Spectroscopic and group theoretical methods in physics

SEMICLASSICAL LIMIT OF RACAH COEFFICIENTS†

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1. Introduction

Almost twenty-five years ago the W-coefficient appeared for the first time in a paper by Racah[1]) as an auxiliary tool for the computation of matrix elements in the theory of complex spectra. Today there is hardly any branch of physics involving angular momenta[2]) where the use of W-coefficients is not needed in order to carry out the simplest computation. Yet we feel that the W-coefficient is something more than an extremely successful computational tool and a beautiful toy for theoretical physicists to play with. In fact, a complete understanding of the properties of this remarkable function may very well yield to a new insight into the theory of angular momenta.

We think, and we know that our view is shared by others[3]), that a complete investigation of the semiclassical limit of W-coefficients and related functions is a prerequisite for a deeper understanding of their properties.

The present paper contains a heuristic derivation of an asymptotic formula, or better, of a set of asymptotic formulae with separate ranges of validity, for the W-coefficient. These formulae are certainly a useful complement to the existing tables of Racah coefficients, since they are remarkably accurate for surprisingly low values of the angular momenta involved.

A similar point of view could be adopted for the Clebsch–Gordan coefficients. However, since their definition depends on the particular labelling

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adopted for the vector basis of the representations, and since, moreover, they can be deduced as a particular limit carried on the W-coefficients, we are definitely tempted to regard them as subsidiary quantities in this paper.

Coming back to the W-coefficient or, rather, to the symmetric version of it, i.e. the $6j$-symbol defined by Wigner[4]), it has been for years a normal practice to associate to it a diagram or graph which exhibits the symmetry properties in a most obvious way. A further advantage of these graphs is that they can be generalized to the higher order $3nj$-symbols defined by Wigner[4]) and others[5,6]). There are at least three different versions[7]) of these graphical algorithms, all having approximately the same content, the

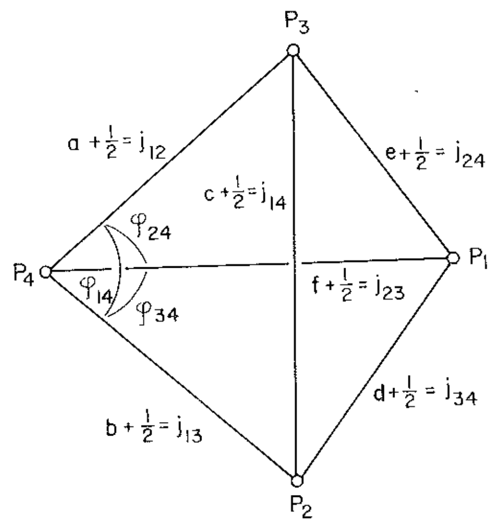


Fig. 1. Three-dimensional representation of the $6j$-symbol $\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$.

translation of one into the others being achieved through some principle of plane or space duality. The reason for choosing any one of them is rather sentimental and largely related to individual habits.

We shall prefer here a three-dimensional representation in which angular momenta appear as vectors satisfying "bona fide" graphical composition rules. In this particular calculus the $6j$-symbol $\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$ is associated to the tetrahedron shown in fig. 1. So far the tetrahedron is just a mnemonical device. However, we may think about a real solid T whose edges are just $a+\frac{1}{2}$, $b+\frac{1}{2}$, etc.[8]). With reference to fig. 1, we shall use also the following

notation for the edges: $j_{12}=a+\frac{1}{2}$, $j_{13}=b+\frac{1}{2}$, $j_{14}=c+\frac{1}{2}$, $j_{34}=d+\frac{1}{2}$, $j_{24}=e+\frac{1}{2}$, $j_{23}=f+\frac{1}{2}$, and $j_{hh}=0$, $j_{hk}=j_{kh}$ $(h, k=1, 2, 3, 4)$.

We shall restrict ourselves to values of the angular momenta which satisfy the triangular inequalities, i.e., $|b-c|\leqslant a\leqslant b+c$, etc. for the triads (abc), (aef), (dbf), (dec). Therefore, in each face there must be an even number of half-integer angular momenta. This entails that the sums $q_1=a+b+c$, $q_2=a+e+f$, $q_3=b+d+f$, $q_4=c+d+e$, $p_1=a+b+d+e$, $p_2=a+c+d+f$, $p_3=b+c+e+f$, are all integer. Moreover, because of the triangular inequalities, we have

$$p_h \geqslant q_k, \qquad h, k = 1, 2, 3, 4. \tag{1.1}$$

While these conditions are in general sufficient to guarantee the existence of a non-vanishing $6j$-symbol, they are not enough to ensure the existence of the tetrahedron T with the given edges. Since the two cases (A) T exists, (B) T does not exist, deserve radically different asymptotic treatments, we must give necessary and sufficient conditions for the existence of T.

It is known since Tartaglia[9]) and Jungius[10]) that the square of the volume of a tetrahedron is given by a polynomial in the square of its edges; in a more symmetrical setting, given by Cayley in his first published paper[11]), we have indeed

$$2^3(3!)^2V^2=\begin{vmatrix} 0 & j_{34}^2 & j_{24}^2 & j_{23}^2 & 1 \\ j_{34}^2 & 0 & j_{14}^2 & j_{13}^2 & 1 \\ j_{24}^2 & j_{14}^2 & 0 & j_{12}^2 & 1 \\ j_{23}^2 & j_{13}^2 & j_{12}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}. \tag{1.2}$$

Therefore we see that $V^2\geqslant 0$ is a necessary condition. It can be proved to be also sufficient. In fact let us keep all edges fixed but, for instance, j_{12} and let $j_{12}=x$. Then V^2 is a second order polynomial in x^2 which will have two roots $(x_<)^2<(x_>)^2$. Since $\partial^2[V^2(x^2)]/[\partial x^2]^2=-(j_{34})^2/72$, we shall have $V^2(x^2)>0$ if $x_<<x<x_>$. A more elaborate discussion would show, in addition, that $x_<>x_m-\frac{1}{2}$ where x_m is the largest between $|b-c|+\frac{1}{2}$ and $|e-f|+\frac{1}{2}$, and that $x_><x_M+\frac{1}{2}$ where x_M is the smallest between $b+c+\frac{1}{2}$ and $e+f+\frac{1}{2}$. Therefore the condition $V^2>0$ is stronger than (1.1). We shall accordingly distinguish between the above mentioned cases: (A) $V^2>0$, and (B) $V^2<0$. The third possibility, $V^2=0$, which would correspond to a flat tetrahedron, is purely academical, for it can be proved that if p_h and q_k are all integer then $V^2\neq 0$ (ref. 12). A "tetrahedron" in (B) will be referred to as a hyperflat tetrahedron. Let us start with case:

(A) We expect T to be relevant in describing the properties of $6j$-symbols.

In fact a result due to Wigner[13]) states that for large angular momenta

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}^2 \sim \frac{1}{24\pi V}, \tag{1.3a}$$

where V is the volume of T. This formula, which has been a guiding principle in the present investigation, is very interesting because it relates the numerical value of the $6j$-symbol directly to a geometric property of T. However, as stressed in the same ref. 13, Wigner's asymptotic estimate cannot be accepted at face value. Inspection of numerical tables shows in fact that the symbol is a rapidly oscillating function of the indices and that the r.h.s. of (1.3a) more correctly approximates the average of the l.h.s. over several contiguous values of the indices.

A correct statement would be

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \simeq \frac{1}{\sqrt{12\pi V}} \mathscr{C}, \tag{1.3b}$$

where $\mathscr{C}$ is a rapidly oscillating function so that the average $\mathscr{C}^2$ over a large enough interval is $\frac{1}{2}$. We claim that

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \simeq \frac{1}{\sqrt{12\pi V}} \cos\left(\sum_{h,k=1}^{4} j_{hk}\theta_{hk} + \tfrac{1}{4}\pi \right), \tag{1.4}$$

where $\theta_{hk}=\theta_{kh}(k \neq h=1, 2, 3, 4)$ are the angles between the outer normals of the two faces which belong to j_{hk}. Let A_h be the area of the face opposite to the vertex h (fig. 1); then we have (appendix B)

$$A_h A_k \sin\theta_{hk} = \tfrac{3}{2} V j_{hk}, \qquad h \neq k = 1, 2, 3, 4. \tag{1.5}$$

(B) Wigner's argument yields

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}^2 \sim 0. \tag{1.6}$$

This result could be loosely described as the impossibility of having six angular momenta forming a non-existing tetrahedral scheme. A closer scrutiny of numerical tables, however, shows that the symbols in (B), although, as a rule, smaller than in (A), are still non-vanishing.

Any attempt to use (1.4) in this region leads to a meaningless result. In fact, relations (1.1) guarantee that A_h are real; since $V^2<0$, V is imaginary and (1.5) implies that $\theta_{hk}=n\pi+\mathrm{i}\,\mathrm{Im}\,\theta_{hk}$. However, as it stands, (1.4) bears a strong resemblance with some formulae familiar from the WKB method[14]). Although we know of no differential equation from which in general (1.4) might be deduced, there are particular instances in which this can be done (section 5). This suggests that we may use the connection formulae of the WKB method to go across the transition points $x_<$, $x_>$. We define

$$\Phi = \sum_{h,k=1}^{4} (j_{hk} - \tfrac{1}{2})\,\mathrm{Re}\,\theta_{hk} \tag{1.7}$$

which, for physical values of j_{hk}, is always an integer multiple of π. According to the WKB connection formulae (appendix G), we find for physical angular momenta

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \simeq \frac{1}{2\sqrt{12\pi|V|}} \cos\Phi \exp\left(- \left| \sum_{h,k=1}^{4} j_{hk}\,\mathrm{Im}\,\theta_{hk} \right| \right), \tag{1.8}$$

where the sign of $\mathrm{Im}\,\theta_{hk}$ must be chosen according to the rules explained in section 5. Also this formula turns out to be in remarkably satisfactory agreement with numerical tables. The exponential decrease shown by (1.8) clearly describes a quantum tunnel effect into the classically forbidden region (B).

We expect (1.4) and (1.8) to be inaccurate in the neighbourhood of $x_{\gtrless}$. In fact, the error is here considerably large, although not disastrous. Transition formulae involving Airy functions have been worked out for this region (section 5) and found to be accurate.

In spite of these numerical checks, a sound proof of our formulae is still missing. However there are other arguments in favour of (1.4) and (1.8). For instance, (1.4) has the right symmetry properties, including the extra symmetries discovered by one of us[15]) and satisfies asymptotically the recursion relations as well as the identities of the $6j$-symbols. It is also consistent with the previously investigated particular cases of asymptotic behaviours[16]).

Relations (1.4) and (1.8), together with a transition formula, solve completely the analysis of $6j$-symbols when all angular momenta are large. However, it is also interesting to investigate the case in which one or more edges remain constant and finite while the others increase. We may picture the limiting process as one in which one or more vertices of T go to infinity either separately or in clusters. Therefore there are as many ways to carry out the process as decompositions of 4 into sums of natural integers, i.e., $1+1+1+1$ (all edges large), $1+1+2$ (one small edge), $2+2$ (two small edges), $1+3$ (three small edges). The $1+1+2$ case has been widely studied[16]), while we have found no reference to $2+2$ and accordingly we solve this uninteresting case in the present paper. Some examples of the $1+3$ case, which yield a connection between $6j$- and $3j$-symbols, have been discussed by Brussaard and Tolhoek[16]). Our treatment, however, is quite general and shows how the symmetries of the 3*j*-symbol can be derived from those of the 6*j*-coefficient. As a by-product of this analysis, we obtain a new asymptotic formula for Clebsch–Gordan coefficients.

2. Asymptotic connection with the Clebsch–Gordan coefficients

Of some interest is the 1+3 case, which occurs when we take the positive integer R large in $\begin{Bmatrix} a & b & c \\ d+R & e+R & f+R \end{Bmatrix}$. The related limits $\begin{Bmatrix} a & b+R & c+R \\ d+R & e & f \end{Bmatrix}$ etc. can be reduced to it by symmetry. The starting point of our discussion is Racah's formula (A.4), which, using $\xi = x-2R$ as summation variable becomes

$$\begin{aligned}\begin{Bmatrix} a & b & c \\ d+R & e+R & f+R \end{Bmatrix} &= [\Delta(abc)\,\Delta(d+R,e+R,c)\,\times \\ &\times \Delta(e+R,f+R,a)\,\Delta(f+R,d+R,b)]^{\frac{1}{2}} \sum_{\xi}(-1)^{\xi+2R}\times \\ &\times(\xi+2R+1)!\,[(a+b+d+e-\xi)!\,(b+c+e+f-\xi)!\,\times \\ &\times(c+a+f+d-\xi)!\,(\xi-a-b-c+2R)!\,\times \\ &\times(\xi-a-e-f)!\,(\xi-d-e-c)!\,(\xi-d-b-f)!]^{-1}. \end{aligned} \tag{2.1}$$

From Stirling's formula we find, for instance, that for large R

$$\frac{(\xi+2R+1)!}{(\xi+2R-a-b-c)!} \simeq \xi^{a+b+c+1}, \tag{2.2}$$

which entails for example

$$\Delta(a,e+R,f+R) \simeq (a+e-f)!\,(a-e+f)!\,(2R)^{-2a-1}. \tag{2.3}$$

By means of (2.2), (2.3) and similar relations, (2.1) transforms into

$$\begin{aligned}\begin{Bmatrix} a & b & c \\ d+R & e+R & f+R \end{Bmatrix} &\simeq \left[\frac{\Delta(abc)}{2R}\right]^{\frac{1}{2}} [(a+e-f)!\,(a-e+f)!\,\times \\ &\times(b+f-d)!\,(b-f+d)!\,(c+d-e)!\,(c-d+e)!]^{\frac{1}{2}}\times \\ &\times\sum_{\xi}(-1)^{\xi}\,[(a+b+d+e-\xi)!\,(b+c+e+f-\xi)!\,\times \\ &\times(c+a+f+d-\xi)!\,(\xi-a-e-f)!\,\times \\ &\times(\xi-b-d-f)!\,(\xi-c-d-e)!]^{-1}. \end{aligned} \tag{2.4}$$

Looking at (A.1) we realize that the r.h.s. of (2.4) can be written in terms of

a 3*j*-symbol (this result is quoted by K. Alder et al.[16])):

$$\begin{Bmatrix} a & b & c \\ d+R & e+R & f+R \end{Bmatrix} \simeq (-1)^{a+b+c+2(d+e+f)}\,(2R)^{-\frac{1}{2}} \begin{pmatrix} a & b & c \\ e-f & f-d & d-e \end{pmatrix} \tag{2.5a}$$

or using the pattern notation of appendix A

$$\begin{aligned}&\left|\begin{matrix} a+b-c & b+f-d & a+f-e \\ b+d-f & b+c-a & c+d-e \\ a+e-f & e+c-d & c+a-b \\ e+d-c+2R & e+f-a+2R & d+f-b+2R \end{matrix}\right| \simeq \\ &\simeq (-1)^{a+b+c+2(d+e+f)}\,(2R)^{-\frac{1}{2}} \left|\begin{matrix} b+c-a & c+a-b & a+b-c \\ a-e+f & b-f+d & c-d+e \\ a+e-f & b+f-d & c+d-e \end{matrix}\right|. \end{aligned} \tag{2.5b}$$

From (2.5b) it is easy to check that the symmetry

$$\begin{aligned}&\begin{Bmatrix} a & b & c \\ d+R & e+R & f+R \end{Bmatrix} = \\ &= \begin{Bmatrix} a & \frac{1}{2}(f+c+b-e) & \frac{1}{2}(b+e+c-f) \\ d+R & \frac{1}{2}(f+c+e-b)+R & \frac{1}{2}(b+e+f-c)+R \end{Bmatrix}\end{aligned}$$

entails

$$\left|\begin{matrix} b+c-a & c+a-b & a+b-c \\ a-e+f & b-f+d & c-d+e \\ a+e-f & b+f-d & c+d-e \end{matrix}\right| = \left|\begin{matrix} b+c-a & a+e-f & a-e+f \\ a+b-c & c+d-e & c-d+e \\ c+a-b & b+f-d & b-f+d \end{matrix}\right|,$$

which is one of the extra symmetries of the 3*j*-symbol pointed out by one of us[15]). Actually, (2.5b) relates the subgroup of R_1 (see appendix A) formed by all the even 36 symmetries of the 3*j*-symbol to the subgroup of R_2 corresponding to permutations of columns and/or of the upper three lines of the pattern in the l.h.s. of (2.5b). The geometrical and physical content of these symmetries is still to be understood and they remain a puzzling feature of the theory of angular momenta. Therefore it is a pleasant result to be able to reduce the problem of their interpretation to the Racah coefficient only.

From (2.5a) we obtain also an expression for the 3*j*-symbol for large quantum numbers. Our derivation of this result is rather heuristic as it involves the exchange of different limiting processes. Just as for (1.4) the formula which we are going to present has a rather "a posteriori" validity, for it satisfies all possible consistency checks. We obtain it from (2.5a) by supposing a, b, c, d, e, f large and finite. Using (1.4) in the l.h.s. of (2.5a) and performing the limit $R\to\infty$, we find

$$\begin{pmatrix} a & b & c \\ m_a & m_b & m_c \end{pmatrix} \simeq (2\pi A)^{-\frac{1}{2}}(-1)^{a+b-c+1} \times \cos\left[(a+\tfrac{1}{2})A+(b+\tfrac{1}{2})B+(c+\tfrac{1}{2})C-m_bD+m_aE+\tfrac{1}{4}\pi\right], \qquad (2.6)$$

where $m_a=e-f$, $m_b=f-d$, $m_c=d-e$ are the third components of a, b, c along the direction in which P_4 was sent to infinity (fig. 2). A is the area of the shaded triangle in fig. 2 which corresponds to the projection of $P_1P_2P_3$ on a plane perpendicular to the z axis. The angles $A, \ldots, D, \ldots,$ defined according to (1.4) are given in terms of m_a, m_b, m_c by

$$\cos A = \frac{2(a+\frac{1}{2})^2 m_c + m_a[(c+\frac{1}{2})^2+(a+\frac{1}{2})^2-(b+\frac{1}{2})^2]}{\left([(a+\frac{1}{2})^2-m_a^2]\{4(c+\frac{1}{2})^2(a+\frac{1}{2})^2-[(c+\frac{1}{2})^2+(a+\frac{1}{2})^2-(b+\frac{1}{2})^2]^2\}\right)^{\frac{1}{2}}}. \qquad (2.7)$$

$$\cos D = \frac{1}{2}\,\frac{(a+\frac{1}{2})^2-(b+\frac{1}{2})^2-(c+\frac{1}{2})^2-2m_bm_c}{\{[(b+\frac{1}{2})^2-m_b^2]\,[(c+\frac{1}{2})^2-m_c^2]\}^{\frac{1}{2}}}, \qquad (2.8)$$

$\cos B$, $\cos C$ and $\cos E$, $\cos F$ are deduced respectively from (2.7), (2,8) by circular permutations of the labels, a, b, c; note that $D+E+F=2\pi$. When

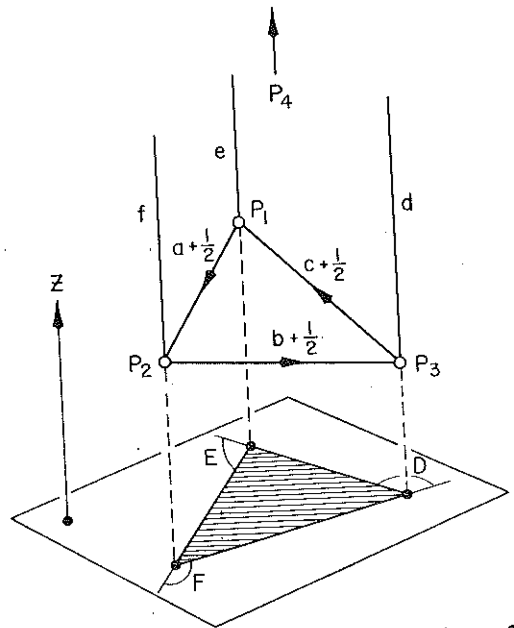


Fig. 2. Limit case in which a $6j$-symbol degenerates into a $3j$-symbol.

