} = e + \frac{1}{2}, j_{34} = f + \frac{1}{2}, \) and \( j_{0} = 0, j_{90} = j_{90}(h, k, l, m, n). \)

We shall restrict ourselves to values of the angular momenta which satisfy the triangular inequalities, i.e., \( |b - c| \leq a + b + c, \) etc. for the triads \( (abc), (ade), (bdf), (dec). \) Therefore, in each case there must be an even number of half-integer angular momenta. This entails that the sums \( q_{1} = a + b + c, q_{2} = a + c + e + f, q_{3} = b + d + f, q_{4} = c + d + e, p_{1} = a + b + d + e, p_{2} = a + c + d + f, p_{3} = b + c + e + f, \) all are integer. Moreover, because of the triangular inequalities, we have

\[
 p_{h} \geq q_{k}, \quad h, k = 1, 2, 3, 4. \tag{1.1}
\]

While these conditions are in general sufficient to guarantee the existence of a non-vanishing 6j-symbol, they are not enough to ensure the existence of the tetrahedron T with the given edges. Since the two cases (A) T exists, (B) T does not exist, deserve radically different asymptotic treatments, we must give necessary and sufficient conditions for the existence of T.

It is known since Tartaglia and Jungius that the square of the volume of a tetrahedron is given by a polynomial in the square of its edges; in a more symmetrical setting, given by Cayley in his first published paper, we have indeed

\[
 2^{3} (3)^{2} V^{2} = \begin{vmatrix} 0 & j_{24} & j_{23} & j_{22} \\ j_{24} & 0 & j_{23} & j_{21} \\ j_{23} & j_{21} & 0 & j_{12} \\ j_{22} & j_{14} & j_{12} & 1 \end{vmatrix}. \tag{1.2}
\]

Therefore we see that \( V^{2} = 0 \) is a necessary condition. It can be proved to be also sufficient. In fact let us keep all edges fixed but, for instance, \( j_{12} \) and let \( j_{13} = x. \) Then \( V^{2} \) is a second order polynomial in \( x^{2} \) which will have two roots \( x_{c} < x_{c} = (x_{c})^{2}. \) Since \( \partial^{2} V^{2}(x^{2})/\partial x^{2} = -(j_{12})^{2}/2, \) we shall have \( V^{2}(x^{2}) = 0 \) if \( x_{c} = x_{c} < x_{c}. \) A more elaborate discussion would show, in addition, that \( x_{c} > x_{c} = \frac{1}{2} \) where \( x_{c} \) is the longest of \( j_{12} = a + b + c + \frac{1}{2}, \) \( j_{13} = e + f + \frac{1}{2}, \) and \( j_{14} = d + f + \frac{1}{2} \) the smallest between \( b + c + e + f + \frac{1}{2}. \) Therefore the condition \( V^{2} = 0 \) is stronger than (1.1). We shall accordingly distinguish between the above mentioned cases: (A) \( V^{2} = 0, \) and (B) \( V^{2} > 0. \) The third possibility, \( V^{2} = 0, \) which would correspond to a flat tetrahedron, is purely academic, for it can be proved that if \( p_{h} \) and \( q_{k} \) are all integer then \( V^{2} = 0. \) A “tetrahedron” in (B) will be referred to as a hyperflat tetrahedron. Let us start with case:

(A) We expect T to be relevant in describing the properties of 6j-symbols.
In fact a result due to Wigner\(^{13}\) states that for large angular momenta

\[
\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \sim \frac{1}{24\pi V}
\]

(1.3a)

where \(V\) is the volume of \(T\). This formula, which has been a guiding principle in the present investigation, is very interesting because it relates the numerical value of the \(6j\)-symbol directly to a geometric property of \(T\). However, as stressed in the same ref. 13, Wigner’s asymptotic estimate cannot be accepted at face value. Inspection of numerical tables shows in fact that the r.h.s. of (1.3a) more correctly approximates the average of the l.h.s. over several contiguous values of the indices.

A correct statement would be

\[
\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \simeq \frac{1}{\sqrt{12\pi V}} \varphi,
\]

(1.3b)

where \(\varphi\) is a rapidly oscillating function so that the average \(\varphi^2\) over a large enough interval is \(\frac{1}{4}\). We claim that

\[
\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \simeq \frac{1}{\sqrt{12\pi V}} \cos \left( \sum_{k=1}^{4} j_{hk} \theta_{hk} + \frac{1}{2} \pi \right),
\]

(1.4)

where \(\theta_{hk} = \theta_{hk}(k \neq h = 1, 2, 3, 4)\) are the angles between the outer normals of the two faces which belong to \(j_{hk}\). Let \(A_k\) be the area of the face opposite to the vertex \(k\) (fig. 1); then we have (appendix B)

\[
A_k A_h \sin \theta_{hk} = \frac{1}{2} V j_{hk}, \quad h \neq k = 1, 2, 3, 4.
\]

(1.5)

(B) Wigner’s argument yields

\[
\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \sim 0.
\]

(1.6)

This result could be loosely described as the impossibility of having six angular momenta forming a non-existing tetrahedral scheme. A closer scrutiny of numerical tables, however, shows that the symbols in (B), although, as a rule, smaller than in (A), are still non-vanishing.

Any attempt to use (1.4) in this region leads to a meaningless result. In fact, relations (1.1) guarantee that \(A_4\) are real; since \(V^2 < 0\), \(V\) is imaginary and (1.5) implies that \(\theta_{hk} = -\pi + \Im \theta_{hk}\). However, as it stands, (1.4) bears a strong resemblance with some formulae familiar from the WKB method\(^{14}\). Although we know of no differential equation from which in general (1.4) might be deduced, there are particular instances in which this can be done (section 5). This suggests that we may use the connection formulae of the WKB method to go across the transition points \(x^-\), \(x^+\). We define

\[
\Phi = \sum_{k=1}^{4} \left( j_{hk} - \frac{1}{2} \right) \Re \theta_{hk}
\]

(1.7)

which, for physical values of \(j_{hk}\), is always an integer multiple of \(\pi\). According to the WKB connection formulae (appendix G), we find for physical angular momenta

\[
\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \simeq \frac{1}{2\sqrt{2\pi |V|}} \cos \Phi \exp \left( - \sum_{k=1}^{4} j_{hk} \Im \theta_{hk} \right),
\]

(1.8)

where the sign of \(\Im \theta_{hk}\) must be chosen according to the rules explained in section 5. Also this formula turns out to be in remarkably satisfactory agreement with numerical tables. The exponential decrease shown by (1.8) clearly describes a quantum tunnel effect into the classically forbidden region (B).

We expect (1.4) and (1.8) to be inaccurate in the neighbourhood of \(x^-\). In fact, the error is here considerably large, although not disastrous. Transition formulae involving Airy functions have been worked out for this region (section 5) and found to be accurate.

In spite of these numerical checks, a sound proof of our formulae is still missing. However there are other arguments in favour of (1.4) and (1.8). For instance, (1.4) has the right symmetry properties, including the extra symmetries discovered by one of us\(^{11}\) and satisfies asymptotically the recursion relations as well as the identities of the \(6j\)-symbols. It is also consistent with the previously investigated particular cases of asymptotic behaviour\(^{14}\).

Relations (1.4) and (1.8), together with a transition formula, solve completely the analysis of \(6j\)-symbols when all angular momenta are large. However, it is also interesting to investigate the case in which one or more edges remain constant and finite while the others increase. We may picture the limiting process as one in which one or more vertices of \(T\) go to infinity either separately or in clusters. Therefore there are as many ways to carry out the process as decompositions of 4 into sums of natural integers, i.e., \(1 + 1 + 1 + 1\) (all edges large), \(1 + 1 + 2\) (one small edge), \(2 + 2\) (two small edges), \(1 + 3\) (three small edges). The \(1 + 1 + 2\) case has been widely studied\(^{15}\), while we have found no reference to \(2 + 2\) and accordingly we solve this interesting case in the present paper. Some examples of the \(1 + 3\) case, which yield a connection between \(6j\)- and \(3j\)-symbols, have been discussed by Brussaard and Tolhoek\(^{16}\). Our treatment, however, is quite general and
shows how the symmetries of the 3j-symbol can be derived from those of the 6j-coefficient. As a by-product of this analysis, we obtain a new asymptotic formula for Clebsch–Gordan coefficients.

2. Asymptotic connection with the Clebsch–Gordan coefficients

Of some interest is the 1 + 3 case, which occurs when we take the positive integer \( R \) large in

\[
\left\{ \begin{array}{l}
a + b + 1 = e + R + f + R \\
d + R + e + R + f + R
\end{array} \right\}
\]

The related limits

\[
\left\{ \begin{array}{l}
a + b + 1 = e + R + f + R \\
d + R + e + f + R
\end{array} \right\}
\]

e etc. can be reduced to it by symmetry. The starting point of our discussion is Racah's formula (A.4), which, using \( \xi = x - 2R \) as summation variable becomes

\[
\left\{ \begin{array}{l}
a + b + c \\
d + R + e + R + f + R
\end{array} \right\} = [A(abc) A(d + R, e + R, c) \times \\
\sum_{\xi} (-1)^{\frac{1}{2} + 2R} \times
\sum_{\xi} (-1)^{\frac{1}{2} - 2R} \times
\sum_{\xi} (-1)^{\frac{1}{2} + 2R} \times
\sum_{\xi} (-1)^{\frac{1}{2} - 2R}
\right]^{-1}.
\]

(2.1)

From Stirling's formula we find, for instance, that for large \( R \)

\[
\frac{\xi + 2R + 1}{\xi + 2R - a - b - c} \approx \xi^{a+b+c+1}.
\]

(2.2)

which entails for example

\[
A(a, e + R, f + R) \approx (a + e - f)! (a - e + f)! (2R)^{-2a-1}.
\]

(2.3)

By means of (2.2), (2.3) and similar relations, (2.1) transforms into

\[
\left\{ \begin{array}{l}
a + b + c \\
d + R + e + R + f + R
\end{array} \right\} \approx \left[ \frac{\Delta(abc)}{2R} \right]^\frac{1}{2} \times
\sum_{\xi} (-1)^{\frac{1}{2} - 2R} \times
\sum_{\xi} (-1)^{\frac{1}{2} + 2R} \times
\sum_{\xi} (-1)^{\frac{1}{2} - 2R} \times
\sum_{\xi} (-1)^{\frac{1}{2} + 2R}
\right]^{-1}.
\]

(2.4)

Looking at (A.1) we realize that the r.h.s. of (2.4) can be written in terms of a 3j-symbol (this result is quoted by K. Alder et al.16):

\[
\left\{ \begin{array}{l}
a + b + c \\
d + R + e + R + f + R
\end{array} \right\} \approx (-1)^{a+b+c+2(a+e+f)} (2R)^{-\frac{3}{2}} \left( \begin{array}{l}
a + b + c \\
d + R + e + f + d + d - e
\end{array} \right)
\]

(2.5a)

or using the pattern notation of appendix A

\[
\left\{ \begin{array}{l}
a + b - c \\
R + e + f + a + e + f + d - e - f - d - e
\end{array} \right\} \approx (-1)^{a+b+c+2(a+e+f)} (2R)^{-\frac{3}{2}} \left( \begin{array}{l}
a + e - f \\
R + e + f + a + e - f - d - e - f - d - e
\end{array} \right)
\]

(2.5b)

From (2.5b) it is easy to check that the symmetry

\[
\left\{ \begin{array}{l}
a + b + c \\
d + R + e + R + f + R
\end{array} \right\} = \frac{1}{2} \left( \begin{array}{l}
1 (f + c + e + b - e) \\
\frac{1}{2} (f + c + e - f)
\end{array} \right)
\]

entails

\[
\left\{ \begin{array}{l}
b + c + a - a + e - f + a + e - f + a + e - f \\
R + e + f + b + f + d + c - d + e
\end{array} \right\} = \frac{1}{2} \left( \begin{array}{l}
b + c - a + a - e + f \\
R + e + f + b + f + d + c - d + e
\end{array} \right)
\]

which is one of the extra symmetries of the 3j-symbol pointed out by one of us16). Actually, (2.5b) relates the subgroup of \( R_3 \) (see appendix A) formed by all the even 36 symmetries of the 3j-symbol to the subgroup of \( R_3 \) corresponding to permutations of columns and/or of the upper three lines of the pattern in the l.h.s. of (2.5b). The geometrical and physical content of these symmetries is still to be understood and they remain a puzzling feature of the theory of angular momenta. Therefore it is a pleasant result to be able to reduce the problem of their interpretation to the Racah coefficient only.

From (2.5a) we obtain also an expression for the 3j-symbol for large quantum numbers. Our derivation of this result is rather heuristic as it involves the exchange of different limiting processes. Just as for (1.4) the formula which we are going to present has a rather "a posteriori" validity, for it satisfies all possible consistency checks. We obtain it from (2.5a) by supposing \( a, b, c, d, e, f \) large and finite. Using (1.4) in the l.h.s. of (2.5a)
and performing the limit $R \to \infty$, we find
\[(m_a m_b m_c (a b c) \simeq (2\pi A)^{-\frac{3}{2}}(-1)^{a+b-c+1} \times \cos \left[\frac{(a + \frac{1}{2}) A + (b + \frac{1}{2}) B + (c + \frac{1}{2}) C - m_e D + m_e E + \frac{1}{4} \pi}{\cos A} \right] \right], (2.6)
\]
where $m_a = e - f$, $m_b = f - d$, $m_c = d - e$ are the third components of $a, b, c$ along the direction in which $P_4$ was sent to infinity (fig. 2). $A$ is the area of the shaded triangle in fig. 2 which corresponds to the projection of $P_1 P_2 P_3$ on a plane perpendicular to $z$ and the angles $A, ..., D, ..., E$ defined according to (1.4) are given in terms of $m_a, m_b, m_c$ by
\[
\cos A = \frac{2(a + \frac{1}{2})^2 m_c + m_a [(a + \frac{1}{2})^3 - (a + \frac{1}{2})^2 - (b + \frac{1}{2})^2]}{((a + \frac{1}{2})^2 - m_a^2) [4(a + \frac{1}{2})^2 (a + \frac{1}{2})^2 - (a + \frac{1}{2})^2 - (b + \frac{1}{2})^2]^{\frac{3}{2}}}, (2.7)
\]
\[
\cos D = \frac{(a + \frac{1}{2})^2 - (b + \frac{1}{2})^2 - (c + \frac{1}{2})^2 - 2 m_a m_c}{((b + \frac{1}{2})^2 - m_a^2) [(c + \frac{1}{2})^2 - m_a^2]^{\frac{3}{2}}}, (2.8)
\]
\]
\[
\cos B, \cos C, \cos E, \cos F \text{ are deduced respectively from (2.7), (2.8) by circular permutations of the labels, } a, b, c; \text{ note that } D + E + F = 2\pi. \text{ When}
\]

![Fig. 2. Limit case in which a 6j-symbol degenerates into a 3j-symbol.](image)

$m_a = m_b = m_c = 0$, the plane $P_1 P_2 P_3$ is perpendicular to $z$ and $A = B = C = \frac{1}{2} \pi$. (2.6) reduces to the known result
\[
\left[\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array}\right] \simeq \frac{1}{2} \left[1 + (-1)^{a+b+c} \right] \left[1 - (-1)^{a+b+c} \right] (2\pi A)^{-\frac{3}{2}}. (2.9)
\]

The 2+2 case can be dealt with in much the same way. Using once more $\zeta = x - 2R$ as summation variable, from (A.4), (2.2) we obtain
\[
\left[\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array}\right] \simeq [a + b - c]! [a - b + c]! [a + e - f]! \times \times \left[(a - e + f)!(d - e + f)!(d - e + f)!(d - b + f)!(f - c - d + e - f)! \times \times (2R)^{-3\alpha - 2d - b - c - e - f - 1} \sum (-1)^{\delta} (2R)^{2\delta} \left[(a + d + b + e - \zeta) \times \times (a + d + c + f - \zeta)! (a - c - b - f)! \times \times (\zeta - a - e - f)! (\zeta - e - d - c)! (\zeta - b - d - f)! \right]^{-1}. (2.10)
\]

For $R$ large the main contribution to this summation comes from the largest allowed value of $\zeta$, i.e. from the minimum between $a + d + b + c$ and $a + d + c + f$; therefore
\[
\left[\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array}\right] \simeq [a + b - c + f]! [a - b + c]! [a + e - f]! \times \times \left[(a - b + c)!(a - e + f)!(d - e + f)!(d - b + f)! \right]^{\text{sign} (c + f - b - e)} \times \times (2R)^{-3\alpha - 2d - b - c - e - f - 1} \left[1 + O(R^{-2}) \right], (2.11)
\]

where sign ($x$) = $+1$, $-1$ according to $x \geq 0$, $< 0$. It must be pointed out that unless $b + c - e + f$, the corresponding tetrahedron becomes hyperflat; in fact it turns out that $144 V^2 = -4(b + c - e - f)^2 R^2 + O(R^4)$.

The remaining particular case $2 + 1 + 1$ will be discussed in some detail in appendix H.