$m_a=m_b=m_c=0$, the plane $P_1P_2P_3$ is perpendicular to z and $A=B=C=\frac{1}{2}\pi$; (2.6) reduces to the known[16]) result

$$\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} \simeq \tfrac{1}{2}[1+(-1)^{a+b+c}](-1)^{\frac{1}{2}(a+b+c)}(2\pi A)^{-\frac{1}{2}}. \qquad (2.9)$$

The $2+2$ case can be dealt with in much the same way. Using once more $\xi=x-2R$ as summation variable, from (A.4), (2.2) we obtain

$$\begin{Bmatrix} a & b+R & c+R \\ d & e+R & f+R \end{Bmatrix} \simeq [(a+b-c)!\,(a-b+c)!\,(a+e-f)!\,\times (a-e+f)!\,(d+e-c)!\,(d-e+c)!\,(d+b-f)!\,(d-b+f)!]^{\frac{1}{2}} \times (2R)^{-2a-2d-b-c-e-f-1}\sum_{\xi}(-1)^{\xi}(2R)^{2\xi}[(a+d+b+e-\xi)!\,\times (a+d+c+f-\xi)!\,(\xi-a-b-c)!\,\times (\xi-a-e-f)!\,(\xi-c-d-e)!\,(\xi-b-d-f)!]^{-1}. \qquad (2.10)$$

For R large the main contribution to this summation comes from the largest allowed value of ξ, i.e. from the minimum between $a+d+b+c$ and $a+d+c+f$; therefore

$$\begin{Bmatrix} a & b+R & c+R \\ d & e+R & f+R \end{Bmatrix} \simeq (-1)^{a+d+\min(b+e,\,c+f)} \times \left[\frac{(a-b+c)!\,(a-e+f)!\,(d-e+c)!\,(d-b+f)!}{(a+b-c)!\,(a+e-f)!\,(d+e-c)!\,(d+b-f)!}\right]^{\frac{1}{2}\mathrm{sign}(c+f-b-e)} \times \frac{(2R)^{-|b+e-c-f|-1}}{|b+e-c-f|!}[1+O(R^{-2})], \qquad (2.11)$$

where $\mathrm{sign}\,(x)=+1, -1$ according to $x\geqslant 0, <0$. It must be pointed out that unless $b+e=c+f$, the corresponding tetrahedron becomes hyperflat; in fact it turns out that $144\,V^2=-4(b+c-e-f)^2R^4+O(R^3)$.

The remaining particular case $2+1+1$ will be discussed in some detail in appendix H.

3. Improvement of Wigner asymptotic formula

According to Wigner[13]), the physical interpretation of the $6j$-symbol is clearly related to its definition as a recoupling coefficient

$$|(j_1,(j_2,j_3)j_{23})J\rangle = \sum_{j_{12}}[(2j_{12}+1)(2j_{23}+1)]^{\frac{1}{2}}(-1)^{j_1+j_2+j_3+J}\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23}\end{Bmatrix} \times |((j_1,j_2)j_{12},j_3)J\rangle\,. \qquad (3.1)$$

It follows that

$$(2j_{12}+1)(2j_{23}+1)\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{Bmatrix}^2 \mathrm{d}j_{23}, \tag{3.2}$$

is the probability that the sum of the angular momenta $\boldsymbol{j}_2$ and $\boldsymbol{j}_3$ has the length in the interval $j_{23}, j_{33}+\mathrm{d}j_{23}$ whenever $\boldsymbol{j}_1$ and $\boldsymbol{j}_2$ have sum of length j_{12} and $\boldsymbol{j}_{12}=\boldsymbol{j}_1+\boldsymbol{j}_2$ is coupled with $\boldsymbol{j}_3$ to a vector $\boldsymbol{J}$ of length J. As anticipated in the introduction, the mutual relationship of these vectors is best seen on a diagram (fig. 3).

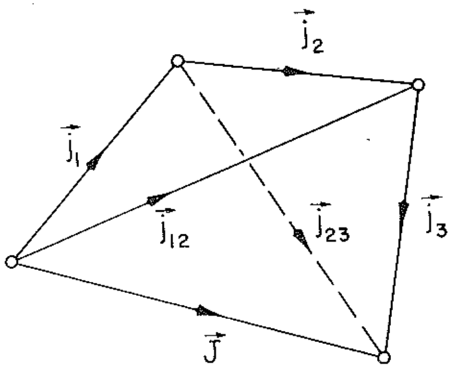


Fig. 3. Recoupling scheme corresponding to (3.1).

Let j_1, j_2, j_3, j_{12}, J (i.e. all the quantum numbers in the l.h.s. of (3.1)) be fixed; then the angle ψ between the plane of the vectors $\boldsymbol{j}_1, \boldsymbol{j}_2$ and the plane of the vectors $\boldsymbol{j}_3, \boldsymbol{J}$ is still undetermined, for both of them can rotate around the common axis $\boldsymbol{j}_{12}$. During this rotation the point P describes a circle; we assume[13]) that every point of this circle has equal probability. Then the probability that the length of $\boldsymbol{j}_{23}$ falls into the interval $j_{23}, j_{23}+\mathrm{d}j_{23}$ is just $\{2|\mathrm{d}\psi/\mathrm{d}j_{23}|/(2\pi)\}\,\mathrm{d}j_{23}$, the factor 2 being needed because there are two configurations corresponding to ψ and $2\pi-\psi$ which yield the same j_{23}. An elementary computation (appendix B) shows that

$$\frac{\mathrm{d}\psi}{\mathrm{d}j_{23}} = \frac{j_{12}j_{23}}{6V}, \tag{3.3}$$

where V is the volume of the tetrahedron in fig. 3. Therefore, for large angular momenta, we obtain Wigner's result (1.3a). If j_{23} is such that $V^2<0$, it is impossible to reach the prescribed value of j_{23} by varying ψ in the real interval $0-2\pi$.

The arguments given so far are clearly heuristic since they assume as granted a uniform probability distribution in ψ. A rigorous justification of this statement would take us too far and would destroy the simplicity of the discussion.

It is, however, interesting to notice that there is a variation to Wigner's argument which seems to have escaped detection so far. Let us suppose that in fig. 3 three of the vertices of the tetrahedron are held fixed, while the remaining one P is allowed to vary. We use J^2, j_{23}^2, j_3^2 as coordinates of P instead of the usual Euclidean coordinates (i.e., for instance, the components of $\boldsymbol{J}$: J_x, J_y, J_z). Notice that there are two points P corresponding to the same set J^2, j_{23}^2, j_3^2. We assume that the "a priori" probability for P to lie in the small volume $\mathrm{d}V=\mathrm{d}J_x\,\mathrm{d}J_y\,\mathrm{d}J_z$ does not depend on P. In this case, the probability that the "tricentrical" coordinates of P: J^2, j_{23}^2, j_3^2 lie in the interval $\mathrm{d}J^2\,\mathrm{d}j_{23}^2\,\mathrm{d}j_3^2$ is

$$2\left|\frac{\partial(J^2, j_{23}^2, j_3^2)}{\partial(J_x, J_y, J_z)}\right|^{-1} \mathrm{d}J^2\,\mathrm{d}j_{23}^2\,\mathrm{d}j_3^2 = 2\mathscr{J}\,\mathrm{d}J^2\,\mathrm{d}j_{23}^2\,\mathrm{d}j_3^2, \tag{3.4}$$

and since

$$\left|\frac{\partial(J^2, j_{23}^2, j_3^2)}{\partial(J_x, J_y, J_z)}\right| = 8\,|\boldsymbol{J}\times\boldsymbol{j}_{23}\cdot\boldsymbol{j}_3| = 48V,$$

using (1.3a) we have

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{Bmatrix}^2 \sim 2\,\frac{\mathscr{J}}{\pi}. \tag{3.5}$$

This second argument has the advantage that it can be formally generalized to higher $3nj$-symbols.

An interesting discussion, which leads to a generalization of Wigner's formula, can be developed by the combined use of the known results[16]) for the 1+1+2 case and of the Biedenharn–Elliott identity[17]). According to Edmonds (eq. (A.2.2) of ref. 16) we have with our conventions of appendix A

$$\begin{Bmatrix} c & a & b \\ f & b+\delta & a+\delta' \end{Bmatrix} \simeq \frac{(-1)^{a+b+c+f+\delta}}{[(2a+1)(2b+1)]^{\frac{1}{2}}}\, d^{(f)}_{\delta,\delta'}(\theta), \tag{3.6}$$

where a, b, c are large in comparison with f, δ, δ' and (see fig. 4)

$$\cos\theta = \frac{a(a+1)+b(b+1)-c(c+1)}{2[a(a+1)b(b+1)]^{\frac{1}{2}}}, \qquad 0\leqslant\theta\leqslant\pi. \tag{3.7}$$

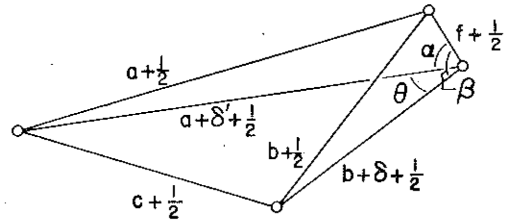


Fig. 4. Particular case in which f is small with respect to the other edges.

Let us recall here the just mentioned identity

$$Y \equiv \begin{Bmatrix} g & h & j \\ e & a & d \end{Bmatrix} \begin{Bmatrix} g & h & j \\ e' & a' & d' \end{Bmatrix} = \sum_x (-1)^{\varphi_x}(2x+1) \begin{Bmatrix} a & a' & x \\ d' & d & g \end{Bmatrix} \begin{Bmatrix} d & d' & x \\ e' & e & h \end{Bmatrix} \begin{Bmatrix} e & e' & x \\ a' & a & j \end{Bmatrix}, \tag{3.8}$$

$$\varphi_x = g + h + j + e + a + d + e' + a' + d' + x .$$

Now we intend to use (3.8) under the following conditions:

i) $g, h, j, e, a, d, e', a', d'$, are large;

ii) $e'-e=\eta$, $a'-a=\alpha$, $d'-d=\delta$ are small with respect to the parameters quoted in i). It follows that the $6j$-symbols which appear under summation in (3.8) are of a form suitable for the use of (3.6); we have:

$$\begin{Bmatrix} a & a' & x \\ d' & d & g \end{Bmatrix} \simeq \frac{(-1)^{a+d+g+\delta+x}}{[(2a+1)(2d+1)]^{\frac{1}{2}}} d^{(x)}_{\delta,\alpha}(\gamma),$$
$$\cos\gamma = \frac{a(a+1)+d(d+1)-g(g+1)}{2[a(a+1)d(d+1)]^{\frac{1}{2}}}, \quad 0 \leqslant \gamma \leqslant \pi, \tag{3.9}$$

$$\begin{Bmatrix} d & d' & x \\ e' & e & h \end{Bmatrix} \simeq \frac{(-1)^{d+e+h+\eta+x}}{[(2d+1)(2e+1)]^{\frac{1}{2}}} d^{(x)}_{\eta,\delta}(\chi),$$
$$\cos\chi = \frac{d(d+1)+e(e+1)-h(h+1)}{2[d(d+1)e(e+1)]^{\frac{1}{2}}}, \quad 0 \leqslant \chi \leqslant \pi, \tag{3.10}$$

$$\begin{Bmatrix} e & e' & x \\ a' & a & j \end{Bmatrix} \simeq \frac{(-1)^{e+a+j+\alpha+x}}{[(2e+1)(2a+1)]^{\frac{1}{2}}} d^{(x)}_{\alpha,\eta}(\iota),$$
$$\cos\iota = \frac{e(e+1)+a(a+1)-j(j+1)}{2[e(e+1)a(a+1)]^{\frac{1}{2}}}, \quad 0 \leqslant \iota \leqslant \pi. \tag{3.11}$$

If we now replace (3.9)–(3.11) into (3.8) and assume that we may perform the asymptotic limit under the infinite summation on x, we obtain an expression for the product of symbols in the l.h.s. of (3.8) where *all* angular momenta are large. Clearly this procedure is incorrect, but, nevertheless,

it turns out that it is very instructive. We find

$$\begin{Bmatrix} g & h & j \\ e & a & d \end{Bmatrix} \begin{Bmatrix} g & h & j \\ e+\eta & a+\alpha & d+\delta \end{Bmatrix} \sim$$
$$\sim [(2a+1)(2e+1)(2d+1)]^{-1} \sum_{x=0}^{\infty} (2x+1)\, d^{(x)}_{\alpha,\eta}(\iota)\, d^{(x)}_{\eta,\delta}(\chi)\, d^{(x)}_{\delta,\alpha}(\gamma). \tag{3.12}$$

The sum in the r.h.s. can be performed by exploiting the group representation properties of the functions $d^{(x)}_{\alpha,\eta}(\iota)$ as shown in appendix C. The result is

$$\begin{Bmatrix} g & h & j \\ e & a & d \end{Bmatrix} \begin{Bmatrix} g & h & j \\ e+\eta & a+\alpha & d+\delta \end{Bmatrix} \sim \frac{\Theta(V^2)}{24\pi V} \cos(\eta E + \alpha A + \delta D), \tag{3.13}$$

where $\Theta(V^2)=0$ or 1 according to $V^2<0$ or $V^2>0$; V is the volume of the tetrahedron T with edges $g+\frac{1}{2}$, $h+\frac{1}{2}$ etc. The angles E, A, D are defined by

$$\cos E = \frac{\cos\chi\cos\iota - \cos\gamma}{\sin\chi\sin\iota},$$
$$\cos A = \frac{\cos\iota\cos\gamma - \cos\chi}{\sin\iota\sin\gamma}, \quad \cos D = \frac{\cos\gamma\cos\chi - \cos\iota}{\sin\gamma\sin\chi} \tag{3.14}$$

and A, for instance, can be interpreted as the angle between outer normals of the two faces of T which have a as common edge. If $\delta=\alpha=\eta=0$, we find once again Wigner's result.

This procedure is in part disappointing because it fails to yield a complete description of the rapidly oscillating term $\mathscr{C}$ in (1.3b). However the result (3.13), when δ, α, $\eta \neq 0$, is very illuminating because the r.h.s. contains the interference term $\cos(\eta E+\alpha A+\delta D)$ which, according to the point of view exposed in section 1, is an average over the product of the two rapidly oscillating factors of the two $6j$-symbols.

In order to reconstruct the original expression, let us introduce the function

$$\Omega'(T) \equiv \Omega'\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \sum_{h,k=1}^{4} j_{hk}\theta_{hk}, \tag{3.15a}$$

with notations defined in section 1. An interesting property of $\Omega'(T)$ is that, with obvious notations[18])

$$\Omega'(T+\delta T) - \Omega'(T) = \sum_{h,k=1}^{4} \delta j_{hk}\theta_{hk} \quad \text{or} \quad \frac{\partial\Omega'}{\partial j_{hk}} = \theta_{hk}, \tag{3.16}$$

i.e. we may vary the parameters in Ω' as if θ_{hk} were constant. Therefore we deduce readily that

$$\delta\Omega' = \Omega'\begin{pmatrix} g & h & j \\ e+\eta & a+\alpha & d+\delta \end{pmatrix} - \Omega'\begin{pmatrix} g & h & j \\ e & a & d \end{pmatrix} = \eta E + \alpha A + \delta D,$$

and the r.h.s. is just the argument of the cosine in (3.13). Notice that V and $\delta\Omega'$ are slowly varying functions of the edges as compared to Ω' itself. This result suggests a formula of the following type:

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \simeq \frac{1}{\sqrt{12\pi V}} \cos(\Omega' + \omega), \tag{3.17}$$

where ω is a yet unknown constant phase. ω can be determined by matching (3.17) to the particular 1+1+2 case studied first by Racah[1]):

$$\begin{Bmatrix} a & b & c \\ b & a & f \end{Bmatrix} \simeq \frac{(-1)^{a+b+c+f}}{\sqrt{(2a+1)(2b+1)}} P_f(\cos\theta), \tag{3.18}$$

where $\cos\theta$ is given by (3.7). If f is large, but small with respect to a, b, c, we may replace the Legendre polynomial with its asymptotic behaviour[19]):

$$P_f(\cos\theta) \simeq \left[\frac{2}{\pi(f+\frac{1}{2})\sin\theta}\right]^{\frac{1}{2}} \cos\left[(f+\tfrac{1}{2})\theta - \tfrac{1}{4}\pi\right], \tag{3.19}$$

from which we deduce:

$$\begin{Bmatrix} a & b & c \\ b & a & f \end{Bmatrix} \simeq \frac{(-1)^{a+b+c+f}}{\sqrt{12\pi V}} \cos\left[(f+\tfrac{1}{2})\theta - \tfrac{1}{4}\pi\right], \tag{3.20a}$$

having noticed that $6V \simeq (a+\frac{1}{2})(b+\frac{1}{2})(f+\frac{1}{2})\sin\theta$. On the other hand, in order to work out how Ω' depends on f, we can write $\Omega'(f+\frac{1}{2})$ explicitly for this particular case. We have (fig. 4 with $\delta=\delta'=0$): $\Omega'(0)=\pi(a+b+c+\frac{3}{2})$ and $(\partial\Omega'/\partial f)_{f+\frac{1}{2}=0} = \pi-\theta$. Therefore,

$$\begin{aligned}\Omega'(f+\tfrac{1}{2}) &\simeq \Omega'(0) + \left(\frac{\partial\Omega}{\partial f}\right)_{f+\frac{1}{2}=0}(f+\tfrac{1}{2}) \simeq \\ &\simeq \pi(a+b+c+\tfrac{3}{2}) + (\pi-\theta)(f+\tfrac{1}{2}).\end{aligned} \tag{3.21}$$

Taking into account these results, we see that (3.20a) can be rewritten as

$$\begin{Bmatrix} a & b & c \\ b & a & f \end{Bmatrix} \simeq \frac{1}{(12\pi V)^{\frac{1}{2}}} \cos(\Omega' + \tfrac{1}{4}\pi), \tag{3.20b}$$

which shows not only that (3.1) is compatible with this particular case, but also tells us that $\omega = \frac{1}{4}\pi$. We have reached, therefore, the important general

formula

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \simeq \frac{1}{(12\pi V)^{\frac{1}{2}}} \cos\Omega, \tag{1.4a}$$

$$\Omega = \sum_{h,k=1}^{4} j_{hk}\theta_{hk} + \tfrac{1}{4}\pi, \tag{3.15b}$$

which must be supplemented with explicit formulae for the angles θ_{hk} (appendix B):

$$\cos\theta_{hk} = -\frac{9}{A_h A_k}\frac{\partial V^2}{\partial(j_{rs}^2)}, \qquad 0 \leqslant \theta_{rs} \leqslant \pi \tag{3.22}$$

where $h \neq k \neq r \neq s = 1, 2, 3, 4$; A_h are defined in (1.5).