3. Improvement of Wigner asymptotic formula

According to Wigner, the physical interpretation of the $6j$-symbol is clearly related to its definition as a recoupling coefficient
\[
\langle (J_1, J_2, J_3) | J_{123} \rangle = \sum_{J_1} \left[ (2J_{12} + 1) (2J_{23} + 1) \right] \left(-1\right)^{J_{12} + J_{23} + J_{123}} \left(J_1 J_2 J_3 \right) \left(J_{123} \right) \times \langle (J_1, J_2) | J_{123} \rangle. (3.1)
\]
It follows that
\[(2j_2 + 1)(2j_3 + 1) \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_2 & j_3 & j_23 \end{array} \right\}^2 \, dj_{23}, \tag{3.2} \]
is the probability that the sum of the angular momenta \(j_2\) and \(j_3\) has the length in the interval \(j_{23}, j_{23} + dj_{23}\) whenever \(j_1\) and \(j_2\) have sum of length \(j_{12} = j_1 + j_2\) is coupled with \(j_3\) to a vector \(J\) of length \(J\). As anticipated in the introduction, the mutual relationship of these vectors is best seen on a diagram (fig. 3).

![Diagram of coupling scheme](image)

**Fig. 3.** Recoupling scheme corresponding to (3.1).

Let \(j_1, j_2, j_3, j_{12}, J\) (i.e. all the quantum numbers in the l.h.s. of (3.1)) be fixed; then the angle \(\psi\) between the plane of the vectors \(j_1, j_2\) and the plane of the vectors \(j_3, J\) is still undetermined, for both of them can rotate around the common axis \(j_{12}\). During this rotation the point \(P\) describes a circle; we assume that every point of this circle has equal probability. Then the probability that the length of \(j_{23}\) falls into the interval \(j_{23}, j_{23} + dj_{23}\) is just \(2(d\psi/dj_{23})/(2\pi) \, dj_{23}\), the factor 2 being needed because there are two configurations corresponding to \(\psi\) and \(2\pi - \psi\) which yield the same \(j_{23}\).

An elementary computation (appendix B) shows that
\[
d\psi = \frac{j_{12} j_{23}}{6V}, \tag{3.3} \]
where \(V\) is the volume of the tetrahedron in fig. 3. Therefore, for large angular momenta, we obtain Wigner’s result (1.3a). If \(j_{23}\) is such that \(V^2 < 0\), it is impossible to reach the prescribed value of \(j_{23}\) by varying \(\psi\) in the real interval \(0 - 2\pi\).

The arguments given so far are clearly heuristic since they assume as granted a uniform probability distribution in \(\psi\). A rigorous justification of this statement would take us too far and would destroy the simplicity of the discussion.

It is, however, interesting to notice that there is a variation to Wigner’s argument which seems to have escaped detection so far. Let us suppose that in fig. 3 three of the vertices of the tetrahedron are held fixed, while the remaining one \(P\) is allowed to vary. We use \(J^2, j_{23}^2, j_3^2\) as coordinates of \(P\) instead of the usual Euclidean coordinates (i.e., for instance, the components of \(J\): \(J_x, J_y, J_z\)). Notice that there are two points \(P\) corresponding to the same set \(J^2, j_{23}^2, j_3^2\). We assume that the "a priori" probability for \(P\) to lie in the small volume \(dV = dJ_x \, dJ_y \, dJ_z\) does not depend on \(P\). In this case, the probability that the "tricentrical" coordinates of \(P\): \(J^2, j_{23}^2, j_3^2\) lie in the interval \(dJ^2 \, dj_{23}^2 \, dj_3^2\) is
\[
2 \left| \frac{\partial (J^2, j_{23}^2, j_3^2)}{\partial (J_x, J_y, J_z)} \right|^{-1} \, dJ^2 \, dj_{23}^2 \, dj_3^2 = 2\pi \, dJ^2 \, dj_{23}^2 \, dj_3^2, \tag{3.4} \]
and since \[
\left| \frac{\partial (J^2, j_{23}^2, j_3^2)}{\partial (J_x, J_y, J_z)} \right| = 8 |J \times j_{23}, j_3| = 48V, \]
using (1.3a) we have
\[
\left\{ \frac{j_{12} j_{23}}{j_3 J j_{23}} \right\}^2 \sim 2 \frac{\pi}{\gamma}. \tag{3.5} \]

This second argument has the advantage that it can be formally generalized to higher \(3n\)-symbols.

An interesting discussion, which leads to a generalization of Wigner’s formula, can be developed by the combined use of the known results\(^{19}\) for the 1+1+1 case and of the Biedenharn–Elliott identity\(^{17}\). According to Edmonds (eq. (A.2.2) of ref. 16) we have with our conventions of appendix A
\[
\left\{ \begin{array}{ccc} a & b & c+\delta \\ f & b+\delta & a+\delta' \end{array} \right\} \approx \frac{(-1)^{\delta+\delta'+f+g}}{[(2a+1)(2b+1)]^2} \, d\theta, \tag{3.6} \]
where \(a, b, c\) are large in comparison with \(f, \delta, \delta'\) and (see fig. 4)
\[
\cos \theta = \frac{a(a+1) + b(b+1) - c(c+1)}{2[a(a+1)(b+1)]^2}, \quad 0 \leq \theta \leq \pi. \tag{3.7} \]
Let us recall here the just mentioned identity

\[ Y = \left\{ g \ h \ j \right\} \left\{ g \ h \ j \right\} = \sum_{x} (-1)^{x} (2x + 1) \left\{ a \ a' \ x \right\} \left\{ g \ d \ h \ j \right\} \left\{ e \ e' \ h \right\} \left\{ a' \ a \ j \right\}, \]

\[ \varphi_{x} = g + h + j + e + a + d + e' + a' + d' + x. \quad (3.8) \]

Now we intend to use (3.8) under the following conditions:

i) \( g, h, j, e, a, d, e', a', d' \), are large;

ii) \( e' - e = \eta, \quad a' - a = \alpha, \quad d' - d = \delta \) are small with respect to the parameters quoted in i). It follows that the 6j-symbols which appear under summation in (3.8) are of a form suitable for the use of (3.6); we have:

\[ \left\{ a \ a' \ x \right\} \sim \left\{ (2a + 1)(2d + 1) \right\}^\frac{1}{2} d_{a, a}(\gamma), \]

\[ \cos \gamma = \frac{a(a + 1) + d(d + 1) - g(g + 1)}{2[a(a + 1) + d(d + 1)]}, \quad 0 \leq \gamma \leq \pi, \quad (3.9) \]

\[ \left\{ d \ d' \ y \right\} \sim \left\{ (2d + 1)(2e + 1) \right\}^\frac{1}{2} d_{a, a}(\chi), \]

\[ \cos \chi = \frac{d(d + 1) + e(e + 1) - h(h + 1)}{2[d(d + 1)e(e + 1)]}, \quad 0 \leq \chi \leq \pi, \quad (3.10) \]

\[ \left\{ e \ e' \ h \right\} \sim \left\{ (2e + 1)(2d + 1) \right\}^\frac{1}{2} d_{a, a}(\iota), \]

\[ \cos \iota = \frac{e(e + 1) + a(a + 1) - j(j + 1)}{2[e(e + 1) + a(a + 1)]}, \quad 0 \leq \iota \leq \pi. \quad (3.11) \]

If we now replace (3.9)-(3.11) into (3.8) and assume that we may perform the asymptotic limit under the infinite summation on \( x \), we obtain an expression for the product of symbols in the l.h.s. of (3.8) where all angular momenta are large. Clearly this procedure is incorrect, but, nevertheless, it turns out that it is very instructive. We find

\[ \left\{ g \ h \ j \right\} \left\{ g \ h \ j \right\} \sim \sum_{x} (2x + 1) d_{a, a}(\gamma) d_{a, a}(\iota) \quad (3.12) \]

The sum in the r.h.s. can be performed by exploiting the group representation properties of the functions \( d_{a, a}(i) \) as shown in appendix C. The result is

\[ \left\{ g \ h \ j \right\} \left\{ g \ h \ j \right\} \sim \frac{\Theta(V^{2})}{24\pi V} \cos(\eta E + \alpha A + \delta D), \quad (3.13) \]

where \( \Theta(v^{2}) = 0 \) or 1 according to \( v^{2} < 0 \) or \( v^{2} > 0 \); \( V \) is the volume of the tetrahedron \( T \) with edges \( g + \frac{1}{3}, \ h + \frac{1}{3} \), etc. The angles \( E, A, D \) are defined by

\[ \cos E = \frac{\cos \chi - \cos \gamma}{\sin \chi \sin \iota}, \]

\[ \cos A = \frac{\cos \iota - \cos \chi}{\sin \chi \sin \iota}, \]

\[ \cos D = \frac{\cos \gamma - \cos \chi}{\sin \gamma \sin \chi} \quad (3.15a) \]

and \( A \), for instance, can be interpreted as the angle between outer normals of the two faces of \( T \) which have \( a \) as common edge. If \( \delta = \alpha = \eta = 0 \), we find once again Wigner's result.

This procedure is in part disappointing because it fails to yield a complete description of the rapidly oscillating term \( \mathcal{O} \) in (1.3b). However the result (3.13), when \( \delta, \alpha, \eta \not= 0 \), is very illuminating because the r.h.s. contains the interference term \( \cos(\eta E + \alpha A + \delta D) \) which, according to the point of view exposed in section 1, is an average over the product of the two rapidly oscillating factors of the two 6j-symbols.

In order to reconstruct the original expression, let us introduce the function

\[ \Omega(T) = \Omega(a \ b \ c) = \sum_{k} j_{a k} \theta_{k}, \quad (3.15a) \]

with notations defined in section 1. An interesting property of \( \Omega(T) \) is that, with obvious notations, this

\[ \Omega(T + \delta T) - \Omega(T) = \sum_{k} \delta j_{a k} \theta_{k} \quad \text{or} \quad \frac{\partial \Omega}{\partial j_{a k}} = \theta_{k}, \quad (3.16) \]

i.e. we may vary the parameters \( \Omega \) as if \( \theta_{k} \) were constant. Therefore we
deduce readily that
\[
\delta \Omega = \Omega \left( \begin{array}{ccc} g & h & j \\ e + \eta & a + \alpha & d + \beta \end{array} \right) - \Omega \left( \begin{array}{ccc} g & h & j \\ e & a & d \end{array} \right) = \eta E + \alpha A + \delta D,
\]
and the r.h.s. is just the argument of the cosine in (3.13). Notice that \( V \) and \( \delta \Omega \) are slowly varying functions of the edges as compared to \( \Omega \) itself. This result suggests a formula of the following type:
\[
\left[ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right] \simeq \frac{1}{\sqrt{12\pi V}} \cos \left( \Omega' + \omega \right), \tag{3.17}
\]
where \( \omega \) is a yet unknown constant phase. \( \omega \) can be determined by matching (3.17) to the particular 1+1+2 case studied first by Racah\(^1\):
\[
\left[ \begin{array}{ccc} a & b & c \\ b & a & f \end{array} \right] \simeq \frac{(-1)^n b + c + f}{\sqrt{2(a+1)(2b+1)}} P_f(\cos \theta), \tag{3.18}
\]
where \( \cos \theta \) is given by (3.7). If \( f \) is large, but small with respect to \( a, b, c \), we may replace the Legendre polynomial with its asymptotic behaviour\(^1\):
\[
P_f(\cos \theta) \approx \left[ \frac{2}{\pi(f + \frac{1}{2}) \sin \theta} \right]^{\frac{1}{2}} \cos \left( (f + \frac{1}{2})\theta - \frac{1}{4}\pi \right), \tag{3.19}
\]
from which we deduce:
\[
\left[ \begin{array}{ccc} a & b & c \\ b & a & f \end{array} \right] \simeq \frac{(-1)^n b + c + f}{\sqrt{12\pi V}} \cos \left( (f + \frac{1}{2})\theta - \frac{1}{4}\pi \right), \tag{3.20a}
\]
having noticed that \( 6V \approx (a + \frac{1}{2})(b + \frac{1}{2})(f + \frac{1}{2}) \sin \theta \). On the other hand, in order to work out how \( \Omega' \) depends on \( f \), we can write \( \Omega'(f + \frac{1}{2}) \) explicitly for this particular case. We have (fig. 4 with \( \delta = \delta' = 0 \)) \( \Omega' = \pi (a + b + c + \frac{1}{2}) \) and \( (\delta \Omega' / \delta f)_{f + \frac{1}{2} = 0} = \pi - \theta \). Therefore,
\[
\Omega'(f + \frac{1}{2}) \approx \Delta \Omega'(0) + \left( \frac{\delta \Omega'}{\delta f} \right)_{f + \frac{1}{2} = 0} (f + \frac{1}{2}) \approx \pi (a + b + c + \frac{1}{2}) + (\pi - \theta) (f + \frac{1}{2}). \tag{3.21}
\]
Taking into account these results, we see that (3.20a) can be rewritten as
\[
\left[ \begin{array}{ccc} a & b & c \\ b & a & f \end{array} \right] \approx \frac{1}{(12\pi V)^{\frac{1}{2}}} \cos \left( \Omega' + \frac{1}{4}\pi \right), \tag{3.20b}
\]
which shows not only that (3.1) is compatible with this particular case, but also tells us that \( \omega = \frac{1}{4}\pi \). We have reached, therefore, the important general

\[
\begin{align*}
\{a & b & c \} \simeq \frac{1}{(12\pi V)^{\frac{1}{2}}} \cos \Omega, \tag{1.4a} \\
\Omega = \sum_{h,k=1}^{4} J_{h} \theta_{hk} + \frac{1}{4}\pi, \tag{3.15b}
\end{align*}
\]

which must be supplemented with explicit formulae for the angles \( \theta_{hk} \) (appendix B):
\[
\cos \theta_{hk} = -\frac{9}{A_{h} A_{k}} \frac{\partial^{2} V}{\partial \theta^{2}}(\frac{1}{4}\pi), \tag{3.22}
\]
where \( h \neq k \neq r \neq s = 1, 2, 3, 4; A_{h} \) are defined in (1.5).

\begin{table}[h]
\begin{tabular}{cccccc}
\hline
\( a \) & \( b \) & \( c \) & \( d \) & \( e \) & Exact value & Approximate value \\
\hline
1 & 1 & 1 & 1/2 & 1/2 & -13333 -00 & -37828 -00 \\
1 & 1 & 1 & 1/2 & 1/2 & 16667 -00 & 16668 -00 \\
7/2 & 7/2 & 7/2 & 12/2 & 5/2 & -47185 -01 & -47520 -01 \\
17/2 & 15/2 & 10 & 15/2 & 15/2 & 16494 -01 & 16422 -01 \\
13/2 & 8 & 9/2 & 13/2 & 6/2 & 25518 -01 & 25506 -01 \\
5 & 8 & 8 & 9 & 7 & -22441 -01 & -22422 -01 \\
9 & 9 & 9 & 9 & 9 & -15647 -01 & -15600 -01 \\
1/2 & 1 & 1/2 & 1/2 & 1/2 & 16667 -00 & 16026 -00 \\
7/2 & 7/2 & 7/2 & 17/2 & 5/2 & -19826 -01 & -19954 -01 \\
4 & 13/2 & 15/2 & 8 & 9/2 & 17/2 & -26020 -01 & -19897 -01 \\
17/2 & 15/2 & 9 & 15/2 & 15/2 & -13891 -01 & -13773 -01 \\
17/2 & 15/2 & 10 & 15/2 & 13/2 & 12 & 99633 -02 & 97944 -02 \\
5 & 8 & 8 & 9 & 7 & -21100 -01 & -19971 -01 \\
8 & 9 & 9 & 9 & 8 & -13420 -01 & -13295 -01 \\
9/2 & 5 & 1/2 & 11/2 & 6 & 3/2 & 13559 -01 \\
17/2 & 6 & 9/2 & 17/2 & 6/7 & 10 & -10386 -02 & -10411 -02 \\
8 & 9 & 13 & 9 & 7 & -19671 -02 & -19726 -02 \\
17/2 & 15/2 & 10 & 15/2 & 13/2 & 14 & 49191 -03 & 49301 -03 \\
8 & 9 & 13 & 9 & 7 & -11052 -04 & -10443 -04 \\
8 & 9 & 17 & 9 & 8 & -59756 -06 & 56161 -06 \\
8 & 9 & 17 & 9 & 8 & -31622 -09 & 28579 -09 \\
\hline
\end{tabular}
\end{table}
4. Arguments in favor of the proposed asymptotic formula

We list here some arguments in favour of (1.4).

i) Our formula is numerically accurate, as can be seen from tables 1–4 and figs. 5–7. On the average, the accuracy improves as the values of the angular momenta increase.