TABLE 1

Numerical examples of 6j-symbols showing the degree of approximation of our asymptotic formulae. The cases denoted with I, II, III correspond to the use of (1.4), (5.6), or (5.7), (1.8) respectively. The indices $a, b, \ldots, f$ are the same as in fig. 1. The exact values of this table as well as of table 2 are taken from A. F. Nikiforov et al., *Tables of Racah coefficients* (New York, 1965); as usual, .1–01, for instance, means 10^{-2}

a	b	c	d	e	f	Exact value	Approximate value	
1	1	1	1/2	1/2	1/2	−.33333–00	−.37828–00	I
1	1	1	1	1	1	.16667–00	.16683–00	I
7/2	7	9/2	17/2	5	5/2	−.41785–01	−.41520–01	I
17/2	15/2	10	15/2	15/2	4	.16494–01	.16422–01	I
13/2	8	9/2	13/2	6	15/2	.25518–01	.25506–01	I
5	8	12	9	7	6	−.22441–01	−.22422–01	I
9	9	9	9	9	9	−.15647–01	−.15640–01	I
1/2	1	1/2	1/2	1	1/2	.16667–00	.16026–00	II
7/2	7	9/2	17/2	5	17/2	.19826–01	.18954–01	II
4	13/2	15/2	8	9/2	17/2	−.20120–01	−.19897–01	II
17/2	15/2	9	15/2	15/2	13	−.13801–01	−.13773–01	II
17/2	15/2	10	15/2	13/2	12	.99633–02	.97944–02	II
5	8	10	9	7	10	−.21100–01	−.19971–01	II
8	9	14	9	8	10	.13420–01	.13296–01	II
9/2	5	1/2	11/2	6	3/2	.13222–01	.13549–01	III
17/2	6	9/2	17/2	6	7/2	.10386–02	.10411–02	III
5	8	13	9	7	9	−.19671–02	−.19726–02	III
17/2	15/2	10	15/2	13/2	14	.49191–03	.49301–03	III
5	8	13	9	7	12	−.11052–04	−.10443–04	III
8	9	17	9	8	13	.59756–06	.56161–06	III
8	9	17	9	8	16	.31622–09	.28579–09	III

4. Arguments in favor of the proposed asymptotic formula

We list here some arguments in favour of (1.4).

i) Our formula is numerically accurate, as can be seen from tables 1–4 and figs. 5–7. On the average, the accuracy improves as the values of the angular momenta increase.

TABLE 2

Numerical results for the 6*j*-symbol with $a=7$, $b=8$, $c=9$, $d=6$, $e=9$ as $f\equiv J$ assumes all permissible values

J	Exact value	Approximate value		J	Exact value	Approximate value	
2	.76018–02	.76923–02	III	9	−.20540–01	−.20578–01	I
3	−.22627–01	−.22171–01	II	10	−.31376–02	−.36479–02	I
4	.29469–01	.30474–01	I	11	.20586–01	.20528–01	I
5	−.13704–01	−.13212–01	I	12	.20068–01	.19019–01	II
6	−.13529–01	−.13944–01	I	13	.84777–02	.78298–02	II
7	.20388–01	.20345–01	I	14	.16637–02	.16638–02	III
8	.34782–02	.37774–02	I				

TABLE 3

Case with $a=13$, $b=15$, $c=24$, $d=29/2$, $e=33/2$ and $f\equiv J$ variable. The exact values of this table as well as of table 4 were obtained by means of the recursion formula (4.3)

J	Exact value	Approximate value		J	Exact value	Approximate value	
7/2	−.17136–01	−.18812–01	I	35/2	−.66442–02	−.66033–02	II
9/2	−.94803–02	−.94381–02	I	37/2	−.38116–02	−.37155–02	II
11/2	.59559–02	.60861–02	I	39/2	−.17980–02	−.18980–02	III
13/2	.11188–01	.11222–01	I	41/2	−.70668–03	−.72809–03	III
15/2	.25847–02	.25778–02	I	43/2	−.23231–03	−.23652–03	III
17/2	−.78677–02	−.78773–02	I	45/2	−.63708–04	−.64401–04	III
19/2	−.85291–02	−.85619–02	I	47/2	−.14450–04	−.14536–04	III
21/2	−.13122–03	−.19186–03	I	49/2	−.26704–05	−.26756–05	III
23/2	.77958–02	.77680–02	I	51/2	−.39262–06	−.39197–06	III
25/2	.81257–02	.82093–02	I	53/2	−.44251–07	−.43997–07	III
27/2	.17590–02	.19590–02	I	55/2	−.35949–08	−.35537–08	III
29/2	−.56229–02	−.54326–02	I	57/2	−.18760–09	−.18332–09	III
31/2	−.95185–02	−.94911–02	II	59/2	−.47264–11	−.44115–11	III
33/2	−.92382–02	−.87903–02	II				

TABLE 4

Case with $a=35/2$, $b=39/2$, $c=9$, $d=31/2$, $e=37/2$ and $f\equiv J$ variable

J	Exact value	Approximate value		J	Exact value	Approximate value	
4	.21149–03	.20909–03	III	20	−.66008–02	−.65600–02	I
5	−.69682–03	−.71066–03	III	21	.29687–02	.29011–02	I
6	.16116–02	.16753–02	III	22	.16649–02	.17306–02	I
7	−.30296–02	−.32386–02	III	23	−.57355–02	−.57788–02	I
8	.49034–02	.47498–02	II	24	.76269–02	.76402–02	I
9	−.70293–02	−.70467–02	II	25	−.63213–02	−.63099–02	I
10	.90494–02	.88460–02	II	26	.20313–02	.20080–02	I
11	−.10502–01	−.99536–02	II	27	.34992–02	.35214–02	I
12	.10918–01	.11351–01	I	28	−.73734–02	−.73873–02	I
13	−.99508–02	−.99826–02	I	29	.68216–02	.68250–02	I
14	.75021–02	.73395–02	I	30	−.12056–02	−.11953–02	I
15	−.38167–02	−.35839–02	I	31	−.61644–02	−.61915–02	I
16	−.49920–03	−.72271–03	I	32	−.83350–02	−.83482–02	I
17	.45603–02	.47270–02	I	33	.10743–03	.24555–03	I
18	−.73894–02	−.74789–02	I	34	−.11326–01	−.11727–01	I
19	.81842–02	.81987–02	I	35	−.58537–02	−.56155–02	II

ii) (1.4) is obviously invariant under the exchange of the vertices of T. As stated before, this is only a subgroup of the full symmetry group R_2 of 6*j*-symbols (appendix A). We checked, however, in a somewhat laborious way, that both V and Ω are actually invariant under R_2. The proof is sketched in appendix D.

iii) It has been shown[20]) that the identities (A.6), (A.7), (2.8) together with the tetrahedral symmetries, are enough to derive all properties as well as the numerical values of 6*j*-symbols, apart from an overall phase. In particular, the Biedenharn–Elliott identity and the recursion relations which follow from it are a distinctive feature of Racah coefficients. Therefore it is a highly significant result that our formula satisfies asymptotically not only these recursion relations, but also the above mentioned identities.

An intuitive understanding of these formulae can be reached by a correspondence principle of the form:

$$\frac{1}{\mathrm{i}}\frac{\partial}{\partial j_{hk}} \sim \theta_{hk},$$

where θ_{hk} are the angles appearing in Ω. By the same token we define the

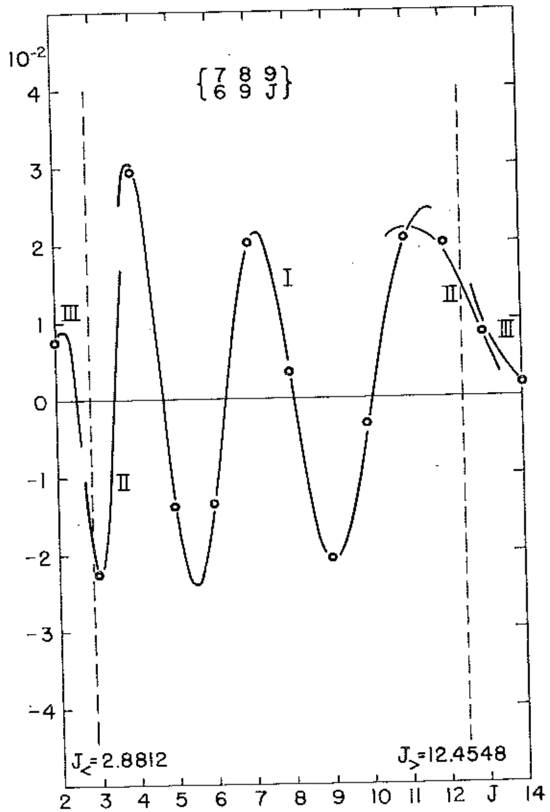


Fig. 5. Here, as well as in figs. 6 and 7, the interpolation between contiguous physical points in the classically forbidden regions is based on (1.8). For numerical values, see table 2. According to the numerical tables, curves labelled with I, II, III correspond to the use of (1.4), (5.6), or (5.7), (1.8) respectively.

operator

$$\mathscr{D}_{j_{hk}} = \exp\left(\frac{1}{2}\frac{\partial}{\partial_{j_{hk}}}\right) \sim \exp\left(\tfrac{1}{2}i\theta_{hk}\right), \tag{4.1}$$

so that, for instance:

$$\mathscr{D}_a^2 \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \begin{Bmatrix} a+1 & b & c \\ d & e & f \end{Bmatrix}$$

and from (4.1)

$$(\mathscr{D}_a^2 + \mathscr{D}_a^{-2}) \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \\ = \begin{Bmatrix} a+1 & b & c \\ d & e & f \end{Bmatrix} + \begin{Bmatrix} a-1 & b & c \\ d & e & f \end{Bmatrix} \sim 2\cos\theta_a \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}. \tag{4.2}$$

Let us consider [21])

$$\begin{aligned} & a\,[(a+b+c+2)(a-b+c+1)(a+b-c+1)\times \\ & \quad \times(b+c-a)(a+e+f+2)(a-e+f+1)(a+e-f+1)\times \\ & \quad \times(e+f-a)]^{\frac{1}{2}} \begin{Bmatrix} a+1 & b & c \\ d & e & f \end{Bmatrix} + (a+1)\,[(a+b+c+1)\times \\ & \quad \times(a-b+c)(a+b-c)(b+c-a+1)(a+e+f+1)\times \\ & \quad \times(a-e+f)(a+e-f)(e+f-a+1)]^{\frac{1}{2}} \begin{Bmatrix} a-1 & b & c \\ d & e & f \end{Bmatrix} = \\ & \quad = (2a+1)\,\{2\,[a(a+1)d(d+1) - b(b+1)e(e+1) - c(c+1)\times \\ & \quad \times f(f+1)] + [a(a+1) - b(b+1) - c(c+1)]\,[a(a+1) + \\ & \quad - e(e+1) - f(f+1)]\} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}; \end{aligned} \tag{4.3a}$$

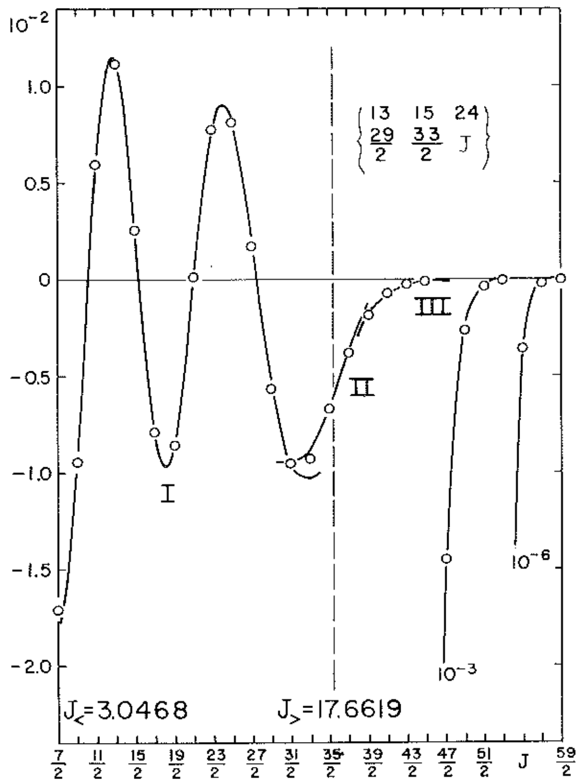


Fig. 6. In this case, which corresponds to table 3, the exponential decrease is particularly emphasized.

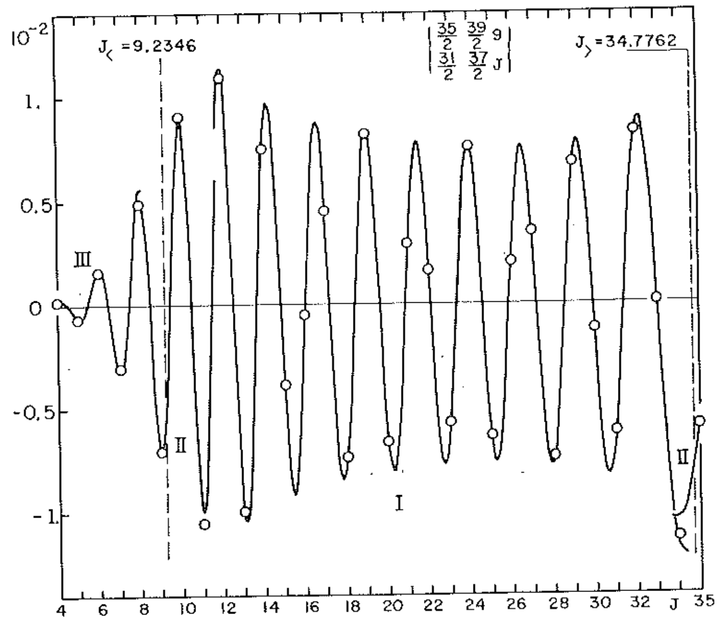


Fig. 7. The corresponding numerical values for physical points are given in table 4.

using notations defined in section 1, we have asymptotically

$$8A_1A_2\left(\begin{Bmatrix} a+1 & b & c \\ d & e & f\end{Bmatrix} + \begin{Bmatrix} a-1 & b & c \\ d & e & f\end{Bmatrix}\right) \simeq [2(j_{12}^2 j_{34}^2 - j_{13}^2 j_{24}^2 - j_{14}^2 j_{23}^2) + \\ + (j_{12}^2 - j_{13}^2 - j_{14}^2)(j_{12}^2 - j_{24}^2 - j_{23}^2)] \begin{Bmatrix} a & b & c \\ d & e & f\end{Bmatrix}.$$

Recalling (B.2) and (4.1) we obtain

$$\begin{Bmatrix} a+1 & b & c \\ d & e & f\end{Bmatrix} + \begin{Bmatrix} a-1 & b & c \\ d & e & f\end{Bmatrix} \simeq 2\cos\theta_{12} \begin{Bmatrix} a & b & c \\ d & e & f\end{Bmatrix}, \tag{4.3b}$$

which is equivalent to the result (4.2) based on the correspondence principle (4.1). Less formally, we want to check that (4.3b) is identically satisfied if we replace the $6j$-symbols of (4.3b) with our asymptotic formula (1.4). Since the volume of T is a slowly varying function of the edges, it can be regarded as constant in the three symbols of (4.3b); in this case (4.3b) transforms into the trivial identity $\cos(\Omega+\theta_{12})+\cos(\Omega-\theta_{12})\simeq 2\cos\theta_{12}\cos\Omega$.

In a similar way, the recursion relation

$$[(a+b+c+1)(b+c-a)(c+d+e+1)(c+d-e)]^{\frac{1}{2}} \begin{Bmatrix} a & b & c \\ d & e & f\end{Bmatrix} + \\ - [(a+b-c+1)(a-b+c)(e+d-c+1)(e+c-d)]^{\frac{1}{2}} \times \\ \times \begin{Bmatrix} a & b & c-1 \\ d & e & f\end{Bmatrix} = -2c[(b+d+f+1)(b+d-f)]^{\frac{1}{2}} \times \\ \times \begin{Bmatrix} a & e & f \\ d-\frac{1}{2} & b-\frac{1}{2} & c-\frac{1}{2}\end{Bmatrix}$$

by means of (1.4) becomes asymptotically

$$[(j_{13}+j_{12}+j_{14})(j_{13}+j_{14}-j_{12})(j_{14}+j_{24}+j_{34}) \times \\ \times (j_{14}+j_{34}-j_{24})]^{\frac{1}{2}} e^{\frac{1}{2}i(\theta_{13}+\theta_{14}+\theta_{34})} - [(j_{12}+j_{13}-j_{14}) \times \\ \times (j_{12}+j_{14}-j_{13})(j_{24}+j_{34}-j_{14})(j_{24}+j_{14}-j_{34})]^{\frac{1}{2}} e^{\frac{1}{2}i(\theta_{13}-\theta_{14}+\theta_{34})} \simeq \\ \simeq -2j_{14}[(j_{13}+j_{34}+j_{23})(j_{13}+j_{34}-j_{23})]^{\frac{1}{2}},$$

which, using Delambre's relations [22]), reduces to simple identities.