<table>
<thead>
<tr>
<th>Table 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numerical results for the 6j-symbol with ( a = 7, b = 8, c = 9, d = 6, e = 9 ) as ( f \equiv J ) assumes all permissible values</td>
</tr>
</tbody>
</table>

| \( J \) | Exact value | Approximate value | \( J \) | Exact value | Approximate value |
|------|---------|----------------|------|---------|----------------|  |
| 2    | .76018-02 | .76923-02 | III | 9       | .20540-01 | .20578-01 |  |
| 3    | -.22627-01 | -.22171-01 | II  | 10      | -.31576-02 | -.36479-02 |  |
| 4    | .29469-01  | .30474-01  | I   | 11      | .20586-01  | .20528-01  |  |
| 5    | -.13704-01 | -.13212-01 | I   | 12      | .20068-01  | .19019-01  |  |
| 6    | -.35291-01 | -.33944-01 | I   | 13      | .84777-02  | .78298-02  |  |
| 7    | .20388-01  | .20345-01  | I   | 14      | .16637-02  | .16638-02  | III |
| 8    | .34782-02  | .37774-02  | I   |          |              |              |  |

<table>
<thead>
<tr>
<th>Table 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case with ( a = 13, b = 15, c = 24, d = 29/2, e = 33/2 ) and ( f \equiv J ) variable. The exact values of this table as well as of table 4 were obtained by means of the recursion formula (4.3)</td>
</tr>
</tbody>
</table>

| \( J \) | Exact value | Approximate value | \( J \) | Exact value | Approximate value |
|------|---------|----------------|------|---------|----------------|  |
| 7/2  | -.17136-01 | -.18812-01 | I   | 35/2   | -.66442-02 | -.66033-02 | II |
| 9/2  | -.94803-02 | -.94381-02 | I   | 37/2   | -.38116-02 | -.37515-02 | II |
| 11/2 | .59559-02  | .60861-02  | I   | 39/2   | -.17980-02 | -.18090-02 | II |
| 13/2 | .11188-01  | .11222-01  | I   | 41/2   | -.20668-03 | -.23689-03 | III |
| 15/2 | .25847-02  | .25778-02  | II  | 43/2   | -.23231-02 | -.23652-02 | III |
| 17/2 | .78677-02  | .78773-02  | II  | 45/2   | -.63708-04 | -.64401-04 | III |
| 19/2 | .85291-02  | .85619-02  | II  | 47/2   | -.14450-04 | -.14536-04 | III |
| 21/2 | .13122-03  | .13186-03  | I   | 49/2   | -.26704-05 | -.26756-05 | III |
| 23/2 | .77958-02  | .77680-02  | I   | 51/2   | -.39262-06 | -.39197-06 | III |
| 25/2 | .81257-02  | .82023-02  | I   | 53/2   | -.44251-07 | -.43997-07 | III |
| 27/2 | .17590-02  | .17590-02  | I   | 55/2   | -.35949-08 | -.35557-08 | III |
| 29/2 | .56229-02  | .54326-02  | I   | 57/2   | -.18760-09 | -.18332-09 | III |
| 31/2 | .95185-02  | .94911-02  | II  | 59/2   | -.47264-11 | -.44115-11 | III |
| 33/2 | .92382-02  | .87903-02  | II  |          |              |              |  |

ii) (1.4) is obviously invariant under the exchange of the vertices of T. As stated before, this is only a subgroup of the full symmetry group \( R_3 \) 6j-symbols (appendix A). We checked, however, in a somewhat laborious way, that both \( V \) and \( \Omega \) are actually invariant under \( R_3 \). The proof is sketched in appendix D.

iii) It has been shown\(^{20}\) that the identities (A.6), (A.7), (2.8) together with the tetrahedral symmetries, are enough to derive all properties as well as the numerical values of 6j-symbols, apart from an overall phase. In particular, the Biedenharn-Elliott identity and the recursion relations which flow from it are a distinctive feature of Racah coefficients. Therefore it is a highly significant result that our formula satisfies asymptotically not only these recursion relations, but also the above mentioned identities.

An intuitive understanding of these formulae can be reached by a correspondence principle of the form:

\[
\frac{1}{i} \frac{\partial}{\partial \theta_{\text{th}}} \sim \theta_{\text{th}},
\]

where \( \theta_{\text{th}} \) are the angles appearing in \( \Omega \). By the same token we define the
Let us consider

\[
\begin{aligned}
&\left( a + b + c + 2 \right) (a - b + c + 1) (a + b - c + 1) \times \\
&\times (b + c - a)(a + e + f + 2)(a - e + f + 1)(a + e - f + 1) \times \\
&\times (e + f - a) \left\{ \begin{array}{l}
a + 1 \\
d & e & f
\end{array} \right\} + (a + 1) \left\{ \begin{array}{l}
a + b + c + 1 \\
d & e & f
\end{array} \right\} \\
&\times (a - b + c)(a + b - c)(b + c - a + 1)(a + e + f + 1) \times \\
&\times (a - e + f)(a + e - f)(e + f - a + 1) \left\{ \begin{array}{l}
a - 1 \\
d & e & f
\end{array} \right\} = \\
&= (2a + 1) \left\{ \begin{array}{l}
[ a(a + 1) d(d + 1) - b(b + 1) e(e + 1) - c(c + 1) \\
& e(f + 1)] + [a(a + 1) - b(b + 1) - c(c + 1)] \right\} \left\{ \begin{array}{l}
a b & c \\
d & e & f
\end{array} \right\} \\
&\sim 2 \cos \theta \left\{ \begin{array}{l}
a b & c \\
d & e & f
\end{array} \right\} .
\end{aligned}
\]

(4.3a)

Fig. 5. Here, as well as in figs. 6 and 7, the interpolation between contiguous physical points in the classically forbidden regions is based on (1.8). For numerical values, see table 2. According to the numerical tables, curves labelled with I, II, III correspond to the use of (1.4), (5.6), or (5.7), (1.8) respectively.

Operator

\[
\mathcal{D}_{J_{MN}} = \exp \left( \frac{1}{2} \frac{\partial}{\partial \theta_{J_{MN}}} \right) \sim \exp \left( \frac{1}{2} \theta_{J_{MN}} \right),
\]

so that, for instance:

\[
\mathcal{D}_a^2 \left\{ \begin{array}{l}
a & b & c \\
de & e & f
\end{array} \right\} = \left\{ \begin{array}{l}
a + 1 & b & c \\
de & e & f
\end{array} \right\}
\]

and from (4.1)

\[
\left( \mathcal{D}_a^2 + \mathcal{D}_a^{-2} \right) \left\{ \begin{array}{l}
a & b & c \\
de & e & f
\end{array} \right\} = \\
\left\{ \begin{array}{l}
a + 1 & b & c \\
de & e & f
\end{array} \right\} + \left\{ \begin{array}{l}
a - 1 & b & c \\
de & e & f
\end{array} \right\} \sim 2 \cos \theta \left\{ \begin{array}{l}
a & b & c \\
de & e & f
\end{array} \right\} .
\]

(4.2)

Fig. 6. In this case, which corresponds to table 3, the exponential decrease is particularly emphasized.
In a similar way, the recursion relation

\[
[(a + b + c + 1)(b + c - a)(c + d + e + 1)(e + d - e)]^3 \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} + \\
- [(a + b - c + 1)(a - b + c)(e + d - c + 1)(e + c - d)]^3 \times \\
\begin{bmatrix} a & b & c - 1 \\ d & e & f \end{bmatrix} = -2c[(b + d + f + 1)(b + d - f)]^3 \times \\
\begin{bmatrix} a & e & f \\ d - \frac{1}{2} & b - \frac{1}{2} & c - \frac{1}{2} \end{bmatrix}
\]

by means of (1.4) becomes asymptotically

\[
[(j_{13} + j_{12} + j_{14})(j_{13} + j_{14} - j_{12})(j_{14} + j_{34} + j_{34}) \times \\
\times (j_{34} + j_{34} - j_{32})]^3 e^{i(\theta_{12} + \theta_{14} + \theta_{34})} - [(j_{12} + j_{13} - j_{34}) \times \\
\times (j_{12} + j_{13} - j_{14})(j_{14} + j_{34} - j_{12})(j_{14} + j_{14} - j_{34})]^3 e^{i(\theta_{12} - \theta_{14} + \theta_{34})} \simeq \\
\simeq -2j_{14}[(j_{13} + j_{34} + j_{34})(j_{13} + j_{34} - j_{32})]^3,
\]

which, using Delambre's relations, reduces to simple identities.

A much more involved computation is needed to show that also the full Biedenharn-Elliott identity is satisfied asymptotically by (1.4). Introducing (1.4) into the r.h.s. of (3.8), we realize that inside the summation over \(x\) there appears a rapidly varying function of \(x\). Its behaviour can be displayed most transparently if we split all cosines into positive and negative frequency parts according to Euler's formula. Let \(T_j (j=1,2,3)\) be the tetrahedra corresponding to the r.h.s. of (3.8); with obvious notations we have

\[
\prod_{j=1}^{3} \cos \Omega_j = 2^{-3} \times \\
\times \left[ e^{i(\theta_{11} + \theta_{12} + \theta_{13})} + e^{i(\theta_{11} + \theta_{12} - \theta_{13})} + e^{i(\theta_{11} - \theta_{12} + \theta_{13})} + e^{i(\theta_{11} - \theta_{12} - \theta_{13})} + \text{c.c.} \right].
\]

Given the heuristic character of our investigation, it is reasonable to assume that the discrete summation over \(x\) can be replaced with an integration whose most important contribution arises from points where the phase \(\Gamma(x) = \Omega_1(x) + \Omega_2(x) + \Omega_3(x)\) and its analogs of (4.4) are slowly varying as functions of \(x\). We are led quite naturally to consider, for example

\[
\frac{\partial \Gamma(x)}{\partial x} = 0.
\]

Denoting the supplementary dihedral angles relative to \(x\) with \(\theta_{11}^2, \theta_{12}^2, \theta_{13}^2\), a heartening result is that, since in any first order variation of the edges we
may consider all angles as constants, we have as necessary condition for (4.5)
\[ \theta_1^2 + \theta_2^2 + \theta_3^2 - \pi = 0, \]
(4.6)
having taken into account the phase \((-1)^{\infty}\) of (3.8). If we consider the other terms in (4.4) as well, we find that the bulk of the contribution to the integral may come only from those values of \(x\) such that
\[ \pm \theta_1^2 \pm \theta_2^2 \pm \theta_3^2 = \pi. \]
(4.7)
The conditions (4.7) have an immediate geometrical interpretation if we look at the diagram in fig. 8a. There one sees that by leaving out in turn any one of the five points \(P_1, \ldots, P_5\), the remaining points form five tetrahedra \(T_1, \ldots, T_5\) which are just those appearing in (3.8). Note that there are ten edges connecting five points in all possible ways and in fact there are ten angular momenta appearing in (3.8) including \(x\). As it has been long known, if five points are imbedded into a three dimensional-Euclidean space, their mutual distances are not independent. The explicit form of their dependence was discovered by Cayley\(^{11}\) and can be written as
\[
-2\pi^2 (41)^2 = \begin{vmatrix}
0 & j_{13}^2 & j_{13}^2 & j_{14}^2 & j_{15}^2 & 1 \\
j_{13}^2 & 0 & j_{23}^2 & j_{24}^2 & j_{25}^2 & 1 \\
j_{14}^2 & j_{24}^2 & 0 & j_{34}^2 & j_{35}^2 & 1 \\
j_{15}^2 & j_{25}^2 & j_{35}^2 & 0 & j_{45}^2 & 1 \\
1 & 1 & 1 & 1 & 1 & 0
\end{vmatrix} = 0,
\]
(4.8)
where \(j_{nk}\) is the distance between \(P_n\) and \(P_k\). It is crucial to understand that (4.8) is in fact equivalent to (4.7). Indeed (4.7) implies that the sum of the internal dihedral angles \(\pi - \theta_k^2\) around the edge \(x\) is a multiple of \(2\pi\), as expected if the diagram is drawn in three dimensions. The ambiguity in the signs of the angles arises from the different possible orientations of the five involved tetrahedra, as exemplified by fig. 8b.

Therefore, the only asymptotical contribution to the integral comes from the configuration of the diagram in fig. 8a and the like, which are three-dimensional, i.e. from those values of \(x\) such that \(J(x^2) = 0\). First we notice that the range of summation is restricted to \(x > 0\) and, "a fortiori," to \(x^2 > 0\). Secondly, \(J(x^2)\) is a quadratic polynomial in \(x^2\) and it has therefore two roots \((x_+)^2\) and \((x_-)^2\). There are rigorous arguments showing that, if \(T_4, T_5\) are physical tetrahedra, i.e. \(V_1^2, V_2^2 > 0\), then \(x_+\) and \(x_-\) are real. To us it will be enough to note that, quite obviously, the smaller root \(x_-\) corresponds to the configuration in which \(P_4, P_5\) lie on the same side of the plane.
P_1, P_2, P_3, as depicted in fig. 9, while \( x_+ \) corresponds to \( P_4, P_5 \) being on opposite sides as in fig. 8a.

Since in general \( \theta^j_i (j = 1, 2, 3) \) are not vanishing, or are equal to some multiple of \( \pi \), only one of the choices of signs in (4.7) is valid for a given root.

It follows that, for \( x = x_+ \), only one of the eight terms arising from the decomposition (4.4), together with its complex conjugate, does actually contribute to the integral. We cannot decide here which term contributes, because this will depend on the values of the other fixed angular momenta. Let us suppose they are such that for both roots \( \theta_1^1 + \theta_2^2 + \theta_3^3 = \pi \).

From (3.8) we have

\[
Y \approx \frac{\pi^{-\frac{3}{2}}}{96 \sqrt{3}} e^{-i\pi(x + \frac{1}{2})} R^2 \int_\mathbb{C} d\xi \xi [V_1(\xi) V_2(\xi) V_3(\xi)]^{-\frac{1}{2}} e^{iR(\xi^2 - x^2)} + \text{c.c.}
\]

(4.11)

having used the fact that \( \omega + x \) is integer; here \( \Gamma(x) = R\Gamma(\xi) + \frac{1}{2} \pi \). In the limit \( R \to \infty \) this integral can be computed with the steepest descent method (4.10), which yields for an integral of the form

\[
\mathcal{F} = \int g(\xi) e^{iRf(\xi)} d\xi,
\]

(4.12)

with \( f \) and \( g \) real, the approximate result

\[
\mathcal{F}_{R \to \infty} \approx \sum_j \left( \frac{2\pi}{R \left| f''(\xi_j) \right|} \right)^{\frac{1}{2}} \theta(\xi_j) e^{iRf(\xi_j) + \frac{1}{2} \pi};
\]

(4.13)

here \( \xi_j \) are such that \( f''(\xi_j) = 0 \) and \( \pi < \xi_j < \beta \). In (4.13) the phases \( \pm \frac{1}{2} \pi \) must be chosen according to \( f'(\xi_j) \geq 0 \). If \( \xi_\infty \) correspond to \( x_\infty \), since we have supposed

\[
\left[ \frac{\partial \left\{ \frac{\Gamma(\xi) - \pi \xi}{\xi} \right\}}{\partial \xi} \right]_{x = x_\infty} = [\theta_1^1 + \theta_2^2 + \theta_3^3 - \pi]_{x = x_\infty} = 0,
\]

(4.14)

we find in our case

\[
f(\xi) = \widetilde{f}(\xi) - \pi \xi; \quad g(\xi) = \xi [V_1(\xi) V_2(\xi) V_3(\xi)]^{-\frac{1}{2}},
\]

\[
f''(\xi) = \frac{\partial \left( \theta_1^1 + \theta_2^2 + \theta_3^3 - \pi \right)}{\partial \xi}.
\]

(4.15)

A naive computation of \( f''(\xi) \) is out of question because of the lengthy and uninspiring algebra involved. We rather take \( P^2(x^2) \), defined in (4.8) as independent variable and write

\[
f''(\xi_\infty) = \lim_{\xi \to \xi_\infty} \left[ \frac{\partial \left( \theta_1^1 + \theta_2^2 + \theta_3^3 - \pi \right)}{\partial P^2(x^2)^3} \right];
\]

(4.16)

some manipulation of determinants (appendix D) shows that

\[
f''(\xi_\infty) = \mp \frac{1}{8} R^3 (\xi_\infty)^3 \left[ \begin{array}{c} V_1 V_5 \\ V_2 V_3 \end{array} \right]_{\xi = \xi_\infty}.
\]

(4.17)
The evaluation of \( Rf(\xi_2) \) is straightforward and yields
\[
Rf(\xi_2) \equiv \frac{1}{4\pi} = \Omega_4 \pm \Omega_5 + \pi (\omega - \frac{1}{2}).
\]
(4.18)

From (4.11)-(4.18) we obtain
\[
Y \approx \frac{1}{4} \frac{1}{12\pi (V_4 V_5)\xi} \left[ e^{i(\vartheta_4 + \vartheta_5)} + e^{i(\vartheta_4 - \vartheta_5)} + \text{c.c.} \right]
\]
(4.19)
\[
Y \approx \frac{1}{12\pi V_4} \cos \Omega_4 \frac{1}{12\pi V_5} \cos \Omega_5,
\]
which is in obvious agreement with the straightforward use of (1.4) in the l.h.s.

We indulged somewhat more than strictly necessary on the proof of the Biedenharn–Elliott identity in the asymptotic limit, because we felt that the mechanism involved is illuminating and more general than shown by this case. We do not discuss whether the other identities of Racah coefficients: (A.6), (A.7) are satisfied by (1.4), since these proofs follow quite easily from the stationary-phase method. The same procedure can be extended in principle to the computation of asymptotic \( j \)-symbols.