A much more involved computation is needed to show that also the full Biedenharn–Elliott identity is satisfied asymptotically by (1.4). Introducing (1.4) into the r.h.s. of (3.8), we realize that inside the summation over x there appears a rapidly varying function of x. Its behaviour can be displayed most transparently if we split all cosines into positive and negative frequency parts according to Euler's formula. Let T_j $(j=1, 2, 3)$ be the tetrahedra corresponding to the r.h.s. of (3.8); with obvious notations we have

$$\prod_{j=1}^{3} \cos\Omega_j = 2^{-3} \times \\ \times [e^{i(\Omega_1+\Omega_2+\Omega_3)} + e^{i(\Omega_1+\Omega_2-\Omega_3)} + e^{i(\Omega_1-\Omega_2+\Omega_3)} + e^{i(\Omega_1-\Omega_2-\Omega_3)} + \text{c.c.}]. \tag{4.4}$$

Given the heuristic character of our investigation, it is reasonable to assume that the discrete summation over x can be replaced with an integration whose most important contribution arises from points where the phase $\Gamma(x)=\Omega_1(x)+\Omega_2(x)+\Omega_3(x)$ and its analogs of (4.4) are slowly varying as functions of x. We are led quite naturally to consider, for example

$$\frac{\partial\Gamma(x)}{\partial x} = 0. \tag{4.5}$$

Denoting the supplementary dihedral angles relative to x with θ_x^1, θ_x^2, θ_x^3, a heartening result is that, since in any first order variation of the edges we may consider all angles as constants, we have as necessary condition for (4.5)

$$\theta_x^1 + \theta_x^2 + \theta_x^3 - \pi = 0\,, \tag{4.6}$$

having taken into account the phase $(-1)^{\varphi_x}$ of (3.8). If we consider the other terms in (4.4) as well, we find that the bulk of the contribution to the integral may come only from those values of x such that

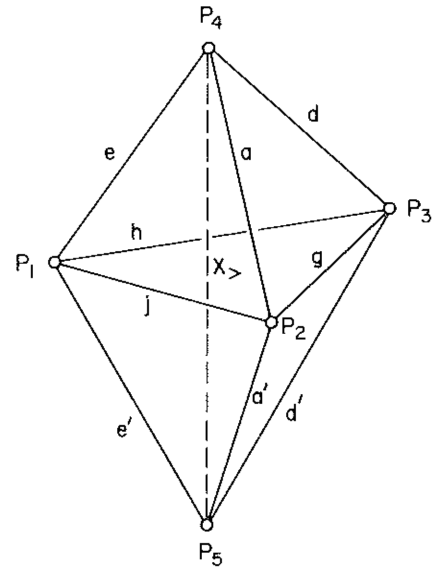
$$\pm\,\theta_x^1 \pm \theta_x^2 \pm \theta_x^3 = \pi\,. \tag{4.7}$$

The conditions (4.7) have an immediate geometrical interpretation if we look at the diagram in fig. 8a. There one sees that by leaving out in turn any one of the five points $P_1, \ldots, P_5$, the remaining points form five tetrahedra $T_1, \ldots, T_5$ which are just those appearing in (3.8). Note that there are ten edges connecting five points in all possible ways and in fact there are ten angular momenta appearing in (3.8) including x. As it has been long known, if five points are imbedded into a three dimensional-Euclidean space, their mutual distances are not independent. The explicit form of their dependence was discovered by Cayley[11]) and can be written as

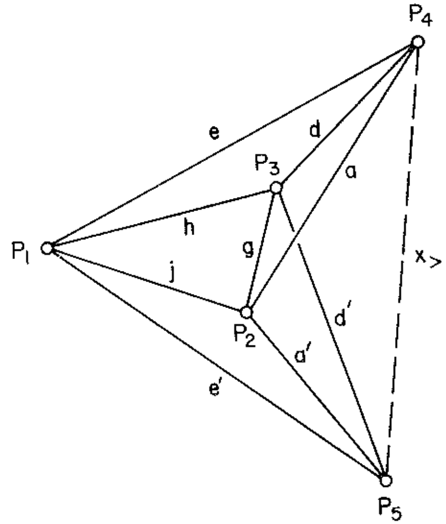
$$-\,2^4(4!)^2 I^2 \equiv \begin{vmatrix} 0 & j_{12}^2 & j_{13}^2 & j_{14}^2 & j_{15}^2 & 1 \\ j_{12}^2 & 0 & j_{23}^2 & j_{24}^2 & j_{25}^2 & 1 \\ j_{13}^2 & j_{23}^2 & 0 & j_{34}^2 & j_{35}^2 & 1 \\ j_{14}^2 & j_{24}^2 & j_{34}^2 & 0 & j_{45}^2 & 1 \\ j_{15}^2 & j_{25}^2 & j_{35}^2 & j_{45}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0\,, \tag{4.8}$$

where j_{hk} is the distance between P_h and P_k. It is crucial to understand that (4.8) is in fact equivalent to (4.7). Indeed (4.7) implies that the sum of the internal dihedral angles $\pi - \theta_x^j$ around the edge x is a multiple of 2π, as expected if the diagram is drawn in three dimensions. The ambiguity in the signs of the angles arises from the different possible orientations of the five involved tetrahedra, as exemplified by fig. 8b.

Therefore, the only asymptotical contribution to the integral comes from the configuration of the diagram in fig. 8a and the like, which are three-dimensional, i.e. from those values of x such that $I(x^2)=0$. First we notice that the range of summation is restricted to $x>0$ and, "a fortiori," to $x^2>0$. Secondly, $I(x^2)$ is a quadratic polynomial in x^2 and it has therefore two roots $(x_<)^2$ and $(x_>)^2$. There are rigorous arguments showing that, if T_4, T_5 are physical tetrahedra, i.e. $V_4^2, V_5^2>0$, then $x_<$ and $x_>$ are real. To us it will be enough to note that, quite obviously, the smaller root $x_<$ corresponds to the configuration in which P_4, P_5 lie on the same side of the plane



(a)



(b)

Fig. 8.

$P_1P_2P_3$ as depicted in fig. 9, while $x_>$ corresponds to P_4,P_5 being on opposite sides as in fig. 8a.

Since in general θ_x^j $(j=1, 2, 3)$ are not vanishing, or are equal to some multiple of π, only one of the choices of signs in (4.7) is valid for a given root.

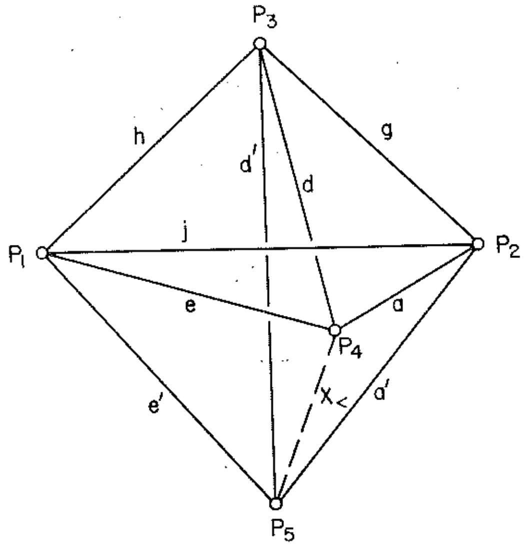


Fig. 9.

It follows that, for $x=x_<$, only one of the eight terms arising from the decomposition (4.4), together with its complex conjugate, does actually contribute to the integral. We cannot decide here which term contributes, because this will depend on the values of the other fixed angular momenta. Let us suppose they are such that for both roots $\theta_x^1+\theta_x^2+\theta_x^3=\pi$.

From (3.8) we have

$$Y \simeq \sum_x \frac{\pi^{-\frac{3}{2}}}{96\sqrt{3}} e^{i\pi(\omega+x)} x [V_1(x)\, V_2(x)\, V_3(x)]^{-\frac{1}{2}} [e^{i\Gamma(x)} + e^{-i\Gamma(x)}], \tag{4.9}$$

where $\omega=a+e+d+a'+e'+d'+g+j+h$. Let us write now

$$x+\tfrac{1}{2}=R\xi;\quad a+\tfrac{1}{2}=R\alpha,\quad e+\tfrac{1}{2}=R\eta,\quad d+\tfrac{1}{2}=R\delta,\quad \ldots,\quad d'+\tfrac{1}{2}=R\delta' \tag{4.10}$$

with R large and ξ, α, η, δ, α', η', δ' finite; note that the angles θ_{hk} of the five tetrahedra are independent of R. The spacing in the summation on ξ is now

R^{-1} and, as anticipated, we may replace the summation with an integral on ξ

$$Y \simeq \frac{\pi^{-\frac{3}{2}}}{96\sqrt{3}} e^{-i\pi(\omega-\frac{3}{4})} R^2 \int_{\Xi} d\xi\, \xi [V_1(\xi)\, V_2(\xi)\, V_3(\xi)]^{-\frac{1}{2}} e^{iR[\tilde{\Gamma}(\xi)-\pi\xi]} + \text{c.c.} \tag{4.11}$$

having used the fact that $\omega+x$ is integer; here $\Gamma(x)=R\tilde{\Gamma}(\xi)+\frac{1}{4}\pi$. In the limit $R\to\infty$ this integral can be computed with the steepest descent method [23]), which yields for an integral of the form

$$\mathcal{J} = \int_\alpha^\beta g(\xi)\, e^{iRf(\xi)}\, d\xi, \tag{4.12}$$

with f and g real, the approximate result

$$\mathcal{J}_{R\to\infty} \cong \sum_j \left(\frac{2\pi}{R\,|f''(\xi_j)|}\right)^{\frac{1}{2}} g(\xi_j)\, e^{i[Rf(\xi_j)\pm\frac{1}{4}\pi]}; \tag{4.13}$$

here ξ_j are such that $f'(\xi_j)=0$ and $\alpha<\xi_j<\beta$. In (4.13) the phases $\pm\frac{1}{4}\pi$ must be chosen according to $f''(\xi_j)\gtrless 0$. If $\xi_\gtrless$ correspond to $x_\gtrless$, since we have supposed

$$\left[\frac{\partial[\tilde{\Gamma}(\xi)-\pi\xi]}{\partial\xi}\right]_{\xi=\xi_\gtrless} = [\theta_x^1+\theta_x^2+\theta_x^3-\pi]_{x=x_\gtrless}=0, \tag{4.14}$$

we find in our case

$$f(\xi)=\tilde{\Gamma}(\xi)-\pi\xi;\quad g(\xi)=\xi[V_1(\xi)\, V_2(\xi)\, V_3(\xi)]^{-\frac{1}{2}},$$
$$f''(\xi)=\frac{\partial(\theta_x^1+\theta_x^2+\theta_x^3-\pi)}{\partial\xi}. \tag{4.15}$$

A naive computation of $f''(\xi)$ is out of question because of the lengthy and uninspiring algebra involved. We rather take $I^2(x^2)$, defined in (4.8) as independent variable and write

$$f''(\xi_\gtrless)=\lim_{\xi\to\xi_\gtrless}\left[\frac{\partial(\theta_x^1+\theta_x^2+\theta_x^3-\pi)}{\partial I^2(x^2)}\frac{\partial I^2(x^2)}{\partial\xi}\right]; \tag{4.16}$$

some manipulation of determinants (appendix D) shows that

$$f''(\xi_\gtrless)=\mp\tfrac{1}{6}R^3(\xi_\gtrless)^2\left[\frac{V_4V_5}{V_1V_2V_3}\right]_{\xi=\xi_\gtrless}. \tag{4.17}$$

The evaluation of $f(\xi_{\gtrless})$ is straightforward and yields

$$Rf\,(\xi_{\gtrless}) \mp \tfrac{1}{4}\pi = \Omega_4 \pm \Omega_5 + \pi(\omega - \tfrac{3}{4}). \tag{4.18}$$

From (4.11)–(4.18) we obtain

$$Y \simeq \frac{1}{4}\frac{1}{12\pi(V_4 V_5)^{\frac{1}{2}}}\,[\mathrm{e}^{\mathrm{i}(\Omega_4+\Omega_5)} + \mathrm{e}^{\mathrm{i}(\Omega_4-\Omega_5)} + \text{c.c.}] \tag{4.19}$$

$$Y \simeq \frac{1}{(12\pi V_4)^{\frac{1}{2}}}\cos\Omega_4 \frac{1}{(12\pi V_5)^{\frac{1}{2}}}\cos\Omega_5\,,$$

which is in obvious agreement with the straightforward use of (1.4) in the l.h.s.

We indulged somewhat more than strictly necessary on the proof of the Biedenharn–Elliott identity in the asymptotic limit, because we felt that the mechanism involved is illuminating and more general than shown by this case. We do not discuss whether the other identities of Racah coefficients: (A.6), (A.7) are satisfied by (1.4), since these proofs follow quite easily from the stationary-phase method. The same procedure can be extended in principle to the computation of asymptotic $9j$-symbols.

5. A formal analogy with the WKB method

In the previous section we strived to provide as many as possible independent checks and counterchecks for the validity of (1.4) in the region $A(V^2>0)$. As stated in our Introduction, a complete description of the behaviour of a $6j$-symbol for large angular momenta must include of necessity a similar formula for the region $B(V^2<0)$ and makes it highly desirable to have one for the transitional region $V^2 \simeq 0$ as well.

The guiding idea in this section is a formal analogy with a WKB approximation for the solution of a differential equation. This analogy was prompted by the actual existence of a differential equation at least in the 1+1+2 case, where in fact the symbol can be expressed as a Jacobi polynomial.

Unfortunately we have not been able to derive a single differential equation valid for unrestricted large parameters, with the possible exception of the transitional region. Our disappointment is somehow mitigated by noticing that, after all, we need a differential equation only in order to fix unambiguously the prosecution of (1.4) through a transition point. This we have achieved and the resulting formulae satisfy properties which are similar to the ones listed under i, ii, iii in section 4. However, it must be stated that the WKB analogy is just a formal device which cannot be accepted as a proof.

Before proceeding further along this analogy, a more detailed investigation of the region B is necessary. From (B.3) we see that if $V^2=0$ and $\prod_{h=1}^{4} A_h \neq 0$, then the angles θ_{hk} are all multiple of π. Since we always choose $0 \leqslant \theta_{hk} \leqslant \pi$, we have that either $\theta_{hk}=0$ or $\theta_{hk}=\pi$; the ambiguity can be settled by means of (3.22). Here, as in the following, we always assume that the areas A_h are represented by positive square roots of radicands like (B.1). Because of the conditions satisfied by angular momenta and taking into account our definition of j_{hk}, it is easy to see that these radicands are indeed always non-negative and may vanish only if $V^2=0$, in which case (see footnote 12), at least one angular momentum assumes a non-physical value.

If $V^2=0$, the four vertices of T lie in the same plane; Let Q be the convex plane set generated by the four vertices. Depending on their relative position, Q may have three or four edges (see figs. 10 and 1). As the reader can

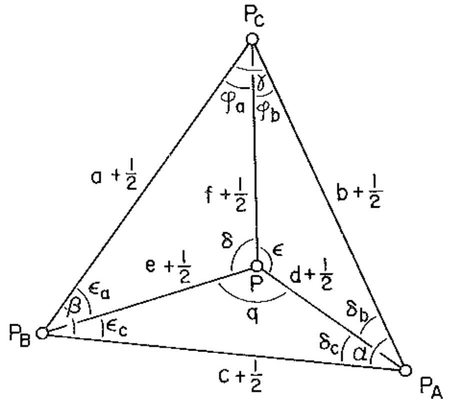


Fig. 10.

easily check, the rule is then: $\theta_{hk}=\pi$ for the edges of Q and $\theta_{hk}=0$ for the others. We may use the symbol $\begin{Bmatrix}\pi & \pi & \pi\\ 0 & 0 & 0\end{Bmatrix}$ to denote the set $\theta_{12}=\theta_{13}=\theta_{14}=\pi$, $\theta_{34}=\theta_{24}=\theta_{23}=0$. The only possibilities are $\begin{Bmatrix}\pi & \pi & \pi\\ 0 & 0 & 0\end{Bmatrix}$, $\begin{Bmatrix}\pi & 0 & 0\\ 0 & \pi & \pi\end{Bmatrix}$, $\begin{Bmatrix}0 & 0 & \pi\\ \pi & \pi & 0\end{Bmatrix}$, $\begin{Bmatrix}0 & \pi & 0\\ \pi & 0 & \pi\end{Bmatrix}$, $\begin{Bmatrix}\pi & \pi & 0\\ \pi & \pi & 0\end{Bmatrix}$, $\begin{Bmatrix}\pi & 0 & \pi\\ \pi & 0 & \pi\end{Bmatrix}$, $\begin{Bmatrix}0 & \pi & \pi\\ 0 & \pi & \pi\end{Bmatrix}$. The subsets with some $A_h=0$ lie on the boundary of the above sets. For instance, the case $j_{12}=j_{13}+j_{14}$ is the common boundary of $\begin{Bmatrix}\pi & 0 & 0\\ 0 & \pi & \pi\end{Bmatrix}$, $\begin{Bmatrix}0 & \pi & \pi\\ 0 & \pi & \pi\end{Bmatrix}$ and, in fact, here θ_{12}, θ_{13}, θ_{14} are discontinuous.

If $V^2<0$, then the angles θ_{hk} are all of the form $n\pi+i\xi_{hk}$. Because of continuity, it follows that $\mathrm{Re}\,\theta_{hk}$ are constants in B whenever $\prod_{h=1}^{4} A_h\neq 0$. Therefore we may subdivide B into non-overlapping regions, all points in the same region having the same values of $\mathrm{Re}\,\theta_{hk}$. If we let $V\to 0$ and $\prod_{h=1}^{4} A_h\neq 0$, we see that $\lim_{V\to 0}\mathrm{Re}\,\theta_{hk}=\theta_{hk}(V=0)$ so the value of $\mathrm{Re}\,\theta_{hk}$ is determined by the limiting process for $V\to 0$ and is the same as listed above. We may subdivide B into regions $B\begin{pmatrix}\pi&\pi&\pi\\0&0&0\end{pmatrix}$, $B\begin{pmatrix}\pi&\pi&0\\\pi&\pi&0\end{pmatrix}$, etc. such that we have, for instance, in $B\begin{pmatrix}\pi&\pi&\pi\\0&0&0\end{pmatrix}$: $\mathrm{Re}\,\theta_{12}=\mathrm{Re}\,\theta_{13}=\mathrm{Re}\,\theta_{14}=\pi$, $\mathrm{Re}\,\theta_{23}=\mathrm{Re}\,\theta_{24}=\mathrm{Re}\,\theta_{34}=0$; it is also convenient to use the following abbreviations:

$$B_3\equiv B\begin{pmatrix}\pi&\pi&\pi\\0&0&0\end{pmatrix}\cup B\begin{pmatrix}\pi&0&0\\0&\pi&\pi\end{pmatrix}\cup B\begin{pmatrix}0&0&\pi\\\pi&\pi&0\end{pmatrix}\cup B\begin{pmatrix}0&\pi&0\\\pi&0&\pi\end{pmatrix},$$

$$B_4\equiv B\begin{pmatrix}\pi&\pi&0\\\pi&\pi&0\end{pmatrix}\cup B\begin{pmatrix}\pi&0&\pi\\\pi&0&\pi\end{pmatrix}\cup B\begin{pmatrix}0&\pi&\pi\\0&\pi&\pi\end{pmatrix}.$$

These regions are not closed, for they have common boundary points in which some A_h's vanish. As anticipated in (1.7), in every such region we define a phase function Φ as follows

$$\Phi=\sum_{h,k=1}^{4}(j_{hk}-\tfrac{1}{2})\,\mathrm{Re}\,\theta_{hk}.$$

For physical values of the angular momenta, Φ is always an integer multiple of π.