5. A formal analogy with the WKB method

In the previous section we strived to provide as many as possible independent checks and counterchecks for the validity of (1.4) in the region \( A(V^2 > 0) \). As stated in our Introduction, a complete description of the behaviour of a \( j \)-symbol for large angular momenta must include of necessity a similar formula for the region \( B(V^2 < 0) \) and makes it highly desirable to have one for the transitional region \( V^2 = 0 \) as well.

The guiding idea in this section is a formal analogy with a WKB approximation for the solution of a differential equation. This analogy was prompted by the actual existence of a differential equation at least in the \( 1+1+2 \) case, where in fact the symbol can be expressed as a Jacobi polynomial.

Unfortunately we have not been able to derive a single differential equation valid for unrestricted large parameters, with the possible exception of the transitional region. Our disappointment is somehow mitigated by noticing that, after all, we need a differential equation only in order to fix unambiguously the proccision of (1.4) through a transition point. This we have achieved and the resulting formulæ satisfy properties which are similar to the ones listed under i, ii, iii in section 4. However, it must be stated that the WKB analogy is just a formal device which cannot be accepted as a proof.

Before proceeding further along this analogy, a more detailed investigation of the region \( B \) is necessary. From (8.3) we see that if \( V^2 = 0 \) and \( \prod_{a=1}^5 A_a \neq 0 \), then the angles \( \theta_{ab} \) are all multiple of \( \pi \). Since we always choose \( 0 \leq \theta_{ab} \leq \pi \), we have that either \( \theta_{ab} = 0 \) or \( \theta_{ab} = \pi \); the ambiguity can be settled by means of (8.22). Here, as in the following, we always assume that the areas \( A_a \) are represented by positive square roots of radically functions like (8.1).

Because of the conditions satisfied by angular momenta and taking into account our definition of \( f_{ab} \), it is easy to see that these radically functions are always non-negative and may vanish only if \( V^2 = 0 \), in which case (see footnote 12), at least one angular momentum assumes a non-physical value.

If \( V^2 = 0 \), the four vertices of \( T \) lie in the same plane; Let \( Q \) be the convex plane set generated by the four vertices. Depending on their relative position, \( Q \) may have three or four edges (see figs. 10 and 1). As the reader can easily check, the rule is then: \( \theta_{kk} = \pi \) for the edges of \( Q \) and \( \theta_{kk} = 0 \) for the others. We may use the symbol \( \begin{pmatrix} \pi & \pi \\ 0 & 0 \end{pmatrix} \) to denote the set \( \theta = \theta_{13} = \theta_{14} = \pi, \theta_{34} = \theta_{24} = \theta_{23} = 0 \). The only possibilities are \( \begin{pmatrix} \pi & \pi \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}, \begin{pmatrix} 0 & \pi \\ \pi & 0 \end{pmatrix}, \begin{pmatrix} \pi & \pi \\ \pi & \pi \end{pmatrix} \). The subsets with some \( A_a = 0 \) lie on the boundary of the above sets. For instance, the case \( f_{12} = f_{13} + f_{14} \) is the common boundary of \( \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix} \) and, in fact, here \( \theta_{12}, \theta_{13}, \theta_{14} \) are discontinuous.
If $V^2 < 0$, then the angles $\theta_{ab}$ are all of the form $\pi n + i\delta_{ab}$. Because of continuity, it follows that $\mathrm{Re} \theta_{ab}$ are constants in $B$ whenever $\prod_{k=1}^{4} A_k \neq 0$. Therefore we may subdivide $B$ into non-overlapping regions, all points in the same region having the same values of $\mathrm{Re} \theta_{ab}$. If we let $V \to 0$ and $\prod_{k=1}^{4} A_k \neq 0$, we see that $\lim_{V \to 0} \theta_{ab} = \theta_{ab}(V = 0)$ so the value of $\mathrm{Re} \theta_{ab}$ is determined by the limiting process for $V \to 0$ and is the same as listed above. We may subdivide $B$ into regions $B \left( \begin{array}{c} \pi \pi \pi \\ 0 0 0 \end{array} \right)$, $B \left( \begin{array}{c} \pi \pi 0 \\ 0 0 0 \end{array} \right)$, etc. such that we have, for instance, in $B \left( \begin{array}{c} \pi \pi 0 \\ 0 0 0 \end{array} \right)$: $\mathrm{Re} \theta_{12} = \mathrm{Re} \theta_{13} = \mathrm{Re} \theta_{14} = \pi$, $\mathrm{Re} \theta_{23} = \mathrm{Re} \theta_{24} = \mathrm{Re} \theta_{34} = 0$; it is also convenient to use the following abbreviations:

$$B_3 = B \left( \begin{array}{c} \pi \pi 0 \\ 0 0 0 \end{array} \right) \cup B \left( \begin{array}{c} \pi 0 0 \\ 0 \pi 0 \end{array} \right) \cup B \left( \begin{array}{c} 0 \pi 0 \\ \pi 0 0 \end{array} \right),$$

$$B_4 = B \left( \begin{array}{c} \pi 0 0 \\ 0 \pi 0 \end{array} \right) \cup B \left( \begin{array}{c} 0 \pi 0 \\ \pi 0 0 \end{array} \right) \cup B \left( \begin{array}{c} 0 0 \pi \\ 0 0 \pi \end{array} \right).$$

These regions are not closed, for they have common boundary points in which some $A_k$'s vanish. As anticipated in (1.7), in every such region we define a phase function $\Phi$ as follows

$$\Phi = \sum_{k=1}^{4} (J_{hk} - \frac{1}{2}) \mathrm{Re} \theta_{hk}.$$ 

For physical values of the angular momenta, $\Phi$ is always an integer multiple of $\pi$.

We fix now the value of all parameters but one. Without loss of generality we may always choose $a$ as variable and, in order to stress this point, we use $x$ instead of $a$. We do not restrict ourselves to physical values of $x$ and of the remaining parameters in order to retain some, albeit nominal, freedom in the final results. If $V^2(x_2) = 0$ we have

$$\Omega(x_2) = \Phi - \frac{i}{2} \pi \quad \text{approaching } B_3,$$

$$\Omega(x_2) = \Phi + \frac{i}{2} \pi \quad \text{approaching } B_4.$$ 

In appendix F it is shown that for $x \to x_2$:

$$\Omega - \Omega(x_2) = \begin{cases} \frac{9}{2} \left( \frac{V}{n} \right)^3 \quad & \text{approaching } B_3, \\ \frac{9}{2} \left( \frac{V}{n} \right)^3 \quad & \text{approaching } B_4. \end{cases}$$

(5.1)

It follows that in the neighbourhood of a transition point we have

$$\left[ \begin{array}{c} a \\ b \\ c \end{array} \right] \approx \frac{1}{\sqrt{12\pi V}} \cos \left( \frac{9}{2} \left( \frac{V^3}{n} \right) + \Phi - \frac{i}{2} \pi \right),$$

(5.3)

obviously this formula is incorrect if one gets too close to the transition points. We notice however that, as it stands, (5.3) is the WKB asymptotic approximation to the following differential equation (see (6.6)):

$$\frac{d}{d\psi} \left( \frac{d}{d(V^2)} \psi \right) = \left( \frac{27}{4} \prod_{h=1}^{4} A_h \right)^2 V^2 \psi,$$

(5.4)

where the independent variable is $V$, $\prod_{h=1}^{4} A_h$ being treated as a constant.

The general solution of (5.4) is (appendix G):

$$\psi \sim V \left[ c_1 J_3 \left( \frac{9}{2} \left( \frac{V^3}{n} \right) \right) + c_2 J_{-3} \left( \frac{9}{2} \left( \frac{V^3}{n} \right) \right) \right]$$

(5.5)

and the one which joins smoothly with (5.3) for large values of $V^2 > 0$ is:

$$\left[ \begin{array}{c} a \\ b \\ c \end{array} \right] \approx 2^{-\frac{3}{4}} \left( \prod_{h=1}^{4} A_h \right)^{-\frac{3}{4}} \times \cos \Phi A\iota \left[ -\left( \frac{3V^2}{4} \right)^{\frac{3}{2}} \right] + \sin \Phi B\iota \left[ -\left( \frac{3V^2}{4} \right)^{\frac{3}{2}} \right],$$

(5.6)

in terms of Airy functions. We assume (5.6) to be the correct asymptotic form of the Racah coefficient in the transitional region. The soundness of this assumption is of course at this stage purely aesthetic. However, our conjecture is borne out by comparison with published tables (see tables 1–4).

According to the standard procedure, we may continue (5.6) into the region $B$; here the resulting formula in the neighbourhood of a transition point is:

$$\left[ \begin{array}{c} a \\ b \\ c \end{array} \right] \approx 2^{-\frac{3}{4}} \left( \prod_{h=1}^{4} A_h \right)^{-\frac{3}{4}} \times \cos \Phi A\iota \left[ \left( \frac{3\left| V \right|^2}{4} \right)^{\frac{3}{2}} \right] + \sin \Phi B\iota \left[ \left( \frac{3\left| V \right|^2}{4} \right)^{\frac{3}{2}} \right],$$

(5.7)
For large values of $|V|^2$ this function joins smoothly with

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \sim \frac{1}{2(12|V|^2)} \left\{ 2 \sin \phi e^{i|\text{Im} \phi|} + \cos \phi e^{-i|\text{Im} \phi|} \right\},$$

(5.8)

where $\text{Im} \phi = \sum a_k \cdot a_{k+1} \cdot \text{Im} \theta_{ab}$. Choosing conventionally in region $B$: $V = i|V|$, the imaginary part of $\theta_{ab}$ can be retrieved by means of the following formulae:

$$\cosh(\text{Im} \theta_{ab}) = -\cos(\text{Re} \theta_{ab}) \frac{9}{A_k A_k} \frac{\partial V^2}{\partial x}, \quad h \neq k \neq r \neq s,$$

(3.2a)

$$\sinh(\text{Im} \theta_{ab}) = \cos(\text{Re} \theta_{ab}) \frac{3 j_{ab}}{2 A_k A_k} |V|, \quad h \neq k.$$  

(1.5a)

According to (5.8), the 6j-symbol would be represented in region $B$ by a superposition of decreasing and increasing exponentials; however, the coefficient $\sin \phi$ of the increasing exponential vanishes at the physical points, where (5.8) reduces to (1.8). It turns out that (5.8) is numerically accurate when applied to physical angular momenta.

It is worth noticing that the coefficient $\cos \phi = (-1)^{q_{a+c}}$ of the decreasing exponential gives instead a determination of the sign of the 6j-symbol. This sign is in complete agreement with numerical tables, as well as the one obtained from the limiting case of a stretched tetrahedron where one edge reaches its maximum permissible value. As an example, let $a = b + c$; from (A.4) we have:

$$\text{sign of} \left\{ \begin{bmatrix} b & c & b \\ d & e & f \end{bmatrix} \right\} = (-1)^{a+c+e+f}.$$ 

(5.9)

Since $b+c > x_a - \frac{1}{2}$, it is a simple exercise to see that if $a$ increases through $x_a$, then we enter either one of the regions $B\left(\frac{0}{\pi} \frac{\pi}{\pi}, \frac{0}{\pi} \frac{0}{\pi}\right)$, $B\left(\frac{\pi}{\pi} \frac{\pi}{\pi}, \frac{0}{\pi} \frac{0}{\pi}\right)$, $B\left(\frac{0}{\pi} \frac{\pi}{\pi}, \frac{0}{\pi} \frac{0}{\pi}\right)$. The region $B\left(\frac{\pi}{\pi} \frac{\pi}{\pi}, \frac{0}{\pi} \frac{0}{\pi}\right)$ is excluded because the point $a = b + c$ is almost on the boundary between $B\left(\frac{0}{\pi} \frac{\pi}{\pi}, \frac{0}{\pi} \frac{0}{\pi}\right)$ and $B\left(\frac{\pi}{\pi} \frac{0}{\pi}, \frac{0}{\pi} \frac{0}{\pi}\right)$; actually, it would be exactly on this boundary if: $a + \frac{1}{2} = b + \frac{1}{2} + c + \frac{1}{2}$, or $a = b + c = \frac{1}{2}$, which is prevented by selection rules. Therefore our phase is $\Phi = \pi(b+c+e+f) = \pi(a+e+f)$ and it agrees with (5.9).

Further evidence in favour of (5.6), (5.8) is offered by the discussion of the particular case: 2+1+1; this analysis is carried out in appendix H where it is shown that the behaviour of (3.6) in region $A$ as well as in $B$ is in agreement with (5.8). We notice also that the arguments of the Airy functions in (5.6), (5.7) as well as $\Phi$ are invariant under $R_2$.

We have now a complete set of conjectured asymptotic formulae valid for every range of large physical angular momenta. Numerical examples deduced from these formulae are shown in tables 1-4 and in figures 5-7.

6. The 3nj-symbols

The problem of extending our results to higher 3nj-symbols is certainly very difficult and we were not able to reach a solution within the frame of this paper. Yet some of the intuitive arguments presented here provide fairly interesting information about the general problem.

In dealing with 3nj-symbols where $n$ is large the use of diagrams becomes imperative. At first the diagrams, as in the current literature, are just a mnemonical device in order to keep track of the growing complexities of the symbols. They provide an information, which is purely combinatorial, on how angular momenta are coupled in a given scheme. In this sense a very natural language for diagrams is provided by combinatorial topology.

A diagram is essentially a shorthand notation for the expansion of a 3nj-symbol in terms of 3j-symbols. Let $[D]$ be the 3nj-symbol corresponding to the diagram $D$. $[D]$ can be expressed as:

$$[D] = \sum_{\text{all}} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \cdots \begin{pmatrix} l_k & l_{k+1} & l_{k+2} \\ m_k & m_{k+1} & m_{k+2} \end{pmatrix} \begin{pmatrix} l'_k & l_p & l_q \\ m'_k & m_p & m_q \end{pmatrix},$$

(6.1)

where

$$\begin{pmatrix} l'_k & l_p & l_q \\ m'_k & m_p & m_q \end{pmatrix} = (-1)^{k-w} \delta_{m'_k, -m_k}.\quad \text{(6.2)}$$

D can be retrieved from (6.1) by means of the following rules:

a) $D$ is a 2-dimensional combinatorial manifold.

b) There is a one-to-one correspondence between 1-simplexes (edges) of $D$ and angular momenta in the r.h.s. of (6.1).

c) There is a one-to-one correspondence between 3j-symbols in (6.1) and 2-simplexes (faces) of $D$.

d) The boundary of a face is the sum of the three edges appearing as angular momenta in the corresponding 3j-symbol.

e) We work for the time being with homology modulo 2, i.e. we forget about the orientation of the simplexes.

From the general structure of $[D]$ we see that there are $2n$ faces and $3n$ edges in $D$. We have no "a priori" conditions on 0-simplexes (vertices) and in
fact they have no physical meaning. This brings in a certain amount of arbitrariness in the construction of $D$ which is lacking in the customary definition of diagrams (figs. 12a, b). We regard as equivalent diagrams those which yield the same symbol. Since $D$ is a manifold, we may define its Euler characteristic:

$$M = -f + e - v + 2 = n - v + 2,$$

where $f$, $e$, $v$ are the number of faces, edges, vertices respectively. Since $M \geq 0$, we have $v \leq n + 2$. For a $6j$-symbol, $M = 0$, while in the case of a $9j$-symbol we have: $n = 3$, $v = 4$ and $M = 1$ according to the example sketched in fig. 11. When $M = 1$ the diagram is the triangulation of a one-sided surface,

![Diagram](image)

Fig. 11. Planar graphical representation of the $9j$-symbol with triads $(abc)$, $(def)$, $(ghi)$, $(aef)$, $(beh)$, $(cfl)$.

In our case the real projective plane. Since the sphere where opposite points are identified is a homeomorph of the projective plane, it is possible to exhibit the $9j$-symbol as a double hexagonal pyramid as shown in fig. 12a; here each edge and face are repeated twice and the whole diagram has a centre of symmetry. We shall prefer this representation to the one given in fig. 11.

![Diagram](image)

Fig. 12. Different three-dimensional diagrams for the $9j$ of fig. 11; note that the edges in case a have the same length and direction as in b.

such a construction exists with prescribed lengths of the edges, we shall speak of a configuration of the diagram.

An interesting phenomenon is that there are several different configurations with identical edges for the same diagram. This ambiguity is connected with the fact that there is uniqueness for a given polyhedron of given topology and given edges only if the polyhedron is convex. A trivial example is a
pair of mirror-like right-handed and left-handed tetrahedra. A less trivial
example arises already with the 9j-symbol and we expect it to become more
and more important for higher symbols. The technical reasons of this
multiplicity can be best seen in the 9j-symbols. Here the geometrical shape
of the object would be completely determined if we knew all the relative
angles of all the edges of the diagram. Elementary theorems tell us that this
can be achieved if the angle we are looking for is the internal angle of some
triangle of which all edges are known; this is true in particular for all faces
in the diagram. A similar attempt to compute, for instance, the angle be-
tween \( b \) and \( f \) (fig. 12a) fails unless we know the length of the diagonal \( x \).
In the particular case of the 9j-symbol it turns out that all angles can be
computed, provided we know this only missing length \( x \); as we shall see,
there may be in principle as much as four different configurations with the
same topology.