We fix now the value of all parameters but one. Without loss of generality we may always choose a as variable and, in order to stress this point, we use x instead of a. We do not restrict ourselves to physical values of x and of the remaining parameters in order to retain some, albeit nominal, freedom in the final results. If $V^2(x_{\gtrless})=0$ we have

$$\begin{aligned}\Omega(x_{\gtrless})&=\Phi-\tfrac{1}{4}\pi\quad\text{approaching } B_3,\\ \Omega(x_{\gtrless})&=\Phi+\tfrac{1}{4}\pi\quad\text{approaching } B_4.\end{aligned}\tag{5.1}$$

In appendix F it is shown that for $x\to x_{\gtrless}$:

$$\Omega-\Omega(x_{\gtrless})=\begin{cases}+\dfrac{9}{2}\dfrac{V^3}{\prod_{h=1}^{4}A_h} & \text{approaching } B_3,\\[2ex] -\dfrac{9}{2}\dfrac{V^3}{\prod_{h=1}^{4}A_h} & \text{approaching } B_4.\end{cases}\tag{5.2}$$

It follows that in the neighbourhood of a transition point we have

$$\begin{Bmatrix}a&b&c\\d&e&f\end{Bmatrix}\simeq\frac{1}{\sqrt{12\pi V}}\cos\left(\frac{9}{2}\frac{V^3}{\prod_{h=1}^{4}A_h}+\Phi-\tfrac{1}{4}\pi\right);\tag{5.3}$$

obviously this formula is incorrect if one gets too close to the transition points. We notice however that, as it stands, (5.3) is the WKB asymptotic approximation to the following differential equation (see (G.6)):

$$\frac{\mathrm{d}}{\mathrm{d}(V^2)}\frac{\mathrm{d}}{\mathrm{d}(V^2)}\psi=\left(\frac{27}{4\prod_{h=1}^{4}A_h}\right)^2V^2\psi,\tag{5.4}$$

where the independent variable is V, $\prod_{h=1}^{4} A_h$ being treated as a constant[24]). The general solution of (5.4) is (appendix G):

$$\psi\sim V\left[c_1J_{\frac{1}{3}}\left(\frac{9}{2}\frac{V^3}{\prod_{h=1}^{4}A_h}\right)+c_2J_{-\frac{1}{3}}\left(\frac{9}{2}\frac{V^3}{\prod_{h=1}^{4}A_h}\right)\right]\tag{5.5}$$

and the one which joins smoothly with (5.3) for large values of $V^2>0$ is:

$$\begin{Bmatrix}a&b&c\\d&e&f\end{Bmatrix}\simeq2^{-\frac{4}{3}}\left(\prod_{h=1}^{4}A_h\right)^{-\frac{1}{6}}\times$$
$$\times\left\{\cos\Phi\,\mathrm{Ai}\left[-\left(\frac{(3V)^2}{\left(4\prod_{h=1}^{4}A_h\right)^{\frac{2}{3}}}\right)\right]+\sin\Phi\,\mathrm{Bi}\left[-\left(\frac{(3V)^2}{\left(4\prod_{h=1}^{4}A_h\right)^{\frac{2}{3}}}\right)\right]\right\}.\tag{5.6}$$

in terms of Airy functions[25]). We assume (5.6) to be the correct asymptotic form of the Racah coefficient in the transitional region. The soundness of this assumption is of course at this stage purely aesthetical. However, our conjecture is borne out by comparison with published tables (see tables 1–4).

According to the standard procedure, we may continue (5.6) into the region B; here the resulting formula in the neighbourhood of a transition point is:

$$\begin{Bmatrix}a&b&c\\d&e&f\end{Bmatrix}\simeq2^{-\frac{4}{3}}\left(\prod_{h=1}^{4}A_h\right)^{-\frac{1}{6}}\times$$
$$\times\left\{\cos\Phi\,\mathrm{Ai}\left[\frac{(3|V|)^2}{\left(4\prod_{h=1}^{4}A_h\right)^{\frac{2}{3}}}\right]+\sin\Phi\,\mathrm{Bi}\left[\frac{(3|V|)^2}{\left(4\prod_{h=1}^{4}A_h\right)^{\frac{2}{3}}}\right]\right\}.\tag{5.7}$$

For large values of $|V|^2$ this function joins smoothly with

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \simeq \frac{1}{2(12\pi|V|)^{\frac{1}{2}}} \{2 \sin \Phi\, e^{|\mathrm{Im}\,\Omega|} + \cos \Phi\, e^{-|\mathrm{Im}\,\Omega|}\}, \qquad (5.8)$$

where $\mathrm{Im}\,\Omega = \sum_{h,k=1}^{4} \mathrm{Im}\,\theta_{hk}$. Choosing conventionally in region B: $V = \mathrm{i}|V|$, the imaginary part of θ_{hk} can be retrieved by means of the following formulae:

$$\cosh(\mathrm{Im}\,\theta_{hk}) = -\cos(\mathrm{Re}\,\theta_{hk}) \frac{9}{A_h A_k} \frac{\partial V^2}{\partial j_{rs}^2}, \quad h \neq k \neq r \neq s, \qquad (3.22a)$$

$$\sinh(\mathrm{Im}\,\theta_{hk}) = \cos(\mathrm{Re}\,\theta_{hk}) \frac{3\, j_{hk}}{2\, A_h A_k} |V|, \quad h \neq k. \qquad (1.5a)$$

According to (5.8), the $6j$-symbol would be represented in region B by a superposition of decreasing and increasing exponentials; however, the coefficient $\sin \Phi$ of the increasing exponential vanishes at the physical points, where (5.8) reduces to (1.8). It turns out that (5.8) is numerically accurate when applied to physical angular momenta.

It is worth noticing that the coefficient $\cos \Phi = (-1)^{\Phi/\pi}$ of the decreasing exponential gives instead a determination of the sign of the $6j$-symbol. This sign is in complete agreement with numerical tables, as well as with the one obtained from the limiting case of a stretched tetrahedron where one edge reaches its maximum permissible value. As an example, let $a = b + c$; from (A.4) we have:

$$\text{sign of} \begin{Bmatrix} b+c & b & c \\ d & e & f \end{Bmatrix} = (-1)^{b+c+e+f}. \qquad (5.9)$$

Since $b + c > x_> - \frac{1}{2}$, it is a simple exercise to see that if a increases through $x_>$, then we enter either one of the regions $B\begin{pmatrix} 0 & \pi & \pi \\ 0 & \pi & \pi \end{pmatrix}$, $B\begin{pmatrix} \pi & 0 & 0 \\ 0 & \pi & \pi \end{pmatrix}$, $B\begin{pmatrix} \pi & \pi & \pi \\ 0 & 0 & 0 \end{pmatrix}$. The region $B\begin{pmatrix} \pi & \pi & \pi \\ 0 & 0 & 0 \end{pmatrix}$ is exluded because the point $a = b + c$ is almost on the boundary between $B\begin{pmatrix} 0 & \pi & \pi \\ 0 & \pi & \pi \end{pmatrix}$ and $B\begin{pmatrix} \pi & 0 & 0 \\ 0 & \pi & \pi \end{pmatrix}$; actually, it would be exactly on this boundary if: $a + \frac{1}{2} = b + \frac{1}{2} + c + \frac{1}{2}$, or $a - b - c = \frac{1}{2}$ which is prevented by selection rules. Therefore our phase is $\Phi = \pi(b + c + e + f) = \pi(a + e + f)$ and it agrees with (5.9).

Further evidence in favour of (5.6), (5.8) is offered by the discussion of the particular case: 2+1+1; this analysis is carried out in appendix H where it is shown that the behaviour of (3.6) in region A as well as in B is in

agreement with (5.8). We notice also that the arguments of the Airy functions in (5.6), (5.7) as well as Φ are invariant under R_2.

We have now a complete set of conjectured asymptotic formulae valid for every range of large *physical* angular momenta. Numerical examples deduced from these formulae are shown in tables 1–4 and in figs. 5–7.

6. The $3nj$-symbols

The problem of extending our results to higher $3nj$-symbols is certainly very difficult and we were not able to reach a solution within the frame of this paper. Yet some of the intuitive arguments presented here provide fairly interesting information about the general problem.

In dealing with $3nj$-symbols where n is large the use of diagrams becomes imperative. At first the diagrams, as in the current literature[7]), are just a mnemonical device in order to keep track of the growing complexities of the symbols. They provide an information, which is purely combinatorial, on how angular momenta are coupled in a given scheme. In this sense a very natural language for diagrams is provided by combinatorial topology.

A diagram is essentially a shorthand notation for the expansion of a $3nj$-symbol in terms of $3j$-symbols. Let [D] be the $3nj$-symbol corresponding to the diagram D. [D] can be expressed as:

$$[\mathrm{D}] = \sum_{\text{all } m} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \cdots \begin{pmatrix} l_i & l_j & l_k \\ m_i & m_j & m_k \end{pmatrix} \begin{pmatrix} & l_k & \\ m_k & & m_k' \end{pmatrix} \begin{pmatrix} l_k & l_p & l_q \\ m_k' & m_p & m_q \end{pmatrix} \cdots, \qquad (6.1)$$

where[4]):

$$\begin{pmatrix} & l_k & \\ m_k & & m_k' \end{pmatrix} = (-1)^{l_k - m_k} \delta_{m_k, -m'_k}. \qquad (6.2)$$

D can be retrieved from (6.1) by means of the following rules:

a) D is a 2-dimensional combinatorial manifold.

b) There is a one-to-one correspondence between 1-simplexes (edges) of D and angular momenta in the r.h.s. of (6.1).

c) There is a one-to-one correspondence between $3j$-symbols in (6.1) and 2-simplexes (faces) of D.

d) The boundary of a face is the sum of the three edges appearing as angular momenta in the corresponding $3j$-symbol.

e) We work for the time being with homology modulo 2, i.e. we forget about the orientation of the simplexes.

From the general structure of [D] we see that there are $2n$ faces and $3n$ edges in D. We have no "a priori" conditions on 0-simplexes (vertices) and in fact they have no physical meaning. This brings in a certain amount of arbitrariness in the construction of D which is lacking in the customary definition of diagrams[6]) (figs. 12a, b). We regard as equivalent diagrams those which yield the same symbol. Since D is a manifold, we may define its Euler characteristic:

$$M = -f + e - v + 2 = n - v + 2,$$

where f, e, v are the number of faces, edges, vertices respectively. Since $M \geqslant 0$, we have $v \leqslant n+2$. For a $6j$-symbol, $M=0$, while in the case of a $9j$-symbol we have: $n=3$, $v=4$ and $M=1$ according to the example sketched in fig. 11. When $M=1$ the diagram is the triangulation of a one-sided surface,

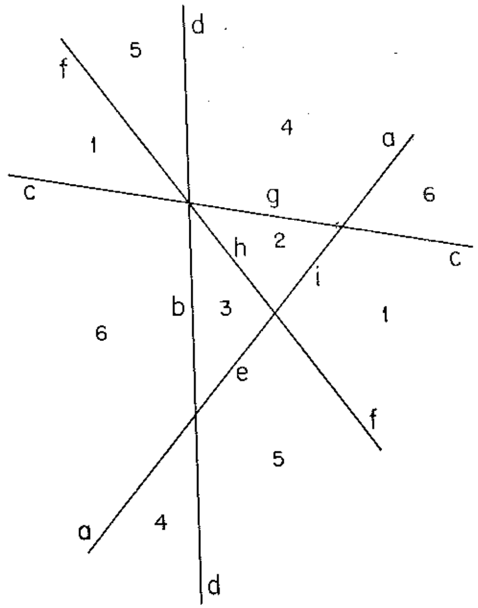


Fig. 11. Planar graphical representation of the $9j$-symbol with triads (abc), (def), (ghi), (adg), (beh), (cfi).

in our case the real projective plane. Since the sphere where opposite points are identified is a homeomorph of the projective plane, it is possible to exhibit the $9j$-symbol as a double hexagonal pyramid as shown in fig. 12a; here each edge and face are repeated twice and the whole diagram has a centre of symmetry. We shall prefer this representation to the one given in fig. 11.

In fact, in dealing with the asymptotic behaviour and with the semiclassical limit of $3nj$-symbols, we expect that it will be possible to represent angular momenta as vectors and the coupling of angular momenta as the addition of classical vectors. It follows that it will be convenient to think of a diagram as a polyhedron imbedded into a 3-dimensional Euclidean space. If one

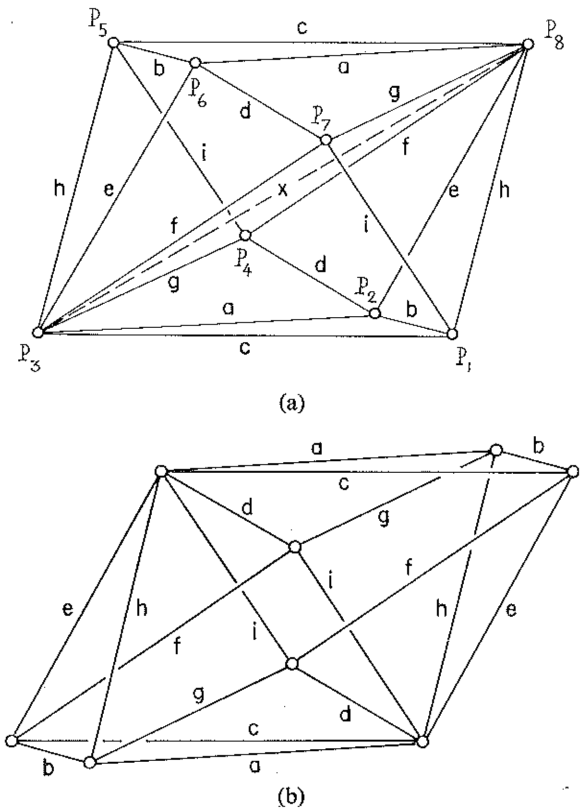


Fig. 12. Different three-dimensional diagrams for the $9j$ of fig. 11; note that the edges in case a have the same length and direction as in b.

such a construction exists with prescribed lengths of the edges, we shall speak of a configuration of the diagram.

An interesting phenomenon is that there are several different configurations with identical edges for the same diagram. This ambiguity is connected with the fact that there is uniqueness for a given polyhedron of given topology and given edges only if the polyhedron is convex. A trivial example is a pair of mirror-like right-handed and left-handed tetrahedra. A less trivial example arises already with the 9*j*-symbol and we expect it to become more and more important for higher symbols. The technical reasons of this multiplicity can be best seen in the 9*j*-symbols. Here the geometrical shape of the object would be completely determined if we knew all the relative angles of all the edges of the diagram. Elementary theorems tell us that this can be achieved if the angle we are looking for is the internal angle of some triangle of which all edges are known; this is true in particular for all faces in the diagram. A similar attempt to compute, for instance, the angle between b and f (fig. 12a) fails unless we know the length of the diagonal x. In the particular case of the 9*j*-symbol it turns out that all angles can be computed, provided we know this only missing length x; as we shall see, there may be in principle as much as four different configurations with the same topology.

Since the classification and the discussion of these configurations is relevant to the asymptotic behaviour of the symbol, it is convenient to refine the so far used language. We shall introduce the word diagram when only the topological properties are considered and in doing so we identify equivalent diagrams. Configuration is instead a diagram with the additional information about the angles needed in order to remove the above ambiguities. We may introduce the additional word orientation if a distinction between left-handed and right-handed configurations is desired.

For complicated 3*nj*-symbols the number of different configurations for the same diagram grows very rapidly. From our discussion it is clear that if one gives the distance between any pair of vertices in the diagram, then the configuration is completely determined. We cannot give here a general set of rules which would allow one to compute the missing lengths. We found, however, that in the simplest cases it is enough to exploit the relations among squared edges which can be obtained as follows:

a) the diagram may contain quadrilaterals with opposite equal edges; in this case the sum of the squared diagonals is twice the sum of the two different squared edges, a known and elementary relation;

b) since the diagram is imbedded into a 3-dimensional space, one can write for any choice of five vertices the Cayley identity (4.8).

Not all these relations are actually independent. Once a complete and consistent set of identities has been written, one finds a set of algebraic equations for the missing lengths; to each solution of these equations we associate a configuration.

We come now to the general problem of computing the 3*nj*-symbols.

In the available literature[7]) explicit formulae are written for any symbol with the aid of diagrams in terms of lower order coefficients. The diagrams used in these previous works are not the same as ours; however it is not difficult to translate one language into the other.

To this purpose we introduce another diagram $\mathscr{D}(\mathrm{D})$ which corresponds to the familiar procedure of dissecting the interior of the polyhedron into tetrahedra; more formally, $\mathscr{D}(\mathrm{D})$ can be conveniently defined as a 3-dimensional combinatorial manifold with boundary D. We name cells the 3-simplexes of $\mathscr{D}$. The edges, faces and vertices of $\mathscr{D}$ will be named external if they belong to D, else internal. Let the set of cells be labelled by T_k, $k=1, 2, \ldots, p$.

From the definition of the symbol [D], we know that it is a function of as many variables as different edges of D. These variables take up integer or half-integer values with the selection rule that the sum of the variables along the boundary of any 2-cycle is always integer. We assume the function [D] to be given by the usual Racah formula (A.4) when D is a tetrahedron. In what follows we shall give a sketchy account of a set of rules which allow the computation of [D] for any D.

In order to evaluate [D], we first construct a given $\mathscr{D}(\mathrm{D})$. We associate variables x_i, $i=1, 2, \ldots, q$ to all internal edges of $\mathscr{D}(\mathrm{D})$ and variables l_j, $j=1, 2, \ldots r$ to the external ones. In this way, to each cell T_k, considered as a diagram, we may associate $[\mathrm{T}_k]$ which is clearly a function of the internal and, possibly, of the external edges of $\mathscr{D}(\mathrm{D})$. Then we form the product:

$$A(x_1, \ldots, x_q) = \prod_{k=1}^{p} [\mathrm{T}_k] \cdot (-1)^{\chi} \prod_{i=1}^{q} (2x_i + 1). \tag{6.3}$$

We found so far no combinatorial rule to construct χ, which applies to any diagram. If D and $\mathscr{D}(\mathrm{D})$ are homeomorphs of a 2-sphere and of a 3-ball, then we have in general:

$$\chi = \sum_{j=1}^{q} (n_j - 2) x_j + \chi_0,$$

where n_j is the number of tetrahedra belonging to x_j and χ_0 is a fixed phase chosen in order to make χ integer. For simplicity we shall limit ourselves to this case. Let us now consider the sum over all internal variables

$$S = \sum_{x_1} \ldots \sum_{x_q} A(x_1, \ldots, x_q). \tag{6.4}$$

If there are no internal vertices, the sum is finite and $S=[\mathrm{D}]$. On the contrary, if there are internal vertices, the sum is infinite but it is still possible to renormalize it in such a way to obtain [D]. When the immediate goal is just the computation of [D] with the simplest possible method, then there is no need to introduce this additional complication, for one can always find a $\mathscr{D}$(D) with no internal vertices. However the more general case is relevant in suggesting a formal analogy with the Feynman summation over histories[26]) in connection with the theory of relativity; we shall discuss this point later.

Coming back to (6.3), (6.4) and supposing that there are no internal vertices, we may attempt to evaluate the summation in (6.4) using the same methods already tested for the Biedenharn–Elliott identity (section 4). In this case we shall replace each $[T_k]$ with its asymptotic behaviour according to (1.4). Moreover, we shall split each cosine according to Euler's formula. The function A will then appear as the sum of 2^p pairwise conjugate terms. It is also convenient to replace each factor $(-1)^{n_j x_j}$ with $\exp[\pm i\,\pi(n_j-2)\,x_j]$. This procedure, which is clearly correct only for integer x_j, can be easily extended also to half-integer summation indices. Therefore A will contain, among the others, a term of the form

$$\prod_{j=1}^{q}(2x_j+1)\exp\left\{\mathrm{i}\left[\left(\sum_{k=1}^{p_j}\theta_j^k\right)-\pi p_j+2\pi\right]x_j\right\}. \tag{6.5}$$

As before, we may try to replace the summation with an integral; we expect that the most important contributions to the integral will arise from the points where the phase is stationary with respect to the q variables x_j. Imposing the stationary phase condition, we find:

$$\sum_{k=1}^{p_j}(\pi-\theta_j^k)=2\pi\,, \tag{6.6}$$

which means that the sum of internal angles around x_j is just 2π. A discussion similar to the one carried out in section 4 brings to the conclusion that (6.6) implies the existence of a configuration, in the sense defined above, imbedded in a 3-dimensional Euclidean space, where the internal and external lengths are well specified. Because of the lack of internal vertices, the internal edges connect external vertices of the diagram and in fact they are sufficient to specify the configuration completely. A similar discussion can be carried out for the other 2^p-1 terms. In this way we see that the final result will be a sum of contributions from each configuration. We do not know of any simple rule to compute the general form of the partial second derivatives:

$$\partial\left[\sum_{k=1}^{p_j}\theta_j^k\right]\Big/\partial x_j \tag{6.7}$$

needed in order to carry out, up to the end, the evaluation of (6.4). This lack of knowledge stops us here short of the final result. However the above discussion shows already that the study of configurations is certainly relevant to a complete understanding of the semiclassical limit of the $3nj$-symbols.

As anticipated, we point out a curious connection between our asymptotic formulae and a simplified quantization "à la Feynman" occurring in a 3-dimensional Euclidean theory of gravitation. The classical counterpart of this theory is trivial because the Einstein field equation for empty space

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=0 \tag{6.8}$$

implies that the complete Riemann tensor vanishes, i.e., the space is flat. However, the connection we point out may be relevant for further generalizations to less academical cases.

We begin by discussing the sum (6.4) when there are internal vertices. As stated, this sum is infinite, but it is rather interesting to see how this infinity actually arises. For simplicity we restrict ourselves to the case when D is a tetrahedron and there is only one internal vertex P_5 (fig. 13). In this case (6.4) reads:

$$S=\sum_{xyzt}(2x+1)(2y+1)(2z+1)(2t+1)(-1)^{x+y+z+t+a+b+c+d+e+f}\times$$
$$\times\begin{Bmatrix}a&b&c\\x&y&z\end{Bmatrix}\begin{Bmatrix}f&e&a\\y&z&t\end{Bmatrix}\begin{Bmatrix}d&b&f\\z&t&x\end{Bmatrix}\begin{Bmatrix}c&e&d\\t&x&y\end{Bmatrix} \tag{6.9}$$

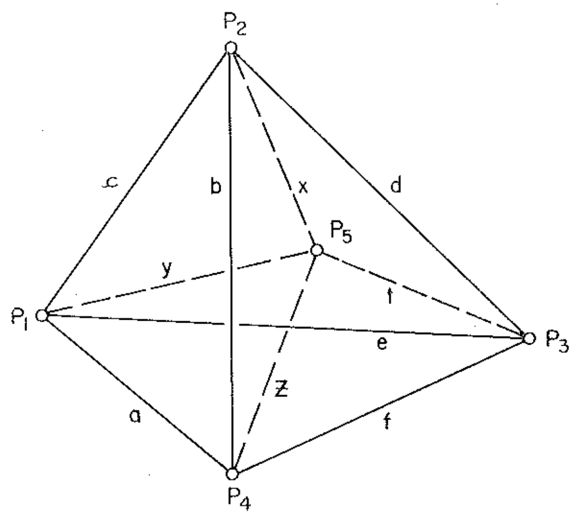


Fig. 13.

having chosen $\chi_0=a+b+c+d+e+f$; the summation is carried out on all x, y, z, t which are compatible with the selection rules. Without loss of generality we may suppose x integer; using (3.8) the summation over t yields:

$$S=\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \sum_{xyz} (2x+1)(2y+1)(2z+1) \begin{Bmatrix} a & b & c \\ x & y & z \end{Bmatrix}^2, \tag{6.10}$$

and from (A.6):

$$S=\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \sum_{xy} \frac{\delta_{xyc}}{2c+1} (2x+1)(2y+1), \tag{6.11}$$

where δ_{xyc} is a triangular delta which is equal to unity if x, y, c satisfy triangular inequalities and zero otherwise. Since

$$\sum_{y=|x-c|}^{x+c} (2y+1)=(2x+1)(2c+1),$$

we have

$$S=\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \sum_{x=0}^{\infty} (2x+1)^2, \tag{6.12}$$

which is infinite and correspondingly meaningless. Let us limit the summation on x up to $x=R$, with R large; in this case we find

$$\mathscr{R}(R) \equiv \sum_{x=0}^{R} (2x+1)^2 \simeq \frac{1}{\pi} \frac{4\pi R^3}{3}. \tag{6.13}$$

This result hints that we may write:

$$\begin{aligned}[\mathrm{D}] \equiv \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \lim_{R\to\infty} (\mathscr{R}(R))^{-1} \sum_{x,y,z,t<R} (-1)^{a+b+c+d+e+f+x+y+z+t} \times \\ \times (2x+1)(2y+1)(2z+1)(2t+1) \times \\ \times \begin{Bmatrix} a & b & c \\ x & y & z \end{Bmatrix} \begin{Bmatrix} f & e & a \\ y & z & t \end{Bmatrix} \begin{Bmatrix} d & b & f \\ z & t & x \end{Bmatrix} \begin{Bmatrix} c & e & d \\ t & x & y \end{Bmatrix}. \end{aligned} \tag{6.14}$$

In fact we have that, for fixed x, the summand vanishes if $z>x+b$, $y>x+c$, $t>x+d$ so that for $x<\min(R-b, R-c, R-d)$ the limitations $y, z, t<R$ have no weight. Therefore we assume that the above limit is correct and expect that in general [D] is given by:

$$[\mathrm{D}] = \lim_{R\to\infty} (\mathscr{R}(R))^{-P} \sum_{x_1<R} \cdots \sum_{x_q<R} A(x_1, \ldots, x_q) \tag{6.15}$$

where P is the number of internal vertices and A is defined by (6.3).

Now let us suppose that the number of vertices and edges in D and in $\mathscr{D}$(D) is very high. Let also the complex $\mathscr{D}$(D) approach a differentiable manifold $\mathscr{M}$ with boundary D. According to a discussion carried out in a previous paper by one of us[27]), the sum $\sum_{j=1}^{q}[\sum_{k=1}^{p_j}(\pi-\theta_j^k)]x_j$ approaches the integral $8\pi\mathscr{L}(\mathscr{M})=\frac{1}{2}\int_{\mathscr{M}} R\, \mathrm{d}V$, where $\mathrm{d}V$ is the volume element on $\mathscr{M}$ and R the scalar Riemann curvature of $\mathscr{M}$. We see therefore that the positive frequency part S^+ of S in some sense looks like:

$$S^+ = \frac{1}{\mathscr{R}^P} \int_{\partial\mathscr{M}=\mathrm{D}} \mathrm{e}^{\mathrm{i}\mathscr{L}(\mathscr{M})}\, \mathrm{d}\mu(\mathscr{M}), \tag{6.16}$$

where the summations over the variables x_j have been interpreted as an integration over all the manifolds $\mathscr{M}$ with fixed boundary D. The measure $\mathrm{d}\mu$ is here not defined in any precise mathematical sense since all the discussion carried out so far is clearly heuristic in character. In this form, S^+ strongly resembles a Feynman summation over histories with density of Lagrangian $\mathscr{L}$ as in a 3-dimensional Einstein theory. In a more conventional 4-dimensional theory with pseudo Euclidean metric, the corresponding summation would be[28]):

$$S(\Sigma_1, \Sigma_2) = \int \mathrm{d}\mu \exp\left(\mathrm{i}\int_{\Sigma_1}^{\Sigma_2} R\, \mathrm{d}^4x\sqrt{-g}\right), \tag{6.17}$$

the integral on the coordinates being performed in the slab between the spacelike hypersurfaces Σ_1, Σ_2. The other terms, other than the positive frequency part, are related to different orientations of the tetrahedra T_j and have a similar interpretation, although their precise meaning is still unclear. It is plausible that in the transition to a smooth manifold $\mathscr{M}$, they will give no essential contribution to the final result.

Finally we report an interesting conjecture over possible extensions of Wigner's result for the 6*j*-symbol. For simplicity we limit ourselves to the 9*j*-symbol, further generalizations being obvious. In the diagram of fig. 12a we keep the vertices P_1, P_2, P_3 of one face fixed; in this way we fix also a, b, c. It can be easily realized that to determine D completely it is enough to give the points P_4 and P_5; in fact, from P_1, P_5 we deduce the symmetry centre $\frac{1}{2}(\mathrm{P}_1+\mathrm{P}_5)$ of D and from it all the remaining vertices. We may use as coordinates for P_4, P_5 either their six Euclidean coordinates $\boldsymbol{r}_4$, $\boldsymbol{r}_5$ or the six remaining lengths d, e, f, g, h, i. Our conjecture, suggested by (3.5), is that:

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix}^2 \sim C \sum_{k=1}^{M} \left| \frac{\partial(\boldsymbol{r}_4, \boldsymbol{r}_5)}{\partial(d^2, e^2, f^2, g^2, h^2, i^2)} \right|_{\boldsymbol{r}_4=\boldsymbol{r}_4^{(k)},\, \boldsymbol{r}_5=\boldsymbol{r}_5^{(k)}}, \tag{6.18}$$

where the summation is carried on all the M different configurations $\boldsymbol{r}_4^{(k)}$,

$r_5^{(k)}$ labelled by k, which correspond to the same values of the lengths appearing in the symbol. Cayley identity (4.8) when applied, for instance, to P_3, P_5, P_6, P_7, P_8, yields a fourth order equation in x^2, which means that, in general, there are four different contributions to (6.18). The constant C can be determined by evaluating

$$\mathfrak{N} = \sum_{a,b,c,d,e,f<R} (2a+1)(2b+1)(2c+1)(2d+1) \times (2e+1)(2f+1) \begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix}^2 \qquad (6.19)$$

by means of the orthogonality of 9j-symbols[6]; following a procedure similar to the one which led to (6.12), we obtain $\mathfrak{N}=(\frac{4}{3}R^3)^2$. On the other hand, we have by means of (6.18)

$$\mathfrak{N} \simeq \int_{a,b,\ldots,f<R} \mathrm{d}(a^2)\,\mathrm{d}(b^2)\ldots\mathrm{d}(f^2) \begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix}^2 \simeq$$
$$\simeq \tfrac{1}{2}C \int_{|r_4|<R} \mathrm{d}^3 r_4 \int_{|r_5|<R} \mathrm{d}^3 r_5 \simeq \tfrac{1}{2}C(\tfrac{4}{3}\pi R^3)^2, \qquad (6.20)$$

from which is follows: $C=2(\pi)^{-2}$. The inverse of the Jacobian appearing in (6.18) can be evaluated easily in terms of volumes of tetrahedra; we hope to present elsewhere more detailed results for this as well as for higher order symbols.

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The authors are indebted to Prof. E. P. Wigner, Prof. F. J. Dyson, Dr. S. Adler and Dr. J. Lascoux for many interesting discussions.

Appendix A. Racah algebra "in nuce" for SU(2)

This appendix will be devoted to a brief summary of formulae and properties of coupling and recoupling coefficients, as well as of matrix representations for the unitary unimodular group in two dimensions[29].

The 3j-symbol is defined by[4]):

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \equiv \begin{vmatrix} b+c-a & c+a-b & a+b-c \\ a-\alpha & b-\beta & c-\gamma \\ a+\alpha & b+\beta & c+\gamma \end{vmatrix} =$$
$$= (-1)^{a-b-\gamma}[\Delta(abc)(a+\alpha)!\,(a-\alpha)!\,(b+\beta)!\,(b-\beta)!\,(c+\gamma)!\,\times$$
$$\times (c-\gamma)!]^{\frac{1}{2}} \sum_x (-1)^x [x!\,(c-b+\alpha+x)!\,(c-a-\beta+x)!\,\times$$
$$\times (a+b-c-x)!\,(a-\alpha-x)!\,(b+\beta-x)!]^{-1}, \qquad (A.1)$$

with

$$\Delta(abc) = \frac{(a+b-c)!\,(b+c-a)!\,(c+a-b)!}{(a+b+c+1)!}. \qquad (A.2)$$

The positive integers or half-integers a, b, c satisfy triangular inequalities: $|a-b| \leqslant c \leqslant a+b$ etc. and $\alpha+\beta+\gamma=0$; $a-|\alpha|$ etc. must be natural integers. If anyone of these conditions is not fulfilled, the value of the symbol is assumed to be zero. The 72 elements[15]) of the symmetry group R_1 of this coefficient are the permutations of lines and/or columns of the square symbol in (A.1) (which yield the phase $(P)^S$ with $S=a+b+c$ and $P=+1,-1$ according to even, odd permutations), and the exchange of lines with columns. For unitarity and orthogonality of 3j-symbols, see Edmonds[16]).

From the definition (3.1) it turns out[4]) that the 6j-symbol is given in terms of 3j-coefficients by:

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \sum_{\substack{\alpha\beta\gamma \\ \delta\varepsilon\varphi}} (-1)^{d+e+f+\delta+\varepsilon+\varphi} \times$$
$$\times \begin{pmatrix} d & e & c \\ \delta & -\varepsilon & \gamma \end{pmatrix} \begin{pmatrix} e & f & a \\ \varepsilon & -\varphi & \alpha \end{pmatrix} \begin{pmatrix} f & d & b \\ \varphi & -\delta & \beta \end{pmatrix} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}. \qquad (A.3)$$

Racah's treatment of this formula[1]) yields:

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = [\Delta(abc)\,\Delta(aef)\,\Delta(cde)\,\Delta(bdf)]^{\frac{1}{2}} \times$$
$$\times \sum_x (-1)^x (x+1)!\,[(a+b+d+e-x)!\,(a+c+d+f-x)!\,\times$$
$$\times (b+c+e+f-x)!\,(x-a-b-c)!\,(x-a-e-f)!\,\times$$
$$\times (x-c-d-e)!\,(x-b-d-f)!]^{-1}. \qquad (A.4)$$

The symbol is assumed to vanish if anyone of the triads (abc), (cde), (aef), (bdf) does not satisfy triangular inequalities. In addition to the well known symmetries of the associated tetrahedron (fig. 1), the 6j-symbol has also the less evident symmetry[15]):

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \equiv \begin{vmatrix} a+b-c & b+f-d & f+a-e \\ d+b-f & b+c-a & c+d-e \\ a+e-f & e+c-d & c+a-b \\ d+e-c & e+f-a & f+d-b \end{vmatrix} =$$
$$= \begin{Bmatrix} a & \frac{1}{2}(c+f+e-b) & \frac{1}{2}(b+e+f-c) \\ d & \frac{1}{2}(c+f+b-e) & \frac{1}{2}(b+e+c-f) \end{Bmatrix}, \qquad (A.5)$$

which entails that this coefficient is invariant under the 144 elements of a group R_2 which correspond to the permutations of lines and/or columns of the 3×4 pattern in (A.5). In addition to the Biedenharn–Elliott identity (3.8), the 6j-symbol satisfies also the following relations[30]:

$$\sum_x (2x+1)\begin{Bmatrix} a & b & x \\ d & e & f \end{Bmatrix}\begin{Bmatrix} a & b & x \\ d & e & f' \end{Bmatrix} = \frac{\delta_{ff'}}{2f+1}, \tag{A.6}$$

$$\sum_x (-1)^{f+g+x}(2x+1)\begin{Bmatrix} a & b & x \\ d & e & f \end{Bmatrix}\begin{Bmatrix} d & e & x \\ b & a & g \end{Bmatrix} = \begin{Bmatrix} a & e & f \\ b & d & g \end{Bmatrix}. \tag{A.7}$$

Wigner's 9j-symbol[4]), which is proportional to the transformation matrix between different coupling schemes of four angular momenta, is given in terms of 6j-coefficients by:

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix} = \sum_x (-1)^{2x}(2x+1)\begin{Bmatrix} a & i & x \\ f & b & c \end{Bmatrix}\begin{Bmatrix} f & b & x \\ h & d & e \end{Bmatrix}\begin{Bmatrix} h & d & x \\ a & i & g \end{Bmatrix}. \tag{A.8}$$

Its known symmetries are formally the same as those mentioned above for the 3j-symbol, with $S=a+b+c+d+e+f+g+h+i$. For a more comprehensive account of relations involving these coefficients, see ref. 6.