Since the classification and the discussion of these configurations is rele-
vant to the asymptotic behaviour of the symbol, it is convenient to refine
the so far used language. We shall introduce the word diagram when only
the topological properties are considered and in doing so we identify equi-
valent diagrams. Configuration is instead a diagram with the additional
information about the angles needed in order to remove the above ambigu-
ities. We may introduce the additional word orientation if a distinction
between left-handed and right-handed configurations is desired.

For complicated 3nj-symbols the number of different configurations for
the same diagram grows very rapidly. From our discussion it is clear that if
one gives the distance between any pair of vertices in the diagram, then
the configuration is completely determined. We cannot give here a general set of
rules which would allow one to compute the missing lengths. We found,
however, that in the simplest cases it is enough to exploit the relations
among squared edges which can be obtained as follows:
a) the diagram may contain quadrilaterals with opposite equal edges; in
this case the sum of the squared diagonals is twice the sum of the two dif-
ferent squared edges, a known and elementary relation;
b) since the diagram is imbedded into a 3-dimensional space, one can write
for any choice of five vertices the Cayley identity (4.8).

Not all these relations are actually independent. Once a complete and
consistent set of identities has been written, one finds a set of algebraic equa-
tions for the missing lengths; to each solution of these equations we asso-
ciate a configuration.

We come now to the general problem of computing the 3nj-symbols.

In the available literature\(^7\) explicit formulae are written for any symbol
with the aid of diagrams in terms of lower order coefficients. The diagrams
used in these previous works are not the same as ours; however it is not
difficult to translate one language into the other.

To this purpose we introduce another diagram \( \mathcal{D}(D) \) which corresponds
to the familiar procedure of dissecting the interior of the polyhedron into
tetrahedra; more formally, \( \mathcal{D}(D) \) can be conveniently defined as a 3-di-
ensional combinatorial manifold with boundary \( D \). We name cells the 3-sim-
plexes of \( \mathcal{D} \). The edges, faces and vertices of \( \mathcal{D} \) will be named external if they
belong to \( D \), else internal. Let the set of cells be labelled by \( T_k \), \( k = 1, 2, \ldots, p \).

From the definition of the symbol \([D]\), we know that it is a function of as
many variables as different edges of \( D \). These variables take up integer or
half-integer values with the selection rule that the sum of the variables along
the boundary of any 2-cycle is always integer. We assume the function \([D]\)
to be given by the usual Racah formula (A.4) when \( D \) is a tetrahedron. In
what follows we shall give a sketchy account of a set of rules which allow
the computation of \([D]\) for any \( D \).

In order to evaluate \([D]\), we first construct a given \( \mathcal{D}(D) \). We associate
variables \( x_i, i = 1, 2, \ldots, q \) to all internal edges of \( \mathcal{D}(D) \) and variables
\( l_j, j = 1, 2, \ldots, r \) to the external ones. In this way, to each cell \( T_k \), considered
as a diagram, we may associate \([T_k]\) which is clearly a function of the internal
and, possibly, of the external edges of \( \mathcal{D}(D) \). Then we form the product:

\[
A(x_1, \ldots, x_q) = \prod_{k=1}^{p} \left[ T_k \right] \cdot (-1)^q \prod_{j=1}^{q} (2x_i + 1). \tag{6.3}
\]

We found so far no combinatorial rule to construct \( \chi \), which applies to any
diagram. If \( D \) and \( \mathcal{D}(D) \) are homeomorphs of a 2-sphere and of a 3-ball,
then we have in general:

\[
\chi = \sum_{j=1}^{q} \left( n_j - 2 \right) x_j + \chi_0,
\]

where \( n_j \) is the number of tetrahedra belonging to \( x_j \) and \( \chi_0 \) is a fixed phase
chosen in order to make \( \chi \) integer. For simplicity we shall limit ourselves
to this case. Let us now consider the sum over all internal variables

\[
S = \sum_{x_1}^{\ldots} \sum_{x_q} A(x_1, \ldots, x_q). \tag{6.4}
\]

If there are no internal vertices, the sum is finite and \( S = [D] \). On the contrary,
if there are internal vertices, the sum is infinite but it is still possible to
renormalize it in such a way to obtain \([D]\). When the immediate goal is just
the computation of $[D]$ with the simplest possible method, then there is no need to introduce this additional complication, for one can always find a $S(D)$ with no internal vertices. However the more general case is relevant in suggesting a formal analogy with the Feynman summation over histories\(^{39}\) in connection with the theory of relativity; we shall discuss this point later.

Coming back to (6.3), (6.4) and supposing that there are no internal vertices, we may attempt to evaluate the summation in (6.4) using the same methods already tested for the Biedenharn–Elliott identity (section 4). In this case we shall replace each $[T]_q$ with its asymptotic behaviour according to (1.4). Moreover, we shall split each cosine according to Euler’s formula. The function $A$ will then appear as the sum of $2^p$ pairwise conjugate terms.

It is also convenient to replace each factor $(-1)^{p_i q_j}$ with $\exp \left[ \pm i \pi (n_j - 2) x_j \right]$. This procedure, which is clearly correct only for integer $x_j$, can be easily extended also to half-integer summation indices. Therefore $A$ will contain, among the others, a term of the form

$$
\prod_{j=1}^{q} (2x_j + 1) \exp \left\{ i \left[ \left( \sum_{k=1}^{p_j} \theta_j^k \right) - \pi p_j + 2\pi \right] x_j \right\}.
$$

As before, we may try to replace the summation with an integral; we expect that the most important contributions to the integral will arise from the points where the phase is stationary with respect to the $q$ variables $x_j$. Imposing the stationary phase condition, we find:

$$
\sum_{k=1}^{p_j} (\pi - \theta_j^k) = 2\pi,
$$

which means that the sum of internal angles around $x_j$ is just $2\pi$. A discussion similar to the one carried out in section 4 brings to the conclusion that (6.6) implies the existence of a configuration, in the sense defined above, imbedded in a 3-dimensional Euclidean space, where the internal and external lengths are well specified. Because of the lack of internal vertices, the internal edges connect external vertices of the diagram and in fact they are sufficient to specify the configuration completely. A similar discussion can be carried out for the other $2^p - 1$ terms. In this way we see that the final result will be a sum of contributions from each configuration. We do not know of any simple rule to compute the general form of the partial second derivatives:

$$
\frac{\partial \left[ \sum_{k=1}^{p_j} \theta_j^k \right]}{\partial x_j}.
$$

needed in order to carry out, up to the end, the evaluation of (6.4). This lack of knowledge stops us here short of the final result. However the above discussion shows already that the study of configurations is certainly relevant to a complete understanding of the semiclassical limit of the 3nj-symbols.

As anticipated, we point out a curious connection between our asymptotic formulae and a simplified quantization “à la Feynman” occurring in a 3-dimensional Euclidean theory of gravitation. The classical counterpart of this theory is trivial because the Einstein field equation for empty space

$$
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = 0
$$

implies that the complete Riemann tensor vanishes, i.e., the space is flat. However, the connection we point out may be relevant for further generalizations to less academical cases.

We begin by discussing the sum (6.4) when there are internal vertices. As stated, this sum is infinite, but it is rather interesting to see how this infinity actually arises. For simplicity we restrict ourselves to the case when $D$ is a tetrahedron and there is only one internal vertex $P_5$ (fig. 13). In this case (6.4) reads:

$$
S = \sum_{x,y,z,t} (2x + 1)(2y + 1)(2z + 1)(2t + 1)(-1)^{x+y+z+t} \times
$$

$$
\times \left\{ \frac{a b c}{x y z} \right\} \left\{ \frac{e f a}{y z t} \right\} \left\{ \frac{c e d}{z t x} \right\} \left\{ \frac{d e f}{t x y} \right\}.
$$

Fig. 13.
having chosen \( \chi_0 = a + b + c + d + e + f \); the summation is carried out on all \( x, y, z, t \) which are compatible with the selection rules. Without loss of generality we may suppose \( x \) integer; using (3.8) the summation over \( t \) yields:

\[
S = \sum_{x \in \mathbb{Z}} \binom{a + b + c}{d + e + f} (2x + 1)(2y + 1)(2z + 1) \binom{a + b + c}{x + y + z},
\]

(6.10)

and from (A.6):

\[
S = \frac{\binom{a + b + c}{d + e + f}}{2y + 1} \sum_{x \in \mathbb{Z}} \delta_{xy} (2x + 1)(2y + 1),
\]

(6.11)

where \( \delta_{xy} \) is a triangular delta which is equal to unity if \( x, y, e \) satisfy triangular inequalities and zero otherwise. Since

\[
\sum_{y = |x|}^{x+1} \frac{1}{2y + 1} = \int_{|x|}^{x+1} \frac{dt}{2t + 1} = \ln(2x + 1) - \ln(2|x| + 1),
\]

we have

\[
S = \frac{\binom{a + b + c}{d + e + f}}{2y + 1} \sum_{x = 0}^{\infty} (2x + 1)^2, \tag{6.12}
\]

which is infinite and correspondingly meaningless. Let us limit the summation on \( x \) up to \( x = R \), with \( R \) large; in this case we find

\[
\mathcal{A}(R) = \sum_{x=0}^{R} (2x + 1)^2 \approx \frac{4\pi R^3}{3}. \tag{6.13}
\]

This result hints that we may write:

\[
[D] = \binom{a + b + c}{d + e + f} = \lim_{R \to \infty} \left( \mathcal{A}(R) \right)^{-1} \sum_{x, y, z, t < R} (-1)^{y+b+c+d+e+f+t+x+y+z+t} \times
\]

\[
\times (2x + 1)(2y + 1)(2z + 1)(2t + 1) \times \binom{a + b + c}{x + y + z} \binom{d + e + f}{x + y + z} \binom{c + e + d}{x + y + z}.
\]

(6.14)

In fact we have that, for fixed \( x \), the summand vanishes if \( z > x + b, y > x + c, t > x + d \) so that for \( x < \min (R - b, R - c, R - d) \) the limitations \( y, z, t < R \) have no weight. Therefore we assume that the above limit is correct and expect that in general \([D] \) is given by:

\[
[D] = \lim_{R \to \infty} \mathcal{A}(R)^{-P} \sum_{x_{1} < R} \cdots \sum_{x_{n} < R} \mathcal{A}(x_{1}, \ldots, x_{n}) \tag{6.15}
\]

where \( P \) is the number of internal vertices and \( A \) is defined by (6.3).

Now let us suppose that the number of vertices and edges in \( D \) and in \( \mathcal{D}(D) \) is very high. Let also the complex \( \mathcal{D}(D) \) approach a differentiable manifold \( \mathcal{M} \) with boundary \( D \). According to a discussion carried out in a previous paper by one of us\(^2\), the sum \( \sum_{j=1}^{n} \binom{p_{j} + 1}{n} \mathcal{A}_{j} \) approaches the integral \( \frac{8\pi \mathcal{F}(-D)}{\rho} \int_{D} dV \), where \( dV \) is the volume element on \( D \) and \( \mathcal{F}(-D) \) the scalar Riemann curvature of \( \mathcal{M} \). We see therefore that the positive frequency part \( S^{+} \) of \( S \) in some sense looks like:

\[
S^{+} = \frac{1}{\mathcal{F}^{-}\rho} \int_{D \setminus \mathcal{M}} e^{i\mathcal{F}(-D)} d\mu(D), \tag{6.16}
\]

where the summations over the variables \( x_{j} \) have been interpreted as an integration over all the manifolds \( D \) with fixed boundary \( D \). The measure \( d\mu(D) \) is here not defined in any precise mathematical sense since all the discussion carried out so far is clearly heuristic in character. In this form, \( S^{+} \) strongly resembles a Feynman summation over histories with density of Lagrangian \( \mathcal{L} \) as in a 3-dimensional Einstein theory. In a more conventional 4-dimensional theory with pseudo Euclidean metric, the corresponding summation would be\(^{28}\):

\[
S(\Sigma_{1}, \Sigma_{2}) = \int d\mu \exp \left( i \int_{\Sigma_{1}} d^{4}x \sqrt{-g} \right), \tag{6.17}
\]

the integral on the coordinates being performed in the slab between the space-like hypersurfaces \( \Sigma_{1}, \Sigma_{2} \). The other terms, other than the positive frequency part, are related to different orientations of the tetrahedra \( \mathcal{T}_{j} \) and have a similar interpretation, although their precise meaning is still unclear. It is plausible that in the transition to a smooth manifold \( \mathcal{M} \), they will give no essential contribution to the final result.

Finally we report an interesting conjecture over possible extensions of Wigner’s result for the 6j-symobol. For simplicity we limit ourselves to the 9j-symbol, further generalizations being obvious. In the diagram of fig. 12a we keep the vertices \( P_{1}, P_{2}, P_{3} \) of one face fixed; in this way we fix also \( a, b, c \). It can be easily realized that to determine \( D \) completely it is enough to give the points \( P_{4} \) and \( P_{5} \); in fact, from \( P_{4}, P_{5} \) we deduce the symmetry centre \( \frac{1}{2}(P_{1} + P_{2}) \) of \( D \) and from it all the remaining vertices. We may use as coordinates for \( P_{4}, P_{5} \) either six Euclidean coordinates \( r_{4}, r_{5} \) or the six remaining lengths \( d, e, f, g, h, i \). Our conjecture, suggested by (3.9), is that:

\[
\binom{a + b + c}{d + e + f} \sim C \sum_{x_{1} = 1}^{M} \frac{\partial \rho_{r_{4}}^{(n)}}{\partial \rho_{r_{5}}^{(n)}} \left| \rho_{r_{4}}^{(n)} \right| \left| \rho_{r_{5}}^{(n)} \right|, \tag{6.18}
\]

where the summation is carried on all the \( M \) different configurations \( r_{4}^{(n)} \).
with

\[ A(abc) = \frac{(a + b + c)! (b + c - a)! (c + a - b)!}{(a + b + c + 1)!}. \] (A.2)

The positive integers or half-integers \(a, b, c\) satisfy triangular inequalities: \(|a - b| \leq c \leq a + b + c\) etc. and \(a + b + c = 0\); \(a - |a|\) etc. must be natural integers.

If one of these conditions is not fulfilled, the value of the symbol is assumed to be zero. The 72 elements of the symmetry group \(R_i\) of this coefficient are the permutations of lines and/or columns of the square symbol in (A.1) (which yield the phase \(P_i\) with \(S = a + b + c\) and \(P = 1, -1\) according to even, odd permutations), and the exchange of lines with columns. For unitarity and orthogonality of 3j-symbols, see Edmonds.