According to the conventions and notations of ref. 30 for basis vectors, angular momentum operators and Euler angles, the rotation operator

$$\hat{D}(\alpha\beta\gamma) = e^{-i\alpha\hat{J}_z}\, e^{-i\beta\hat{J}_y}\, e^{-i\gamma\hat{J}_z} \tag{A.9}$$

has matrix elements in the $2J+1$ dimensional representation which can be written as

$$D^{(J)}_{MM'}(\alpha\beta\gamma) \equiv \langle JM|\hat{D}(\alpha\beta\gamma)|JM'\rangle = e^{-i\alpha M}\, d^{(J)}_{MM'}(\beta)\, e^{-i\gamma M'} \tag{A.10}$$

in terms of the real matrix: $d^{(J)}_{MM'}(\beta) = D^{(J)}_{MM'}(0\beta 0)$. The following properties hold

$$[D^{(J)}(\alpha\beta\gamma)]^+ = [D^{(J)}(\alpha\beta\gamma)]^{-1} = D^{(J)}(-\gamma, -\beta, -\alpha), \tag{A.11}$$

$$\sum_M D^{(J)}_{MM'}\, D^{(J)*}_{MM''} = \delta_{M'M''}; \qquad \sum_M D^{(J)}_{M'M}\, D^{(J)*}_{M''M} = \delta_{M'M''}, \tag{A.12}$$

$$\frac{1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi} d\beta \sin\beta\, D^{(J)*}_{M_1M_2}(\alpha\beta\gamma)\, D^{(J')}_{M_1'M_2'}(\alpha\beta\gamma) = \frac{\delta_{JJ'}\,\delta_{M_1M_1'}\,\delta_{M_2M_2'}}{2J+1}, \tag{A.13}$$

in addition to the symmetry

$$D^{(J)*}_{MM'} = (-1)^{M-M'}\, D^{(J)}_{-M,-M'}. \tag{A.14}$$

In general[4])

$$d^{(J)}_{-M,M'}(\beta) = \sum_x (-1)^x \frac{[(J-M)!\,(J+M)!\,(J-M')!\,(J+M')!]^{\frac12}}{(J-M-x)!\,(J-M'-x)\,x!\,(x+M+M')!} \times$$
$$\times (\cos\tfrac12\beta)^{2J-M-M'-2x}(\sin\tfrac12\beta)^{2x+M+M'}, \tag{A.15}$$

which leads in particular to

$$d^{(L)}_{M,0}(\beta) = (-1)^M\left[\frac{(L-M)!}{(L+M)!}\right]^{\frac12} P^M_L(\cos\beta); \tag{A.16}$$

the following symmetries are very useful

$$d^{(J)}_{M,M'}(\beta) = (-1)^{M-M'}\, d^{(J)}_{-M,-M'}(\beta) =$$
$$= (-1)^{M-M'}\, d^{(J)}_{M',M}(\beta) = (-1)^{J-M'}\, d^{(J)}_{-M',M}(\pi-\beta). \tag{A.17}$$

The connection with Jacobi polynomials is given by

$$d^{(J)}_{MM'}(\beta) = (-1)^{M-M'}\left[\frac{(J+M)!\,(J-M)!}{(J+M')!\,(J-M')!}\right]^{\frac12} \times$$
$$\times (\cos\tfrac12\beta)^{M+M'}(\sin\tfrac12\beta)^{M-M'}\, P^{(M-M',\,M+M')}_{J-M}(\cos\beta). \tag{A.18}$$

Appendix B. Elementary geometry of tetrahedra

Heron's formula for the content A of a triangle of edges j_1, j_2, j_3 can be written as

$$A^2 = \tfrac{1}{16}(j_1+j_2+j_3)(j_1+j_2-j_3)(j_1-j_2+j_3)(-j_1+j_2+j_3) =$$
$$= -\frac{1}{16}\begin{vmatrix} 0 & j_1^2 & j_2^2 & 1 \\ j_1^2 & 0 & j_3^2 & 1 \\ j_2^2 & j_3^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}. \tag{B.1}$$

We have already given in (1.2) Cayley's form for the content V of a tetrahedron. Performing in (1.2) the derivation of V^2 with respect to $(j_{rs})^2$ and denoting the (r, s) algebraic minor of the Cayley determinant with C_{rs}, the relation[31])

$$-C_{rs} = 16 A_h A_k \cos\theta_{hk}, \qquad r \neq s \neq h \neq k \tag{B.2}$$

yields (3.22); in (B.2) h, k, r, s are any permutation of 1, 2, 3, 4. Using (3.22) and the obvious identity

$$A_h A_k \sin\theta_{hk} = \tfrac32 V j_{hk} \tag{B.3}$$

we obtain the differentiation formulae

$$-\frac{\mathrm{d}\theta_{hk}}{\mathrm{d}j_{rs}} = \frac{j_{hk}j_{rs}}{6V}, \qquad h \neq k \neq r \neq s \tag{B.4}$$

Looking at fig. 1 we see that

$$V = \tfrac{1}{6} j_{12} j_{23} j_{13} \sin\varphi_{14} \sin\varphi_{24} \sin\theta_{12} \tag{B.5}$$

or

$$V = \tfrac{1}{6} j_{12} j_{13} j_{23} \begin{vmatrix} 1 & \cos\varphi_{14} & \cos\varphi_{24} \\ \cos\varphi_{14} & 1 & \cos\varphi_{34} \\ \cos\varphi_{24} & \cos\varphi_{34} & 1 \end{vmatrix}^{\frac{1}{2}} \tag{B.6}$$

since $\cos\varphi_{34} = \cos\varphi_{14}\cos\varphi_{24} - \sin\varphi_{14}\sin\varphi_{24}\cos\theta_{12}$. Now, if we define with reference to P_4:

$$2\sigma_4 = \theta_{12} + \theta_{13} + \theta_{23}, \tag{B.7}$$

$$\Sigma_4 = [\sin\sigma_4 \sin(\sigma_4 - \theta_{12}) \sin(\sigma_4 - \theta_{13}) \sin(\sigma_4 - \theta_{23})]^{\frac{1}{2}}, \tag{B.8}$$

then it follows[22]):

$$2\Sigma_4 = \sin\theta_{12}\sin\theta_{23}\sin\varphi_{24} = \sin\theta_{13}\sin\theta_{23}\sin\varphi_{34} = \sin\theta_{12}\sin\theta_{13}\sin\varphi_{14}, \tag{B.9}$$

and

$$\begin{aligned} \Sigma_4 \operatorname{tg} \tfrac{1}{2}\varphi_{14} &= \sin\sigma_4 \sin(\sigma_4 - \theta_{23}), \\ \Sigma_4 \operatorname{tg} \tfrac{1}{2}\varphi_{24} &= \sin\sigma_4 \sin(\sigma_4 - \theta_{13}), \\ \Sigma_4 \operatorname{tg} \tfrac{1}{2}\varphi_{34} &= \sin\sigma_4 \sin(\sigma_4 - \theta_{12}). \end{aligned} \tag{B.10}$$

It turns out that $K = \Sigma_h / A_h$ is independent of h; therefore, from (B.5), (B.9):

$$V = \tfrac{2}{3} \left(\prod_{h=1}^{4} A_h \right)^{\frac{1}{2}} K^{\frac{1}{2}}. \tag{B.11}$$

Appendix C. A summation property of Jacobi polynomials

The purpose of this appendix is to prove the relation

$$\sum_{L=0}^{\infty} (2L+1)\, d^{(L)}_{M_1M_2}(\beta_3)\, d^{(L)}_{M_2M_3}(\beta_1)\, d^{(L)}_{M_3M_1}(\beta_2) = \frac{2\Theta(\mathscr{B})}{\pi[\mathscr{B}]^{\frac{1}{2}}} \cos\left(\sum_{i=1}^{3} M_i\delta_i\right), \tag{C.1}$$

where Θ is the step function used in (3.13), while the angles β_i, δ_i ($i=1, 2, 3$)

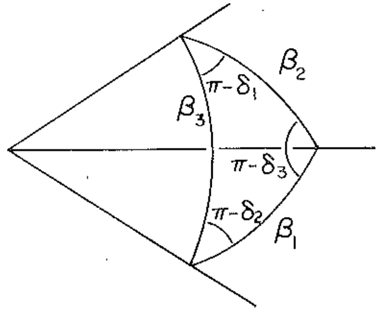

Fig. C1.

are shown in fig. C1 and

$$\mathscr{B} = \begin{vmatrix} 1 & \cos\beta_3 & \cos\beta_2 \\ \cos\beta_3 & 1 & \cos\beta_1 \\ \cos\beta_2 & \cos\beta_1 & 1 \end{vmatrix}. \tag{C.2}$$

The addition property of the rotation group yields:

$$D^{(L)}_{M_1M_2}(\alpha_3\beta_3\gamma_3) = \sum_{M_3} D^{(L)}_{M_1M_3}(\alpha_2\beta_2\gamma_2)\, D^{(L)}_{M_3M_2}(\alpha_1\beta_1\gamma_1), \tag{C.3}$$

from which, using unitarity

$$D^{(L)}_{M_3M_2}(\alpha_1\beta_1\gamma_1) = \sum_{M_1} D^{(L)}_{M_1M_2}(\alpha_3\beta_3\gamma_3)\, D^{(L)*}_{M_1M_3}(\alpha_2\beta_2\gamma_2), \tag{C.4}$$

$$D^{(L)}_{M_1M_3}(\alpha_2\beta_2\gamma_2) = \sum_{M_2} D^{(L)*}_{M_3M_2}(\alpha_1\beta_1\gamma_1)\, D^{(L)}_{M_1M_2}(\alpha_3\beta_3\gamma_3). \tag{C.5}$$

By exploiting (C.3)–(C.5) for low values of L, we obtain

$$\cos\beta_i = \cos\beta_j\cos\beta_k - \sin\beta_j\sin\beta_k\cos\delta_i, \quad i \neq j \neq k = 1, 2, 3 \tag{C.6}$$

$$\sin\beta_i / \sin\delta_i = \sin\beta_j / \sin\delta_j, \quad i \neq j \tag{C.7}$$

where

$$\delta_1 = \alpha_2 - \alpha_3 + \pi, \quad \delta_2 = \gamma_1 - \gamma_3 - \pi, \quad \delta_3 = \alpha_1 + \gamma_2. \tag{C.8}$$

By means of (A.13) and (C.3) we find

$$\int_0^{2\pi} \mathrm{d}\alpha_2 \int_0^{2\pi} \mathrm{d}\gamma_2 \int_0^{\pi} \mathrm{d}\beta_2 \sin\beta_2\, D^{(L')*}_{M_1'M_3'}(\alpha_2\beta_2\gamma_2)\, D^{(L)}_{M_1M_2}(\alpha_3\beta_3\gamma_3) = \frac{8\pi^2}{2L+1} \delta_{LL'}\, \delta_{M_1M_1'}\, \delta_{M_3M_3'}\, D^{(L)}_{M_3M_2}(\alpha_1\beta_1\gamma_1); \tag{C.9a}$$

(A.10) and the reality of functions $d^{(L)}_{MM'}$ give

$$\int_0^{2\pi} d\alpha_2 \int_0^{2\pi} d\gamma_2 \int_0^{\pi} d\beta_2 \sin\beta_2 \cos\left(\sum_{j=1}^{3} M_j\delta_j\right) d^{(L)}_{M_1M_3}(\beta_2)\, d^{(L)}_{M_1M_2}(\beta_3) = \frac{8\pi^2}{2L+1}(-1)^{M_1-M_2}\, d^{(L)}_{M_3M_2}(\beta_1). \qquad \text{(C.9b)}$$

We notice that, once $\alpha_1, \beta_1, \gamma_1, \beta_2, \gamma_2$ have been fixed, β_3 and δ_i $(i=1, 2, 3)$ are independent of α_2; therefore:

$$\int_0^{2\pi} d\gamma_2 \int_0^{\pi} d\beta_2 \sin\beta_2 \cos\left(\sum_{j=1}^{3} M_j\delta_j\right) d^{(L)}_{M_1M_3}(\beta_2)\, d^{(L)}_{M_1M_2}(\beta_3) = \frac{4\pi}{2L+1}(-1)^{M_1-M_2}\, d^{(L)}_{M_3M_2}(\beta_1). \qquad \text{(C.10)}$$

Going from the integration variables β_2, γ_2 to $\cos\beta_2, \cos\beta_3$, we find:

$$\frac{\partial(\beta_2, \gamma_2)}{\partial(\cos\beta_2, \cos\beta_3)} = (\sin\beta_2)^{-1}\mathscr{B}^{-\frac{1}{2}} \qquad \text{(C.11)}$$

and

$$\int_{-1}^{+1} d(\cos\beta_2) \int_{-1}^{+1} d(\cos\beta_3)\, \frac{\Theta(\mathscr{B})}{\mathscr{B}^{\frac{1}{2}}} \cos\left(\sum_{j=1}^{3} M_j\delta_j\right) d^{(L)}_{M_1M_3}(\beta_2) \times d^{(L)}_{M_1M_2}(\beta_3) = \frac{2\pi}{2L+1}(-1)^{M_1-M_2}\, d^{(L)}_{M_3M_2}(\beta_1), \qquad \text{(C.12)}$$

having noticed that a given point of the integration domain in (C.12) corresponds to two different points in (C.10). Since by means of the orthogonal polynomials $d^{(L)}_{MM'}$ we can define the following expansions:

$$f_{M_1M_2M_1'M_2'}(\beta, \beta') = \sum_{L, L'} \frac{2L+1}{2}\,\frac{2L'+1}{2}\, f^{M_1M_2M_1'M_2'}_{LL'}\, d^{(L)}_{M_1M_2}(\beta)\, d^{(L')}_{M_1'M_2'}(\beta'), \qquad \text{(C.13)}$$

with coefficients:

$$f^{M_1M_2M_1'M_2'}_{LL'} = \int_{-1}^{+1} d(\cos\beta) \int_{-1}^{+1} d(\cos\beta')\, f_{M_1M_2M_1'M_2'}(\beta, \beta') \times d^{(L)}_{M_1M_2}(\beta)\, d^{(L)}_{M_1'M_2'}(\beta'), \qquad \text{(C.14)}$$

which are non-vanishing only if $L \geqslant \max(|M_1|, M_2|)$, $L' \geqslant \max(|M_1'|, |M_2'|)$, we have from (C.12)

$$\frac{\Theta(\mathscr{B})}{\mathscr{B}^{\frac{1}{2}}} \cos\left(\sum_{j=1}^{3} M_j\delta_j\right)(-1)^{M_1-M_2} = \tfrac{1}{2}\pi \sum_{L=0}^{\infty} (2L+1)\, d^{(L)}_{M_3M_2}(\beta_1)\, d^{(L)}_{M_1M_3}(\beta_2)\, d^{(L)}_{M_1M_2}(\beta_3).$$

The symmetries (A.17) lead us to (C.1).

In order to complete the proof of (3.13), we have just to recall (B.6).

Appendix D. Invariance of Ω under R_2

It is straightforward, though very tedious, to verify that the volume of T is invariant also under the symmetry (A.5). This can be worked out by expanding (1.2) and replacing j_{13} with $\frac{1}{2}(j_{14}+j_{24}+j_{23}-j_{13})$ etc. Incidentally, we note that in the same way one can check also the invariance of $\prod_{h=1}^{4} A_h$ under R_2. Here we prefer to sketch how the symmetry of Ω can be proved.

Let us multiply eqs. (B.10) among themselves. We have for instance

$$(\sin\sigma_4)^2 = \Sigma_4 \operatorname{tg}\tfrac{1}{2}\varphi_{14} \operatorname{tg}\tfrac{1}{2}\varphi_{24} \operatorname{tg}\tfrac{1}{2}\varphi_{34}; \qquad \text{(D.1)}$$

then choosing for example the first of (B.10), we have

$$[\sin\tfrac{1}{2}(\theta_{12}+\theta_{13}-\theta_{23})]^2 = \Sigma_4 \operatorname{tg}\tfrac{1}{2}\varphi_{14} \operatorname{cotg}\tfrac{1}{2}\varphi_{24} \operatorname{cotg}\tfrac{1}{2}\varphi_{34}. \qquad \text{(D.2)}$$

In order to compute the r.h.s. of (D.2) it is convenient to introduce, with reference to fig. D1, the following notations

$$q_1 = j_{12}+j_{13}+j_{14}, \quad q_2 = j_{12}+j_{23}+j_{24}, \quad q_3 = j_{13}+j_{23}+j_{34}, \quad q_4 = j_{14}+j_{24}+j_{34}; \qquad \text{(D.3)}$$

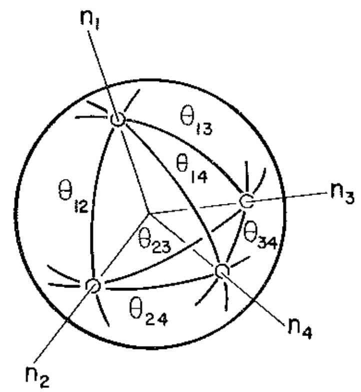

Fig. D1.

$$p_{14} = p_{23} = j_{12} + j_{13} + j_{34} + j_{24}, \quad p_{13} = p_{24} = j_{12} + j_{34} + j_{14} + j_{23},$$
$$p_{12} = p_{34} = j_{13} + j_{24} + j_{14} + j_{23}; \tag{D.4}$$

as well as the two patterns

$$r_{st} \equiv \begin{bmatrix} p_{14}-q_1 & p_{13}-q_1 & p_{12}-q_1 & q_1 \\ p_{14}-q_2 & p_{13}-q_2 & p_{12}-q_2 & q_2 \\ p_{14}-q_3 & p_{13}-q_3 & p_{12}-q_3 & q_3 \\ p_{14}-q_4 & p_{13}-q_4 & p_{12}-q_4 & q_4 \end{bmatrix};$$
$$\tfrac{1}{2}\vartheta_{st} \equiv \begin{bmatrix} \sigma_4-\theta_{23} & \sigma_3-\theta_{24} & \sigma_2-\theta_{34} & \sigma_1 \\ \sigma_3-\theta_{14} & \sigma_4-\theta_{13} & \sigma_1-\theta_{34} & \sigma_2 \\ \sigma_2-\theta_{14} & \sigma_1-\theta_{24} & \sigma_4-\theta_{12} & \sigma_3 \\ \sigma_1-\theta_{23} & \sigma_2-\theta_{13} & \sigma_3-\theta_{12} & \sigma_4 \end{bmatrix}. \tag{D.5}$$

We note that from (B.11)

$$K = \frac{\Sigma_4}{A_4} = \frac{9}{4}\frac{V^2}{\prod_{h=1}^{4} A_h}. \tag{D.6}$$

Since, for example

$$\operatorname{tg} \tfrac{1}{2}\varphi_{14} = \left[\frac{(p_{13}-q_3)(p_{12}-q_2)}{q_1(p_{14}-q_4)}\right]^{\frac{1}{2}},$$

(D.2) becomes

$$\cos(\theta_{12}+\theta_{13}-\theta_{23}) = 1 - 2^5 3^2 V^2 \,[(p_{13}-q_1)\times$$
$$\times (p_{12}-q_1) q_1 (p_{14}-q_2)(p_{14}-q_3)(p_{14}-q_4)]^{-1}$$

and generally for $s \neq t = 1, 2, 3, 4$

$$\cos\vartheta_{st} = 1 - 2^3 (3!)^2 V^2 \left\{\left(\prod_{n=1}^{4} r_{sn}\right)\left(\prod_{n=1}^{4} r_{nt}\right)\right\}^{-1} (r_{st})^2, \tag{D.7}$$

which gives, correctly, for a flat tetrahedron, $\vartheta_{st}=0$ or $\vartheta_{st}=2\pi$.