From the definition (3.1) it turns out that the 6j-symbol is given in terms of 3j-coefficients by:

\[ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \sum_{\varphi \in \Phi} (-1)^{\varphi e + f + d + e + f} \times \]

\[ \times \begin{bmatrix} d & e & f \\ \delta - e & \gamma & a \end{bmatrix} \begin{bmatrix} e & f & a \\ c - \varphi & \alpha & a \end{bmatrix} \begin{bmatrix} f & d & b \\ \varphi - \delta & \beta & \gamma \end{bmatrix}. \] (A.3)

Racah’s treatment of this formula yields:

\[ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = [A(abc) A(aef) A(cde) A(bdf)]^2 \times \]

\[ \times \sum_{x} (-1)^{x} \frac{(a + d + e - x)! (a + e + f - x)! x!}{x!} \times \]

\[ \times (b + c + e + f - x)! (x - a - b - c)! (x - a - e - f)! \times \]

\[ \times \frac{1}{(x - c - d - e)!} (x - b - d - f)! \]^{-1}. \] (A.4)

The symbol is assumed to vanish if anyone of the triads \((abc), (cde), (aef)\), \((bdf)\) does not satisfy triangular inequalities. In addition to the well known symmetries of the associated tetrahedron (fig. 1), the 6j-symbol has also the less evident symmetry:

\[ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \Rightarrow \begin{bmatrix} a + b - c & b + f - d & f + a - e \\ d + b - f & b + c - a & c + d - e \end{bmatrix} = \]

\[ \begin{bmatrix} a + e - f & e + c - d & c + a - b \\ d + e - c & e + f - a & f + d - b \end{bmatrix} \]

\[ \Rightarrow \begin{bmatrix} a \frac{1}{2} (c + f + e - b) & \frac{1}{2} (b + e + f - c) \\ d \frac{1}{2} (c + f + b - e) & \frac{1}{2} (b + e + c - f) \end{bmatrix}. \] (A.5)
which entails that this coefficient is invariant under the 144 elements of a group $R_2$ which correspond to the permutations of lines and/or columns of the $3 \times 4$ pattern in (A.5). In addition to the Biedenharn–Elliott identity (3.8), the $6j$-symbol satisfies also the following relations\(^{39}\):

\[
\sum_x (2x + 1) \begin{vmatrix} a & b & x \\ d & e & f \end{vmatrix} \begin{vmatrix} a & b & x \\ d & e & f' \end{vmatrix} = \frac{\delta_{f'f}}{2J + 1},
\]

\[
\sum_{x} (2x + 1) \begin{vmatrix} a & b & x \\ d & e & x \end{vmatrix} \begin{vmatrix} a & e & f \\ d & e & f \end{vmatrix} = \begin{vmatrix} a & e & f \\ d & e & f \end{vmatrix} \begin{vmatrix} a & e & f \\ d & e & x \end{vmatrix}.
\]

Wigner's $9j$-symbol\(^{4}\), which is proportional to the transformation matrix between different coupling schemes of four angular momenta, is given in terms of $6j$-coefficients by:

\[
\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \sum_{x} (-1)^x (2x + 1) \begin{vmatrix} a & i & x \\ f & b & x \end{vmatrix} \begin{vmatrix} h & d & x \\ b & e & x \end{vmatrix} \begin{vmatrix} a & i & g \\ f & b & c \end{vmatrix} \begin{vmatrix} h & d & e \end{vmatrix}.
\]

Its known symmetries are formally the same as those mentioned above for the $3j$-symbol, with $S = a + b + c + d + e + f + g + h + i$. For a more comprehensive account of relations involving these coefficients, see ref. 6. According to the conventions and notations of ref. 30 for basis vectors, angular momentum operators and Euler angles, the rotation operator

\[
\bar{D}(\alpha \beta \gamma) = e^{-i\alpha \alpha} e^{-i\beta \beta} e^{-i\gamma \gamma}
\]

has matrix elements in the $2J+1$ dimensional representation which can be written as

\[
D_{MM'}^{(J)}(\alpha \beta \gamma) = \langle J M | \bar{D}(\alpha \beta \gamma) | J M' \rangle = e^{-i\alpha M} D_{MM'}^{(J)}(\beta) e^{-i\beta M'},
\]

in terms of the real matrix: $d_{MM'}^{(J)}(\beta) = D_{MM'}^{(J)}(0,0,0)$. The following properties hold

\[
[D^{(J)}(\alpha \beta \gamma)]^* = [D^{(J)}(\alpha \beta \gamma)]^{-1} = D^{(J)}(- \gamma, - \beta, - \alpha),
\]

\[
\sum_{M} D_{MM'}^{(J)} D_{M' M''}^{(J)} = \delta_{M'M''}, \quad \sum_{M} D_{MM'}^{(J)} D_{M' M''}^{(J)} = \delta_{M'M''},
\]

\[
\frac{1}{8\pi^2} \int_0^{2\pi} d\phi \int_0^{2\pi} d\beta \sin \beta \ D_{M'M''}^{(J)}(\alpha \beta \gamma) \ D_{M' M''}^{(J)}(\alpha \beta \gamma) = \frac{\delta_{J, J'} \delta_{M, M'} \delta_{M, M''}}{2J + 1},
\]

in addition to the symmetry

\[
D_{MM'}^{(J)}(\alpha \beta \gamma) = (-1)^{M' - M} D_{MM'}^{(J)}(- \alpha \beta \gamma).
\]

In general\(^{4}\)

\[
d_{MM', M''}^{(J)}(\beta) = \sum_{x} (-1)^x \left\{ [(J - M)! (J + M)! (J - M')! (J + M')!]^{1/2} \times \right.
\]

\[
\left. \frac{(J - M - x)! (J - M' - x)! x! (x + M + M')!}{(J - M - x)! (J - M' - x)! x! (x + M + M')!} \times \right.
\]

\[
\left. \cos \frac{\beta}{2} \cos \frac{M - M'}{2} \sin \frac{x}{2} \right\}^{1/2} (J + M + M')!
\]

which leads in particular to

\[
d_{MM', 0}^{(J)}(\beta) = (-1)^M \left\{ \frac{(L - M)!}{(L + M)!} \right\}^{1/2} P_L^M (\cos \beta);
\]

the following symmetries are very useful

\[
d_{MM', M'}^{(J)}(\beta) = (-1)^{M' - M} d_{-M', M}^{(J)}(\beta) = (-1)^{M - M'} d_{MM', -M}^{(J)}(\pi - \beta).
\]

The connection with Jacobi polynomials is given by

\[
d_{MM', M}^{(J)}(\beta) = (-1)^{M - M'} \left\{ \frac{(J + M)! (J - M)!}{(J + M')! (J - M')!} \right\}^{1/2} \times
\]

\[
\cos \frac{\beta}{2} \cos \frac{M - M'}{2} \sin \frac{\beta}{2} \right\}^{1/2} (J + M + M')!
\]

Appendix B. Elementary geometry of tetrahedra

Heron's formula for the content $A$ of a triangle of edges $j_1, j_2, j_3$ can be written as

\[
A^2 = \frac{1}{15} \left( j_1 + j_2 + j_3 \right) \left( j_1 + j_2 - j_3 \right) \left( j_1 - j_2 + j_3 \right) \left( - j_1 + j_2 + j_3 \right) =
\]

\[
= \frac{1}{15} \left| \begin{array}{ccc} 0 & j_1 & j_2 & j_3 \\ j_1 & 0 & j_2 & j_3 \\ j_2 & j_3 & 0 & 1 \\ j_3 & 1 & 1 & 0 \end{array} \right|.
\]

We have already given in (1.2) Cayley's form for the content $V$ of a tetrahedron. Performing in (1.2) the derivation of $V^2$ with respect to $(j_n)^2$ and denoting the $(r, s)$ algebraic minor of the Cayley determinant with $C_{rs}$, the relation\(^{31}\)

\[
-C_{rs} = 16A_{r}A_{s} \cos \theta_{hk}, \quad r \neq s \neq h \neq k
\]

yields (3.22); in (B.2) $h, k, r, s$ are any permutation of $1, 2, 3, 4$. Using (3.22) and the obvious identity

\[
A_{k} \sin \theta_{hk} = \frac{1}{2} V j_{hk}
\]

(B.3)
we obtain the differentiation formulae

\[ \frac{d\theta_k}{d\gamma_r} = \frac{f_{kr}f_{rs}}{6V}; \quad h \neq k \neq r \neq s \]  

(B.4)

Looking at fig. 1 we see that

\[ V = \frac{1}{6} j_{j_1} j_{j_2} j_{j_3} \sin \phi_{14} \sin \phi_{24} \sin \theta_{12} \]

(B.5)

or

\[ V = \frac{1}{6} j_{j_1} j_{j_2} j_{j_3} \begin{vmatrix} 1 & \cos \varphi_{14} \cos \varphi_{24} \\ \cos \varphi_{14} & 1 \cos \varphi_{34} \\ \cos \varphi_{24} & \cos \varphi_{34} \end{vmatrix}^{\frac{1}{2}} \]

(B.6)

since \( \cos \varphi_{34} = \cos \varphi_{14} \cos \varphi_{24} - \sin \varphi_{14} \sin \varphi_{24} \cos \theta_{12} \). Now, if we define

with reference to \( P_4 \):

\[ 2\sigma_4 = \theta_{12} + \theta_{13} + \theta_{23}, \]

(B.7)

\[ \Sigma_4 = [\sin \sigma_4 \sin (\sigma_4 - \theta_{12}) \sin (\sigma_4 - \theta_{13}) \sin (\sigma_4 - \theta_{23})]^b, \]

(B.8)

then it follows:

\[ 2\Sigma_4 = \sin \theta_{12} \sin \theta_{23} \sin \phi_{24} = \sin \theta_{13} \sin \theta_{23} \sin \phi_{34} = \sin \theta_{12} \sin \theta_{13} \sin \phi_{14}, \]

(B.9)

and

\[ \Sigma_4 \cos \frac{1}{2} \phi_{14} = \sin \sigma_4 \cos (\sigma_4 - \theta_{23}), \]

\[ \Sigma_4 \cos \frac{1}{2} \phi_{24} = \sin \sigma_4 \cos (\sigma_4 - \theta_{13}), \]

\[ \Sigma_4 \cos \frac{1}{2} \phi_{34} = \sin \sigma_4 \cos (\sigma_4 - \theta_{12}). \]

(B.10)

It turns out that \( K = \Sigma_4/A_4 \) is independent of \( h \); therefore, from (B.5), (B.9):

\[ V = \frac{1}{6} \left( \prod_{A_4} \right) \frac{1}{2} K^4. \]

(B.11)

Appendix C. A summation property of Jacobi polynomials

The purpose of this appendix is to prove the relation

\[ \sum_{L=0}^{\infty} (2L + 1) \frac{d^{(L)}_{M,M_1} (\beta_3)}{d\beta_3} \frac{d^{(L)}_{M,M_2} (\beta_1)}{d\beta_1} \frac{d^{(L)}_{M,M_3} (\beta_2)}{d\beta_2} = 2\Theta(\theta) \frac{1}{\pi [d\theta]^3} \cos \left( \sum_{i=1}^{3} M_i \delta_i \right), \]

(C.1)

where \( \Theta \) is the step function used in (3.13), while the angles \( \beta_i, \delta_i (i=1, 2, 3) \) are shown in fig. C1 and

\[ \begin{vmatrix} 1 & \cos \beta_3 & \cos \beta_2 \\ \cos \beta_3 & 1 & \cos \beta_1 \\ \cos \beta_2 & \cos \beta_1 & 1 \end{vmatrix}. \]  

(C.2)

The addition property of the rotation group yields:

\[ D^{(L)}_{M,M_1} (\alpha_2 \beta_3 \gamma_3) = \sum_{M} D^{(L)}_{M,M_1} (\alpha_3 \beta_3 \gamma_3) D^{(L)}_{M,M_2} (\alpha_1 \beta_1 \gamma_1), \]

(C.3)

from which, using unitarity

\[ D^{(L)}_{M,M_1} (\alpha_1 \beta_1 \gamma_1) = \sum_{M} D^{(L)}_{M,M_1} (\alpha_2 \beta_3 \gamma_3) D^{(L)}_{M,M_2} (\alpha_2 \beta_2 \gamma_2), \]

(C.4)

\[ D^{(L)}_{M,M_1} (\alpha_2 \beta_2 \gamma_2) = \sum_{M} D^{(L)}_{M,M_1} (\alpha_1 \beta_1 \gamma_1) D^{(L)}_{M,M_2} (\alpha_2 \beta_2 \gamma_2). \]

(C.5)

By exploiting (C.3)-(C.5) for low values of \( L \), we obtain

\[ \cos \beta_i = \cos \beta_j \sin \beta_k - \sin \beta_j \sin \beta_k \cos \delta_i, \quad i \neq j \neq k = 1, 2, 3 \]

(C.6)

\[ \sin \beta_i / \sin \delta_i = \sin \beta_j / \sin \delta_j, \quad i \neq j \]

(C.7)

where

\[ \delta_1 = \alpha_2 - \alpha_3 + \pi, \quad \delta_2 = \gamma_1 - \gamma_3 - \pi, \quad \delta_3 = \alpha_1 + \gamma_2. \]

(C.8)

By means of (A.13) and (C.3) we find

\[ \int \int d\alpha_3 d\gamma_3 \int d\beta_2 \sin \beta_2 D^{(L)}_{M,M_1} (\alpha_3 \beta_3 \gamma_3) D^{(L)}_{M,M_2} (\alpha_2 \beta_2 \gamma_2) = \]

\[ \frac{8\pi^2}{2L + 1} \delta_{M,M_1} \delta_{M,M_2} D^{(L)}_{M,M_3} (\alpha_1 \beta_1 \gamma_1); \]

(C.9a)
(A.10) and the reality of functions \( d_{M, M'}^{(L)} \), give
\[
\int_0^{2\pi} d\beta_2 \int_0^{2\pi} d\beta_3 \sin\beta_2 \cos\left(\sum_{j=1}^3 M_j \beta_j\right) \frac{d}_{M, M_1}^{(L)}(\beta_2) \frac{d}_{M, M_2}^{(L)}(\beta_3) = \frac{8\pi^2}{2L + 1} (-1)^{M_1 - M_3} \frac{d}_{M, M_1}^{(L)}(\beta_1). \tag{C.9b}
\]
We notice that, once \( \alpha_1, \beta_1, \gamma_1, \beta_2, \gamma_2 \) have been fixed, \( \beta_3, \) and \( \delta_i (i = 1, 2, 3) \) are independent of \( \alpha_2; \) therefore:
\[
\int_0^{2\pi} d\beta_2 \int_0^{2\pi} d\beta_3 \sin\beta_2 \cos\left(\sum_{j=1}^3 M_j \beta_j\right) \frac{d}_{M, M_1}^{(L)}(\beta_2) \frac{d}_{M, M_2}^{(L)}(\beta_3) = \frac{4\pi}{2L + 1} (-1)^{M_1 - M_3} \frac{d}_{M, M_1}^{(L)}(\beta_1). \tag{C.10}
\]
Going from the integration variables \( \beta_2, \gamma_2 \) to \( \cos \beta_2, \cos \beta_3, \) we find:
\[
\frac{\partial(\beta_2, \gamma_2)}{\partial(\cos \beta_2, \cos \beta_3)} = (\sin \beta_2)^{-1} \frac{\partial}{\partial \beta_2}
\]
and
\[
\int_{-1}^{+1} d(\cos \beta_2) \int_{-1}^{+1} d(\cos \beta_3) \frac{\partial}{\partial \beta_2} \cos\left(\sum_{j=1}^3 M_j \beta_j\right) \frac{d}_{M, M_1}^{(L)}(\beta_2) \times \frac{d}_{M, M_2}^{(L)}(\beta_3) = \frac{2\pi}{2L + 1} (-1)^{M_1 - M_3} \frac{d}_{M, M_1}^{(L)}(\beta_1), \tag{C.12}
\]
which are non-vanishing only if \( L \geq \max(|M_1|, |M_2|), L \geq \max(|M_1|, |M_2|) \), we have from (C.12)
\[
\Theta(\beta) \cos\left(\sum_{j=1}^3 M_j \beta_j\right) (-1)^{M_1 - M_3} = \frac{\pi}{2L + 1} \sum_{L=0}^{\infty} (2L + 1) \frac{d}_{M, M_1}^{(L)}(\beta_1) \frac{d}_{M, M_2}^{(L)}(\beta_2) \frac{d}_{M, M_3}^{(L)}(\beta_3).
\]
The symmetries (A.17) lead us to (C.1).
In order to complete the proof of (3.13), we have just to recall (B.6).

**Appendix D. Invariance of \( \Omega \) under \( R_2 \)**

It is straightforward, though very tedious, to verify that the volume of \( T \) is invariant also under the symmetry (A.5). This can be worked out by expanding (1.2) and replacing \( j_{13} \) with \( \frac{1}{2}(j_{14} + j_{24} + j_{23} - j_{13}) \) etc. Incidentally, we note that in the same way one can check also the invariance of \( \prod_{a=1}^{b} \frac{A_a}{A_b} \) under \( R_2 \). Here we prefer to sketch how the symmetry of \( \Omega \) can be proved.

Let us multiply eqs. (B.10) among themselves. We have for instance
\[
(\sin \sigma_2) = \Sigma_a \frac{1}{2} \varphi_{14} \varphi_{24} \varphi_{34}; \tag{D.1}
\]
and then choosing for example the first of (B.10), we have
\[
[ \sin \frac{1}{2}(\theta_{12} + \theta_{13} - \theta_{23}) ]^2 = \Sigma_a \frac{1}{2} \varphi_{14} \varphi_{24} \varphi_{34}. \tag{D.2}
\]
In order to compute the r.h.s. of (D.2) it is convenient to introduce, with reference to fig. D1, the following notations
\[
\begin{align*}
q_1 & = j_{12} + j_{13} + j_{14}, \\
q_2 & = j_{12} + j_{23} + j_{24}, \\
q_3 & = j_{13} + j_{23} + j_{34}, \\
q_4 & = j_{14} + j_{24} + j_{34}.
\end{align*}
\tag{D.3}
\]

![Fig. D1.](image-url)
\[ p_{14} = p_{23} = j_{12} + j_{13} + j_{34} + j_{24}, \quad p_{13} = p_{24} = j_{12} + j_{34} + j_{14} + j_{23}, \]
\[ p_{12} = p_{34} = j_{13} + j_{24} + j_{14} + j_{23}; \quad (D.4) \]
as well as the two patterns
\[ r_m = \begin{bmatrix}
p_{14} - q_1 & p_{13} - q_1 & p_{12} - q_1 & q_1 \\
p_{14} - q_2 & p_{13} - q_2 & p_{12} - q_2 & q_2 \\
p_{14} - q_3 & p_{13} - q_3 & p_{12} - q_3 & q_3 \\
p_{14} - q_4 & p_{13} - q_4 & p_{12} - q_4 & q_4
\end{bmatrix}, \quad \frac{1}{2} \theta_m = \begin{bmatrix}
\sigma_3 - \theta_{23} & \sigma_3 - \theta_{24} & \sigma_2 - \theta_{34} & \sigma_1 \\
\sigma_3 - \theta_{14} & \sigma_3 - \theta_{13} & \sigma_4 - \theta_{34} & \sigma_2 \\
\sigma_3 - \theta_{14} & \sigma_1 - \theta_{24} & \sigma_4 - \theta_{12} & \sigma_3 \\
\sigma_1 - \theta_{23} & \sigma_3 - \theta_{13} & \sigma_2 - \theta_{12} & \sigma_4
\end{bmatrix}. \quad (D.5) \]

We note that from (B.11)
\[ K = \frac{\Sigma_4}{A_4} = \frac{9}{4} \frac{V^2}{\prod_{h=1}^{A_4} A_h}. \quad (D.6) \]

Since, for example
\[ \frac{1}{2} \theta_{14} = \left[ \frac{(p_{14} - q_1)(p_{13} - q_2)}{q_1(p_{14} - q_4)} \right]^\frac{1}{2}, \]
(D.2) becomes
\[ \cos \left( \theta_{14} + \theta_{13} - \theta_{23} \right) = 1 - 2^\frac{3}{2} V^2 \left[ (p_{14} - q_1) \times (p_{13} - q_2) (p_{14} - q_3) (p_{14} - q_4) \right]^{-1} \]
and generally for \( s \neq t = 1, 2, 3, 4 \)
\[ \cos \theta_m = 1 - 2^\frac{3}{2} (31)^2 V^2 \left\{ \left( \prod_{m=1}^4 \frac{r_m}{r_m} \right) \left( \prod_{m=1}^{A_4} r_m \right) \right\}^{-1} \left( r_m \right)^2, \quad (D.7) \]

which gives, correctly, for a flat tetrahedron, \( \theta_m = 0 \) or \( \theta_m = 2 \pi. \)

It is easy to check that, under the symmetry (A.5)
\[ q_1 = q_2, \quad q_2 = q_3, \quad q_3 = q_4, \quad q_4 = q_1; \quad p_{14} = p_{13}, \quad p_{13} = p_{14}, \quad p_{12} = p_{13}, \quad (D.8) \]

which, taking into account account (D.5), (D.7), entails for instance
\[ \begin{align*}
\cos (\theta_{14} + \theta_{13} - \theta_{23}) &= \cos (\theta_{14} + \theta_{13} - \theta_{23}), \\
\cos (\theta_{14} + \theta_{23} - \theta_{34}) &= \cos (\theta_{14} + \theta_{23} - \theta_{12}), \\
\cos (\theta_{14} + \theta_{23} - \theta_{13}) &= \cos (\theta_{14} + \theta_{13} - \theta_{23}).
\end{align*} \quad (D.9) \]

Let us suppose the tetrahedron to be almost regular; asymptotically, this property is not destroyed by the symmetry (A.5); in this case: \( 0 \leq \theta_{ab} \leq \frac{\pi}{16} \) and similarly for the transformed \( \theta_{m}^\prime. \) Moreover, looking at fig. D1 we see for example that \( \theta_{12}, \theta_{13}, \theta_{23} \) satisfy spherical triangular inequalities; therefore \( 0 \leq \theta_m^\prime < \pi, 0 \leq \theta_m < \pi. \) Then from (D.9) and similar relations, we obtain
\[ \begin{align*}
\theta_{14}^\prime &= \frac{1}{2} \left( \theta_{14} + \theta_{24} + \theta_{23} - \theta_{13} \right), \\
\theta_{14} &= \frac{1}{2} \left( \theta_{13} + \theta_{24} + \theta_{24} - \theta_{14} \right), \quad \theta_{12} = \theta_{24}, \\
\theta_{23}^\prime &= \frac{1}{2} \left( \theta_{14} + \theta_{24} + \theta_{23} - \theta_{12} \right), \\
\theta_{23} &= \theta_{24}.
\end{align*} \quad (D.10) \]

We conclude that (A.5) induces the same linear transformation on \( j_{ab} \) as well as on \( \theta_{ab}; \) the unitarity of this transformation entails the invariance of \( Q = \sum_{a,b} \theta_{ab} + \frac{1}{4} \pi \) under \( R_2. \) Finally, the constraint \( \theta_{ab} < \frac{\pi}{16} \) can be dropped by invoking analytical continuation in \( \theta_{ab}. \)

**Appendix E. Evaluation of** \[ \partial (x^4 + \theta_{14}^2 + \theta_{23}^2) / \partial (x^3)|_{x^3 = (x_3)} \]

**According to** (4.16), first let us calculate \( \partial^2 (x^3) / \partial (x^3)|_{x^3 = (x_3)}. \) Write (4.8) as follows:
\[ C = c_2 (x^3)^2 + c_1 (x^2) + c_0 = 0, \quad (E.1) \]
where \(- 2^4 (4!^2) F^2 (x^2) = C; \) then the solutions of (E.1) are obviously
\[ (x_3)^2 = \pm \sqrt{c_1^2 - 4c_0c_2} = \frac{c_1}{2c_2}, \quad (E.2) \]
where \( c_0, c_1, c_2 \) are determinants extracted from \( C \) which, for the sake of brevity, we do not write explicitly. Since the tetrahedra \( T_4, T_3 \) are supposed to be physical, it follows \( (x_3)^2 > 0. \) Then
\[ \left[ \frac{\partial C}{\partial (x^3)} \right]_{x^3 = (x_3)} = \pm \sqrt{c_1^2 - 4c_0c_2}. \quad (E.3) \]

From known properties of determinants \( \left( x^3 \right) \), it turns out that
\[ \frac{1}{2} (c_1^2 - 4c_0c_2) = 2^6 (31)^4 V_4^2 V_5^2, \quad (E.4) \]
therefore
\[ \left[ \frac{\partial^2 (x^3)}{\partial (x^3)^2} \right]_{x^3 = (x_3)} = \mp \frac{V_4 V_5}{16}. \quad (E.5) \]

In order to evaluate \( \partial^2 (x^3) / \partial (\theta_{14}^2 + \theta_{23}^2)|_{\theta_{14}^2 + \theta_{23}^2}, \) suppose to imbed the simplex \( P_1, P_2, P_3, P_4, P_5 \) of fig. 8a into a 4-dimensional Euclidean space.
The content $I$ of this simplex is

$$I = \begin{vmatrix} x & 0 & 0 & 0 \\ 0 & h_{11} & h_{12} & h_{13} \\ 0 & h_{21} & h_{22} & h_{23} \\ 0 & h_{31} & h_{32} & h_{33} \end{vmatrix}^2$$  \hspace{1cm} (E.6)$$

where $h_{ij}, h_{jk}, h_{ik} (i, j, k = 1, 2, 3)$ are the components of the distances from $P_1, P_2, P_3$ to $x$. The direction of $x$ has been chosen as the first axis of reference in this hyperspace. Let $H$ be the volume of the tetrahedron defined by $h_{1}, h_{2}, h_{3}$ (see fig. E1). Then we are led to the simple formula

$$I = \frac{1}{4}xH.$$  \hspace{1cm} (E.7)$$

Now we notice that the angle between, for instance, $h_1$ and $h_2$ is $\pi - \theta_x$. Therefore from (B.6) we have

$$\delta (H^2) = \frac{1}{8} (h_1 h_2 h_3)^2 \left[ \sin \theta_x^1 (\cos \theta_x^2 \cos \theta_x^3 + \cos \theta_x^3) \delta \theta_x^1 + \sin \theta_x^2 (\cos \theta_x^3 \cos \theta_x^1 + \cos \theta_x^1) \delta \theta_x^2 + \sin \theta_x^3 (\cos \theta_x^1 \cos \theta_x^2 + \cos \theta_x^2) \delta \theta_x^3 \right].$$  \hspace{1cm} (E.8)$$

Since (E.8) must be evaluated when $\theta_x^1 + \theta_x^2 + \theta_x^3 = \pi$, i.e. when $h_1, h_2, h_3$ are coplanar, we find

$$\left[ \frac{\partial (H^2)}{\partial (\theta_x^1 + \theta_x^2 + \theta_x^3)} \right]_{\theta_x=0} = \frac{1}{8} (h_1 h_2 h_3)^2 \sin \theta_x^1 \sin \theta_x^2 \sin \theta_x^3.$$  \hspace{1cm} (E.9)$$

Moreover, from (B.5) we obtain

$$V_i = \frac{1}{2} h_i h_2 h_3 x \frac{\sin \theta_x^i}{h_i}, \quad i = 1, 2, 3$$  \hspace{1cm} (E.10)$$

and finally, by means of (E.7)

$$\left[ \frac{\partial I^2 (x^2)}{\partial (\theta_x^1 + \theta_x^2 + \theta_x^3)} \right]_{\theta_x=0} = \frac{3}{4} \left[ \frac{V_1 V_2 V_3}{x} \right]^{\theta_x=0}.$$  \hspace{1cm} (E.11)$$

Relations (E.5) and (E.11) yield (4.17).

**Appendix F. Evaluation of $\lim_{V^2 \to 0} \Omega (V^2)$**

The behaviour of $\Omega$ in the neighbourhood of a transition point deserves a rather careful investigation. To begin with, let us suppose that we approach the region $B_3$ as $V^2 \to 0$. In fig. 10 we show the notations which will be used in the sequel; moreover we indicate with $A, B, C$ the internal dihedral angles between faces belonging to $a+\frac{1}{2}, b+\frac{1}{2}, c+\frac{1}{2}$ and with $\pi - D, \pi - E, \pi - F$ the corresponding ones relative to $d+\frac{1}{2}, e+\frac{1}{2}, f+\frac{1}{2}$. Also the following shorthand notation will be convenient:

$$\mu_{bc} = \cos \alpha, \quad \mu_{bd} = \cos \beta, \quad \mu_{cd} = \cos \gamma, \quad \mu_{ce} = \cos \delta_e, \quad \mu_{de} = \cos \delta_d, \quad \mu_{ef} = \cos \phi_e, \quad \mu_{df} = \cos \phi_d.$$  \hspace{1cm} (F.1)$$

When $P$ lies very near to the plane of the triangle $P_2 P_3 P_C$ and within its boundary we have $\mu_0 = \cos (\delta_e + \delta_d), \mu_0 = \cos (\delta_e + \delta_d)$, and $A = B = C = D = E = F = 0$. Since $\frac{a+b+c+\frac{1}{2}}{4} = \frac{(d+e+f+\frac{1}{2})}{4} \mu_{bc} + \mu_{bd} + \mu_{cd}$ etc., we obtain in general from the definition (3.15b)

$$\Omega = \pi (a+b+c+\frac{1}{2} + (d+\frac{1}{2}) + (e+\frac{1}{2}) + (f+\frac{1}{2})) \Psi_A + (e+\frac{1}{2}) \Psi_E + (f+\frac{1}{2}) \Psi_C,$$  \hspace{1cm} (F.2)$$

where

$$\Psi_A = D - B \mu_{bd} - C \mu_{cd}, \quad \Psi_E = E - C \mu_{de} - A \mu_{ae}, \quad \Psi_C = F - A \mu_{df} - B \mu_{ef}.$$  \hspace{1cm} (F.3)$$

In order to evaluate $\Omega$ when $V^2 \to 0$, it is useful to exploit the following integral representation for $\Psi_A$:

$$\Psi_A = \int_{\mu_{bc} = \mu_{bd} = \mu_{cd} = \Phi} (1 - \Phi^2)^{-1} \left[ 1 + 2 \mu_{bc} \mu_{bd} \Phi^2 - \mu_{cd} - \mu_{bd} \Phi^2 \right] d\Phi,$$  \hspace{1cm} (F.4)$$

and similar ones for $\Psi_B, \Psi_C$. Noticing that $\Psi_A (\mu_{bc} = \mu_{bd} = \Phi) = 0$ and using the relations

$$D = \arccos \left\{ \frac{\mu_{bc} \mu_{bd} \mu_{cd} - \mu_{bd}}{\left( (1 - \mu_{cd}) (1 - \mu_{bc}) \right)^2} \right\}, \quad B = \arccos \left\{ \frac{\mu_{cd} \mu_{bd} \mu_{de} - \mu_{de}}{\left( (1 - \mu_{bc}) (1 - \mu_{de}) \right)^2} \right\}, \quad C = \arccos \left\{ \frac{\mu_{de} \mu_{bc} \mu_{df} - \mu_{df}}{\left( (1 - \mu_{cd}) (1 - \mu_{de}) \right)^2} \right\}.$$  \hspace{1cm} (F.5)
it is easy to verify that the derivatives with respect to $\mu_{bc}$ of both sides of (F.4) are equal. Now let us put
\begin{equation}
1 + 2\mu_{bd}\mu_{cd} - \mu_{bc}^2 - \xi^2 = (\xi - \mu_{bc}) (\mu_{bc} - \xi),
\end{equation}
where $\mu_{bc}$ is the value assumed by $\mu_{bc}$ when the tetradron is flat with P outside $P_{a}P_{b}P_{c}$. In this case $\alpha = \delta_b - \delta_c$ and $\mu_{bc} > \mu_{bc}$. In the neighbourhood of $\mu_{bc} = \mu_{bc}^0$ we can approximate the integrand of (F.4) as follows:
\begin{equation}
\Psi_A \approx \int_{\mu_{bc}^0}^{\mu_{bc}} \frac{2}{3} \left( \frac{\mu_{bc} - \mu_{bc}^0}{1 - (\mu_{bc})^2} \right)^{\alpha} \frac{d\xi}{(\xi - \mu_{bc})^2} \left( \mu_{bc} - \mu_{bc}^0 \right)^{\frac{1}{2}},
\end{equation}
and using (F.6)
\begin{equation}
\Psi_A \approx \frac{3}{2} (\mu_{bc} - \mu_{bc}^0) \left[ \frac{1}{(\mu_{bc})^2} \right]^{-1} \left( 1 + 2\mu_{bd}\mu_{cd}/\mu_{bc} - \mu_{bc}^2 - \mu_{cd}^2 - \mu_{bd}^2 \right)^{\frac{1}{2}}.
\end{equation}
Since $\mu_{bc} - \mu_{bc}^0 = 2\sin \frac{1}{2}(\delta_b + \delta_c)\sin \frac{1}{2}(\delta_b - \delta_c) \sin \alpha$, we obtain from (F.8), (B.6), (B.5)
\begin{equation}
\Psi_A \approx \frac{3}{2} \sin \alpha \left( \frac{6V}{(h + \frac{1}{2})(e + \frac{1}{2})(d + \frac{1}{2})} \right) \sin \alpha \sin \delta_c \sin C.
\end{equation}
If $h$ is the distance of P from the plane $P_{a}P_{b}P_{c}$ we have also
\begin{equation}
(d + \frac{1}{2}) \Psi_A \approx \frac{3}{2} h (\delta_b + \delta_c - \alpha).
\end{equation}
Therefore from (F.10) and similar relations for $\Psi_B, \Psi_C$ we find
\begin{equation}
\Omega \approx \pi (a + b + c - \frac{1}{2}) + \frac{3}{2} h (2\pi - \delta - \epsilon - \varphi).
\end{equation}
Our result (E.9) yields in the neighbourhood of $V^2 = 0$
\begin{equation}
V^2 = \frac{1}{16} \left[ (d + \frac{1}{2})(e + \frac{1}{2})(f + \frac{1}{2}) \right]^2 \sin \delta \sin \epsilon \sin \varphi (2\pi - \delta - \epsilon - \varphi),
\end{equation}
having taken into account the different definition of $\delta, \epsilon, \varphi$. Using the proportionality between $h$ and $V$, we obtain finally from (E.11), (F.12)
\begin{equation}
\Omega \approx \pi (a + b + c - \frac{1}{2}) + \frac{3}{2} V^3 \left( \prod_{k=1}^{4} A_k \right)^{-1}.
\end{equation}
In a similar way, when we approach $B_4$ as $V^2 \to 0$, we find in the case corresponding to fig. 1
\begin{equation}
\Omega \approx \pi (a + b + d + e + \frac{1}{2}) - \frac{3}{2} V^3 \left( \prod_{k=1}^{4} A_k \right)^{-1}.
\end{equation}

**Appendix G. WKB approximation for $d_0^{(g)}(\theta)$**

Let us recall here the main features of the WKB solutions of
\begin{equation}
\frac{d^2 g}{dz^2} + Q^2(z) g = 0,
\end{equation}
without discussing their degree of approximation\^{39}; in (G.1) $Q^2(z) = \delta^2(z) (z - z_1)(z - z_2)$ with $z_1 < z_2$ and $\delta^2 > 0$. We know that the function
\begin{equation}
z \leq z_1, \quad g_1(z) = \frac{2}{\pi} \left( \frac{|\text{Im} t_1|}{|Q|} \right)^{\frac{1}{2}} \times \{ \pi \sin \eta_1 I \left( \text{Im} t_1 \right) + \cos (\frac{\pi}{4} - \eta_1) K_0 \left( \text{Im} t_1 \right) \}.
\end{equation}

Joining smoothly with
\begin{equation}
g_2(z) = \left( \frac{4 t_2}{3 Q} \right)^{\frac{1}{2}} \{ \cos (\frac{3\pi}{4} + \eta_2) J_2(t_2) + \cos (\frac{\pi}{4} - \eta_2) J_{-2}(t_2) \},
\end{equation}
\begin{equation}
z_1 < z \leq z_2,
\end{equation}
if we choose for $z \leq z_1$: $Q(z) = \frac{1}{2} Q(z)$, $|\text{Im} t_1| = \frac{1}{2} |Q(z)|$; far from $z_1$ we have
\begin{equation}
g_1(z) \approx (2\pi|Q|)^{-\frac{1}{2}} \{ 2 \sin \eta_1 e^{i\text{Im} t_1} + \cos \eta_1 e^{-i\text{Im} t_1} \},
\end{equation}
\begin{equation}
g_2(z) \approx \left( \frac{2}{\pi Q} \right)^{\frac{1}{2}} \cos (t_2 + \eta_1 - \frac{3}{4} \pi).
\end{equation}
Similarly, continuity through $z_2$ can be achieved by means of
\begin{equation}
g'_2(z) = \left( \frac{4 t_2}{3 Q} \right)^{\frac{1}{2}} \{ \cos (\frac{3\pi}{4} + \eta_2) J_2(t_2) + \cos (\frac{\pi}{4} - \eta_2) J_{-2}(t_2) \},
\end{equation}
\begin{equation}
z_1 < z \leq z_2,
\end{equation}
\begin{equation}
t_2(z) = \int_{z_2}^{z} \frac{Q(z)}{Q} \, d\xi, \quad Q(z) = [Q^2(z)]^{\frac{1}{2}},
\end{equation}
\begin{equation}
z_2 < z, \quad g_3(z) = \frac{2}{\pi} \left( \frac{|\text{Im} t_2|}{|Q|} \right)^{\frac{1}{2}} \times \{ \pi \sin \eta_2 I \left( \text{Im} t_2 \right) + \cos (\frac{\pi}{4} - \eta_2) K_0 \left( \text{Im} t_2 \right) \}.
\end{equation}
choosing for
\[ z_2 < z : \quad Q(z) = iQ(z), \quad \text{Im} \, t_2 = \int_{z_2}^{z} \frac{Q(\xi)}{Q'} \, d\xi; \]
for large \(|z - z_2|\) these solutions behave according to (G.6), (G.5) with the
obvious replacements: \(t_1 \rightarrow t_2, \eta_1 \rightarrow \eta_2\). We note incidentally that for
instance \(g_2(z)\) can be written in terms of Airy functions\(^{20}\):
\[
g_2(z) = \left( \frac{M}{i} \right)^\frac{3}{2} \cos \eta_1 Ai(-y) + \sin \eta_1 Bi(-y) \right) \quad (G.10)
\]
\[ Ai(-y) = \left( \frac{3}{2} \right)^{1/3} \left[ J_{-1}(t_1) + J_{1}(t_1) \right], \quad Bi(-y) = \left( \frac{3}{2} \right)^{1/3} \left[ J_{-1}(t_1) - J_{1}(t_1) \right], \quad (G.11) \]
where \(y = (\frac{3}{2}t_1)^{1/3}\). We see from (G.6) that the consistency condition \(g_2(z) = g_2'(z)\) can be fulfilled for \(z_1 < z < z_2\) if
\[
\int_{z_1}^{z_2} Q(\xi) \, d\xi = -\eta_1 - \eta_2 + (2N + 1) \pi, \quad N \gg 1 \quad (G.12)
\]
which provides a relation between \(\eta_1\) and \(\eta_2\).

We know (Brusseard and Tolhoek\(^{19}\)) that, for large \(f\), \(d_{\delta\sigma}^{(2)}(\theta)\) satisfies
\[
\frac{d}{d(\cos \theta)} \sin^2 \theta \frac{d}{d(\cos \theta)} + \left( f + \frac{1}{2} \right) \sin^2 \theta \left[ (1 - \mu^2) \times \right.
\]
\[ \times (1 - y^2) - \left( \cos \theta - \mu y \right)^2 \right] \right) \, d_{\delta\sigma}^{(2)}(\theta) = 0, \quad (G.13)
\]
where \((f + \frac{1}{2}) \mu = \delta', (f + \frac{1}{2}) \nu = \delta;\) we shall consider only the domain \(|\cos \theta| < 1\). If we put \(\cos \theta = \tgh z, \, d_{\delta\sigma}^{(2)}(0) = g(z)\), then we obtain (G.1) with
\[
Q^2(z) = \left( f + \frac{1}{2} \right) \left[ (1 - \mu^2) (1 - y^2) - (\tgh z - \mu y)^2 \right], \quad (G.14)
\]
and transition points
\[
\cos \theta_1, 2 = \tgh z_1, 2 = \mu \sqrt{1 - \mu^2} \sqrt{(1 - y^2)}; \quad (G.15)
\]
when \(\mu = \nu, \, z_2\) corresponds to \(\cos \theta_2 = 1\); therefore we shall limit ourselves to \(\delta \neq \delta'\).

In order to apply (G.2)–(G.9), we must provide \(t_1, t_2, \eta_1\) and \(\eta_2\) explicitly.

From (G.4), (G.14) and (F.4) we have
\[
\int_{z_1}^{z_2} (f + \frac{1}{2}) \left[ 1 - \mu^2 - y^2 + 2\mu \tgh z - (\tgh z)^2 \right] \, dz =
\]
\[ -(f + \frac{1}{2}) \left[ F(z) - \mu A(z) - vB(z) \right], \quad (G.16)
\]
where \(\pi - F, A, B\) are the internal dihedral angles between the planes belonging
respectively to \(f, a + \delta', b + \delta\) (fig. 4 with \(\mu = \cos \alpha, v = \cos \beta\)). We notice
from (G.15) that \(z_1\) corresponds to \(\theta_1 = \alpha + \beta, A = B = F = 0\). On the other
hand \(z_2\) is associated to the case \(\theta_2 = |\alpha - \beta|\); we have for \(\alpha > \beta : B = F = \pi, A = 0\) and for \(\alpha < \beta : A = F = \pi, B = 0\). Therefore\(^{6}\):
\[
t_1(\theta) = t = (f + \frac{1}{2})(F - A\mu - B\nu), \quad (G.17)
\]
\[ F = \text{arc cos} \frac{\mu - \cos \theta}{\sin \alpha \sin \beta}, \quad A = \text{arc cos} \frac{\nu - \cos \theta}{\sin \alpha \sin \beta}, \quad B = \text{arc cos} \frac{\mu - \cos \theta}{\sin \beta \sin \theta}, \quad (G.17)
\]
and similarly for \(t_2\). Furthermore (G.12) yields now:
\[
- \eta_1 - \eta_2 + (2M + 1) \pi = \int_{z_1}^{z_2} Q(\xi) \, d\xi =
\]
\[ \int_{z_1}^{z_2} \frac{Q(\xi)}{Q'} \, d\xi =
\]
\[ \{ (f + \frac{1}{2}) (\pi - \pi \nu), \quad \alpha > \beta, \quad (\delta > \delta')\}
\]
\[ = \{ (f + \frac{1}{2}) (\pi - \pi \nu), \quad \alpha < \beta, \quad (\delta < \delta')\}. \quad (G.18)
\]

From the asymptotic behaviour of Jacobi polynomials\(^{19}\) and relation
(A.18) it is easy to obtain:
\[
d_{\delta\sigma}^{(2)}(\theta) \approx \frac{2}{\pi (f + \frac{1}{2}) \sin \theta} \cos \left( (f + \frac{1}{2}) \theta + \frac{1}{2} \pi (\delta - \delta') - \frac{1}{2} \pi \right), \quad (G.19)
\]
valid when \(f\) is large and \(|\delta|, |\delta'| < f\); performing this limit in (G.17), we have:
\(A \approx B \approx \frac{\pi}{2}, F \approx \pi \times \theta\), and consequently \(t_1(\theta) \approx (f + \frac{1}{2}) (\pi - \theta) - \frac{1}{2} \pi (\delta + \delta')\). Using this result and identifying the arguments of the cosines in (G.6) and (G.19), we obtain
\[
\eta_1 = \pi (\delta' - f), \quad (G.20)
\]

Therefore from (G.18)
\[
\eta_2 = \begin{cases} 
\pi (\delta - \delta') & \delta > \delta', \\
0 & \delta < \delta'; \quad (G.21)
\end{cases}
\]

Obviously \(\eta_1\) and \(\eta_2\) are determined modulo \(2\pi\). It must be stressed that
only for "physical" values of \(\delta, \delta'\) the phases \(\eta_1, \eta_2\) (except when \(\delta < \delta'\))
are integer multiples of \(\pi\); in this case the exponentially increasing term in
(G.5) is ruled out.

The overall normalization in (G.2), (G.3) and (G.9) has been chosen in
agreement with (A.15).
Appendix H. The $1 + 1 + 2$ case

In this last appendix we want to show that (3.6) is in agreement with our asymptotic formulae (1.4), (1.8). To this end we consider the limiting case in which $f, \delta, \delta'$, though large, are still small with respect to $a, b, c$; therefore we may use in (3.6) our results of appendix G.

In the oscillatory region we deduce from (3.6), (G.6), (G.17), (G.20)

$$\begin{align*}
&\left\{ \frac{c}{f} \right\} \sim -\left( -1 \right)^{a+b+c+f+\delta} \times \\
&\left\{ \frac{1}{\left[ \left( 2a + 1 \right) \left( 2b + 1 \right) \right]^2} \right\} \times \\
&\left( \left( 1 - \mu^2 \right) \left( 1 - \nu^2 \right) - \left( \cos \theta - \mu \nu \right) \right)^{-1} \cos \left( t + \pi \delta' - f - \frac{1}{2} \pi \right)
\end{align*}$$

and recalling (B.6):

$$\begin{align*}
&\left\{ \frac{c}{f} \right\} \sim -\left( -1 \right)^{a+b+c+f+\delta} \times \\
&\left\{ \frac{1}{12\pi \left[ \left| \mathbf{F} \right| \right]^2} \right\} \cos \left( t - \frac{1}{2} \pi \right).
\end{align*}
$$

(H.1)

On the other hand we note that as $a, b, c$ increase in the tetrahedron of fig. 4, the two faces belonging to $c$ become almost parallel; therefore:

$$\begin{align*}
&\theta_c = \pi, \quad \theta_a = \theta_b = 0, \quad \theta = \pi - \theta_a - \theta_b
\end{align*}$$

from the definition (3.15b) we obtain

$$\begin{align*}
\Omega = (a + b + c + \delta + \delta' - \frac{1}{2}) \pi + \left( f + \frac{1}{2} \right) F - \delta' A - \delta B = \\
= t + (a + b + c + \delta + \delta' - \frac{1}{2}) \pi,
\end{align*}
$$

(H.2)

which, when introduced into (1.4a) leads to (H.1).

Let us now consider the classically forbidden regions. First we notice that the transition points $z_1, z_2$ relative to $d_{ij}^2(\theta)$ discussed in appendix G correspond to configurations in which the tetrahedron of fig. 4 becomes flat; this can be checked by means of (G.14) and (B.6). More precisely, since in $z_1$ $\theta = \pi + \beta$, the tetrahedron enters the region $B_3$ with

$$z_1 \Phi = \left( a + b + c + \delta + \delta' \right) \pi,$$

(H.3)

while in $z_2$, $\theta = \pi - \beta$ and it enters $B_4$ in either one of the two ways

$$z_2 \Phi = \begin{cases} 
(a + b + c + f + \delta') \pi & \delta > \delta', \\
(a + b + c + f + \delta) \pi & \delta < \delta'.
\end{cases}
$$

(H.4)

On the other hand, from (G.5) and (3.6) we have for the forbidden region relative to $z_1$

$$\begin{align*}
&\left\{ \frac{c}{f} \right\} \sim -\left( -1 \right)^{a+b+c+f+\delta} \times \\
&\left\{ \frac{2}{12\pi \left| \mathbf{F} \right|^2} \right\} \times \\
&\left\{ \sin \left( \delta' - f \right) e^{\text{Im} t} + \cos \left( \delta' - f \right) e^{-\text{Im} t} \right\}.
\end{align*}
$$

(H.5)

Noticing that $|\text{Im} t| = |\text{Im} \Omega|$, we see from (5.8), (H.2), (H.3) and (H.5) that for physical values of the angular moments (H.5) and (5.8) become identical even in sign. The same holds also for the forbidden region relative to $z_2$; in fact we have from (G.21):

$$\begin{align*}
&\left\{ \frac{c}{f} \right\} \sim -\left( -1 \right)^{a+b+c+f+\delta} \times \\
&\left\{ \frac{2}{12\pi \left| \mathbf{F} \right|^2} \right\} \times \\
&\left\{ \sin \left( \delta' - f \right) e^{\text{Im} t} + \cos \left( \delta' - f \right) e^{-\text{Im} t} \right\}.
\end{align*}
$$

in agreement with (H.4).

References and footnotes

1) G. Racah, Phys. Rev. 61, 186 (1942); 62, 438 (1942); 63, 367 (1943); 76, 1352 (1949).
3) L. C. Biedenharn, private communication to one of us (T.R.), 1958.
4) E. P. Wigner, Am. J. Math. 63, 57 (1941); On the Matrices which Reduce the Kronecker Products of Representations of S. R. Groups (unpublished, 1940); hectographed paper (Princeton, 1951) (see ref. 2).
7) In addition to the three-dimensional representation which will be used in this paper, we recall the graphical techniques developed by A. Yutis and coworkers* as well as its dual, exemplified by U. Fano and G. Racah, Irreducible Tensorial Sets (Academic Press, New York, 1959) appendices.
8) We introduce $a + i$ rather than $a$...length of the edge corresponding to $a$...because, for high quantum numbers, the length $|a(a+1)|$ of the angular momentum vector is closer to $a + i$ in the semiclassical limit; it turns out that any other choice is inconsistent with numerical results and with existing formulae.
9) N. F. Tartaglia, General trattato di numeri et misure (Venezia, 1560).
10) See the footnote 142 of ref. 22.
12) S. Adler has kindly pointed out to us that:

$$\begin{align*}
&2^{n}(3)^{3/2} = -2(a+1)(2a+1)(2a+1) - 2(2a+1)(2a+1)(2a+1) + 1, \mod. 8
\end{align*}$$

where $a, q, q_4$ are defined above and $|A| = 0, 1$ if $A$ is even, odd;

$$\begin{align*}
&\alpha = (2a+1)(2a+1), \quad \gamma = (2a+1)(2a+1), \quad e = (2a+1)(2a+1)
\end{align*}$$

It follows that if all the $q$s are integer and $\alpha, \gamma, e = 0, 1$, or 1:

$$\begin{align*}
&2^{n}(3)^{3/2} = -2(a+1)(2a+1)(2a+1) - 1 = -2, -4 \mod. 8;
\end{align*}$$

therefore $\mathbf{F} \neq 0$. 


ON RACAH COEFFICIENTS AS COUPLING COEFFICIENTS FOR THE VECTOR SPACE OF WIGNER OPERATORS

L. C. BIEDENHARN*
Duke University, Durham, North Carolina 27706, U.S.A.

1. Introduction and summary

One of the most important contributions made by Giulio Racah to theoretical physics was the systematic development of the theory of tensor operators for angular momentum; the results of this development — called, in honor of its principal creators, the Racah–Wigner calculus — and its application, are now part of the equipment of every working physicist.1)

There are two quite distinct reasons that underlie the success of this work. The first is the fundamental nature of angular momentum in quantum mechanics, and this stems ultimately from symmetry (isotropy) of spacetime (Poincaré group) which is the deepest presently known foundation for quantum mechanics. This may be termed the 'physical reason' for the importance of the Racah–Wigner calculus. The second reason is mathematical, and provides the ultimate source for the very existence of the Racah–Wigner calculus: the angular momentum group (SU3) has the property of being simply reducible.2) This is the property which guarantees that the Racah–Wigner calculus is uniquely defined by the group and contains no inherent ambiguities. The really essential part of Wigner’s definition of simple reducibility is that in reducing the Kronecker product of two irreducible representations (‘irreps’), a given irrep occurs either once or not at all. One is forced by the structure of quantum mechanics (tensor operators acting on states) to ‘multiply’ irreps; if this ‘product’ upon being reduced into elements (irreps) has a given irrep occurring more than once, an inherent ambiguity may occur which is not decidable within the original symmetry group.

* Supported in part by the U.S. Army Research Office (Durham) and the National Science Foundation.

18. This property stems from a particular case of a theorem by Schläfli on elliptic tetrahedra; see H. S. M. Coxeter, *Non-Euclidean Geometry* (Toronto, 1957).
20. A discussion of this theorem, due originally to G. Racah, can be found in the appendix to the book by U. Fano and G. Racah, ref. 7.
21. See for instance eq. (16.8) ref. 6.
24. We do not expect to be true in general that the 6j-symbol is a function of the variables \( V \) and \( \Pi_{\alpha_4} \) only.
34. The discussion which is carried out here as well as in appendix H corresponds actually to \( \delta \geq 0, \delta' \geq 0 \); however, it can be easily extended to the other cases yielding slight modifications in \( t_{\alpha}, t_{\alpha}', t_{\beta}, t_{\beta}' \). For instance, when \( \delta < 0, \delta' < 0 \) one would find \( t_{\alpha} = 2\pi - \alpha - \beta, t_{\beta} = (\pi + \delta/2) - (\alpha - \beta) \mu + (\beta - \alpha) \), \( t_{\alpha}' = -\pi(f + \delta), \) while \( t_{\beta}' \) would be the same as in (G.21) for "physical" values of \( \delta, \delta' \).