It is easy to check that, under the symmetry (A.5)

$$q_1' = q_2, \quad q_2' = q_1, \quad q_3' = q_4, \quad q_4' = q_3;$$
$$p_{14}' = p_{13}, \quad p_{13}' = p_{14}, \quad p_{12}' = p_{12}, \tag{D.8}$$

which, taking into account (D.5), (D.7), entails for instance

$$\cos(\theta_{12}'+\theta_{13}'-\theta_{23}') = \cos(\theta_{12}+\theta_{13}-\theta_{23}),$$
$$\cos(\theta_{13}'+\theta_{23}'-\theta_{12}') = \cos(\theta_{14}+\theta_{24}-\theta_{12}),$$
$$\cos(\theta_{12}'+\theta_{23}'-\theta_{13}') = \cos(\theta_{12}+\theta_{13}-\theta_{23}). \tag{D.9}$$

Let us suppose the tetrahedron to be almost regular; asymptotically, this property is not destroyed by the symmetry (A.5); in this case: $0 \leqslant \theta_{hk} \leqslant \frac{1}{2}\pi$ and similarly for the transformed θ_{hk}'. Moreover, looking at fig. D1 we see for example that θ_{12}, θ_{13}, θ_{23} satisfy spherical triangular inequalities; therefore $0 \leqslant \vartheta_{st} < \pi$, $0 \leqslant \vartheta_{st}' < \pi$. Then from (D.9) and similar relations, we obtain

$$\theta_{13}' = \tfrac{1}{2}(\theta_{14}+\theta_{24}+\theta_{23}-\theta_{13}),$$
$$\theta_{14}' = \tfrac{1}{2}(\theta_{13}+\theta_{23}+\theta_{24}-\theta_{14}), \qquad \theta_{12}' = \theta_{12},$$
$$\theta_{24}' = \tfrac{1}{2}(\theta_{13}+\theta_{23}+\theta_{14}-\theta_{24}),$$
$$\theta_{23}' = \tfrac{1}{2}(\theta_{14}+\theta_{24}+\theta_{13}-\theta_{23}), \qquad \theta_{34}' = \theta_{34}. \tag{D.10}$$

We conclude that (A.5) induces the same linear transformation on j_{hk} as well as on θ_{hk}; the unitarity of this transformation entails the invariance of $\Omega = \sum_{h,k} j_{hk}\,\theta_{hk} + \frac{1}{4}\pi$ under R_2. Finally, the constraint $\theta_{hk} < \frac{1}{2}\pi$ can be dropped by invoking analytical continuation in θ_{hk}.

Appendix E. Evaluation of $[\partial(\theta_x^1+\theta_x^2+\theta_x^3)/\partial(x^2)]_{x^2=(x_{\gtrless})^2}$

According to (4.16), first let us calculate $\partial I^2(x^2)/\partial(x^2)|_{x^2=(x_{\gtrless})^2}$. Write (4.8) as follows:

$$C = c_2(x^2)^2 + c_1(x^2) + c_0 = 0, \tag{E.1}$$

where $-2^4(4!)^2 I^2(x^2) = C$; then the solutions of (E.1) are obviously

$$(x_{\gtrless})^2 = \frac{\pm\sqrt{c_1^2 - 4c_0c_2} - c_1}{2c_2}, \tag{E.2}$$

where c_0, c_1, c_2 are determinants extracted from C which, for the sake of brevity, we do not write explicitly. Since the tetrahedra T_4, T_5 are supposed to be physical, it follows $(x_{\gtrless})^2 > 0$. Then

$$\left[\frac{\partial C}{\partial(x^2)}\right]_{x^2=(x_{\gtrless})^2} = \pm\sqrt{c_1^2 - 4c_0c_2}. \tag{E.3}$$

From known properties of determinants[32]), it turns out that

$$\tfrac{1}{4}(c_1^2 - 4c_0c_2) = 2^6(3!)^4 V_4^2 V_5^2, \tag{E.4}$$

therefore

$$\left[\frac{\partial I^2(x^2)}{\partial(x^2)}\right]_{x^2=(x_{\gtrless})^2} = \mp\frac{V_4V_5}{16}. \tag{E.5}$$

In order to evaluate $\partial^2 I^2(x^2)/\partial(\theta_x^1+\theta_x^2+\theta_x^3)|_{x^2=(x_{\gtrless})^2}$, suppose to imbed the simplex P_1, P_2, P_3, P_4, P_5 of fig. 8a into a 4-dimensional Euclidean space.

The content I of this simplex is

$$(4!)^2 I^2 = \begin{vmatrix} x & 0 & 0 & 0 \\ 0 & h_{11} & h_{12} & h_{13} \\ 0 & h_{21} & h_{22} & h_{23} \\ 0 & h_{31} & h_{32} & h_{33} \end{vmatrix}^2 \tag{E.6}$$

where $h_{1j}, h_{2j}, h_{3j} (j=1, 2, 3)$ are the components of the distances from P_1, P_2, P_3 to x. The direction of x has been chosen as the first axis of reference in this hyperspace. Let H be the volume of the tetrahedron defined by h_1, h_2, h_3 (see fig. E1). Then we are led to the simple formula

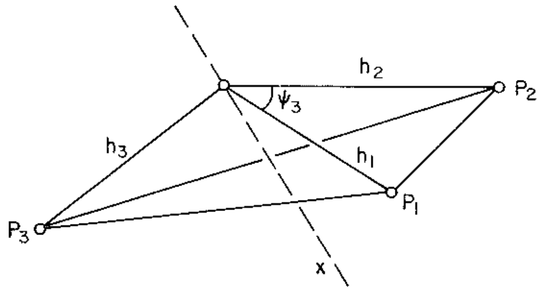

Fig. E1.

$$I = \tfrac{1}{4} x H\,. \tag{E.7}$$

Now we notice that the angle between, for instance, h_1 and h_2 is $\pi - \theta_x^3$. Therefore from (B.6) we have

$$\begin{aligned} \delta(H^2) = \tfrac{1}{18}(h_1h_2h_3)^2 [&\sin\theta_x^1(\cos\theta_x^2\cos\theta_x^3 + \cos\theta_x^1)\,\delta\theta_x^1 + \\ &+ \sin\theta_x^2(\cos\theta_x^3\cos\theta_x^1 + \cos\theta_x^2)\,\delta\theta_x^2 + \\ &+ \sin\theta_x^3(\cos\theta_x^1\cos\theta_x^2 + \cos\theta_x^3)\,\delta\theta_x^3]\,. \end{aligned} \tag{E.8}$$

Since (E.8) must be evaluated when $\theta_x^1+\theta_x^2+\theta_x^3=\pi$, i.e. when h_1, h_2, h_3 are coplanar, we find

$$\left[\frac{\partial(H^2)}{\partial(\theta_x^1+\theta_x^2+\theta_x^3)}\right]_{x^2=(x\gtrless)^2} = \tfrac{1}{18}(h_1h_2h_3)^2 \sin\theta_x^1 \sin\theta_x^2 \sin\theta_x^3\,. \tag{E.9}$$

Moreover, from (B.5) we obtain

$$V_i = \tfrac{1}{6} h_1h_2h_3 x\,\frac{\sin\theta_x^i}{h_i}\,, \qquad i = 1, 2, 3 \tag{E.10}$$

and finally, by means of (E.7)

$$\left[\frac{\partial I^2(x^2)}{\partial(\theta_x^1+\theta_x^2+\theta_x^3)}\right]_{x^2=(x\gtrless)^2} = \frac{3}{4}\left[\frac{V_1V_2V_3}{x}\right]_{x^2=(x\gtrless)^2}. \tag{E.11}$$

Relations (E.5) and (E.11) yield (4.17).

Appendix F. Evaluation of $\lim_{V^2\to 0}\Omega(V^2)$

The behaviour of Ω in the neighbourhood of a transition point deserves a rather careful investigation. To begin with, let us suppose that we approach the region B_3 as $V^2 \to 0$. In fig. 10 we show the notations which will be used in the sequel; moreover we indicate with A, B, C the *internal* dihedral angles between faces belonging to $a+\frac{1}{2}, b+\frac{1}{2}, c+\frac{1}{2}$ and with $\pi-D, \pi-E, \pi-F$ the corresponding ones relative to $d+\frac{1}{2}, e+\frac{1}{2}, f+\frac{1}{2}$. Also the following shorthand notation will be convenient:

$$\begin{aligned} \mu_{bc} &= \cos\alpha\,, & \mu_{bd} &= \cos\delta_b\,, & \mu_{cd} &= \cos\delta_c\,, \\ \mu_{ca} &= \cos\beta\,, & \mu_{ce} &= \cos\varepsilon_c\,, & \mu_{ae} &= \cos\varepsilon_a\,, \\ \mu_{ab} &= \cos\gamma\,, & \mu_{af} &= \cos\varphi_a\,, & \mu_{bf} &= \cos\varphi_b\,. \end{aligned} \tag{F.1}$$

When P lies very near to the plane of the triangle $P_AP_BP_C$ and within its boundary we have $\mu_{bc}^0=\cos(\delta_c+\delta_b)$, $\mu_{ca}^0=\cos(\varepsilon_a+\varepsilon_c)$, $\mu_{ab}^0=\cos(\varphi_a+\varphi_b)$ and $A=B=C=D=E=F=0$. Since $a+\frac{1}{2}=(e+\frac{1}{2})\,\mu_{ae}+(f+\frac{1}{2})\,\mu_{af}$ etc., we obtain in general from the definition (3.15b)

$$\Omega = \pi(a+b+c+\tfrac{7}{4}) + (d+\tfrac{1}{2})\Psi_A + (e+\tfrac{1}{2})\Psi_B + (f+\tfrac{1}{2})\Psi_C\,, \tag{F.2}$$

where

$$\begin{aligned} \Psi_A = D - B\mu_{bd} - C\mu_{cd}\,, \quad \Psi_B &= E - C\mu_{ce} - A\mu_{ae}\,, \\ \Psi_C = F - A\mu_{af} - B\mu_{bf}\,. \end{aligned} \tag{F.3}$$

In order to evaluate Ω when $V^2 \sim 0$, it is useful to exploit the following integral representation for Ψ_A:

$$\Psi_A = \int_{\mu_{bc}^0}^{\mu_{bc}} (1-\xi^2)^{-1}(1+2\mu_{bd}\mu_{cd}\xi - \mu_{bd}^2 - \mu_{cd}^2 - \xi^2)^{\frac{1}{2}}\,d\xi\,, \tag{F.4}$$

and similar ones for Ψ_B, Ψ_C. Noticing that $\Psi_A(\mu_{bc}=\mu_{bc}^0)=0$ and using the relations

$$\begin{aligned} D &= \arccos\left\{\frac{\mu_{bd}\mu_{cd}-\mu_{bc}}{[(1-\mu_{bd}^2)(1-\mu_{cd}^2)]^{\frac{1}{2}}}\right\}, \quad B = \arccos\left\{\frac{\mu_{cd}-\mu_{bd}\mu_{bc}}{[(1-\mu_{bd}^2)(1-\mu_{bc}^2)]^{\frac{1}{2}}}\right\}, \\ C &= \arccos\left\{\frac{\mu_{bd}-\mu_{cd}\mu_{bc}}{[(1-\mu_{cd}^2)(1-\mu_{bc}^2)]^{\frac{1}{2}}}\right\}, \end{aligned} \tag{F.5}$$

it is easy to verify that the derivatives with respect to μ_{bc} of both sides of (F.4) are equal. Now let us put

$$1 + 2\mu_{bd}\mu_{cd}\xi - \mu_{bd}^2 - \mu_{cd}^2 - \xi^2 = (\xi - \mu_{bc}^0)(\mu_{bc}^1 - \xi), \tag{F.6}$$

where μ_{bc}^1 is the value assumed by μ_{bc} when the tetrahedron is flat with P outside $P_AP_BP_C$. In this case $\alpha = |\delta_b - \delta_c|$ and $\mu_{bc}^1 > \mu_{bc}^0$. In the neighbourhood of $\mu_{bc} = \mu_{bc}^0$ we can approximate the integrand of (F.4) as follows:

$$\Psi_A \simeq \int_{\mu_{bc}^0}^{\mu_{bc}} (1 - \xi^2)^{-1} [(\xi - \mu_{bc}^0)(\mu_{bc}^1 - \mu_{bc}^0)]^{\frac{1}{2}} \, d\xi \simeq \frac{2}{3} \frac{(\mu_{bc} - \mu_{bc}^0)^{\frac{3}{2}}}{1 - (\mu_{bc}^0)^2} (\mu_{bc}^1 - \mu_{bc}^0)^{\frac{1}{2}}, \tag{F.7}$$

and using (F.6)

$$\Psi_A \simeq \tfrac{2}{3}(\mu_{bc} - \mu_{bc}^0)[1 - (\mu_{bc}^0)^2]^{-1}(1 + 2\mu_{bd}\mu_{cd}\mu_{bc} - \mu_{bd}^2 - \mu_{cd}^2 - \mu_{bc}^2)^{\frac{1}{2}}. \tag{F.8}$$

Since $\mu_{bc} - \mu_{bc}^0 = 2 \sin \frac{1}{2}(\alpha + \delta_b + \delta_c) \sin \frac{1}{2}(-\alpha + \delta_b + \delta_c) \simeq -(\alpha - \delta_b - \delta_c) \sin \alpha$, we obtain from (F.8), (B.6), (B.5)

$$\Psi_A \simeq \tfrac{2}{3} \sin \alpha (\delta_b + \delta_c - \alpha) \frac{6V}{(b + \frac{1}{2})(c + \frac{1}{2})(d + \frac{1}{2})} (\sin \alpha)^{-2} \simeq$$
$$\simeq \tfrac{2}{3}(\delta_b + \delta_c - \alpha) \sin \delta_c \sin C. \tag{F.9}$$

If h is the distance of P from the plane $P_AP_BP_C$ we have also

$$(d + \tfrac{1}{2}) \Psi_A \simeq \tfrac{2}{3} h (\delta_b + \delta_c - \alpha). \tag{F.10}$$

Therefore from (F.10) and similar relations for Ψ_B, Ψ_C we find

$$\Omega \simeq \pi(a + b + c - \tfrac{1}{4}) + \tfrac{2}{3} h (2\pi - \delta - \varepsilon - \varphi). \tag{F.11}$$

Our result (E.9) yields in the neighbourhood of $V^2 = 0$

$$V^2 = \tfrac{1}{18}[(d + \tfrac{1}{2})(e + \tfrac{1}{2})(f + \tfrac{1}{2})]^2 \sin \delta \sin \varepsilon \sin \varphi \, (2\pi - \delta - \varepsilon - \varphi), \tag{F.12}$$

having taken into account the different definition of δ, ε, φ. Using the proportionality between h and V, we obtain finally from (F.11), (F. 12)

$$\Omega \simeq \pi(a + b + c - \tfrac{1}{4}) + \tfrac{9}{2} V^3 \left(\prod_{k=1}^{4} A_k \right)^{-1}. \tag{F.13}$$

In a similar way, when we approach B_4 as $V^2 \to 0$, we find in the case corresponding to fig. 1

$$\Omega \simeq \pi(a + b + d + e + \tfrac{1}{4}) - \tfrac{9}{2} V^3 \left(\prod_{k=1}^{4} A_k \right)^{-1}. \tag{F.14}$$

Appendix G. WKB approximation for $d_{\delta\delta'}^{(f)}(\theta)$

Let us recall here the main features of the WKB solutions of

$$\frac{d^2 g}{dz^2} + Q^2(z)\, g = 0, \tag{G.1}$$

without discussing their degree of approximation[33]); in (G.1) $Q^2(z) = \mathcal{Q}^2(z)(z - z_1)(z_2 - z)$ with $z_1 < z_2$ and $\mathcal{Q}^2 > 0$. We know that the function

$$z \leqslant z_1, \quad g_1(z) = \frac{2}{\pi} \left(\frac{|\mathrm{Im}\, t_1|}{|Q|} \right)^{\frac{1}{2}} \times$$
$$\times \{\pi \sin \eta_1 I_{\frac{1}{3}}(|\mathrm{Im}\, t_1|) + \cos(\tfrac{1}{3}\pi - \eta_1) K_{\frac{1}{3}}(|\mathrm{Im}\, t_1|)\} \tag{G.2}$$

joins smoothly with

$$z_1 < z \leqslant z_2 \begin{cases} g_2(z) = \left(\dfrac{4t_1}{3Q} \right)^{\frac{1}{2}} \{\cos(\tfrac{1}{3}\pi + \eta_1) J_{\frac{1}{3}}(t_1) + \cos(\tfrac{1}{3}\pi - \eta_1) J_{-\frac{1}{3}}(t_1)\}, & \text{(G.3)} \\ t_1(z) = \displaystyle\int_{z_1}^{z} Q(\xi)\, d\xi, \quad Q(\xi) = +[Q^2(z)]^{\frac{1}{2}}, & \text{(G.4)} \end{cases}$$

if we choose for $z \leqslant z_1$: $Q(z) = i|Q(z)|$, $|\mathrm{Im}\, t_1| = \int_z^{z_1} |Q(\xi)|\, d\xi$; far from z_1 we have

$$g_1(z) \simeq (2\pi|Q|)^{-\frac{1}{2}} \{2 \sin \eta_1 \, e^{|\mathrm{Im}\, t_1|} + \cos \eta_1 \, e^{-|\mathrm{Im}\, t_1|}\}, \tag{G.5}$$

$$g_2(z) \simeq \left(\frac{2}{\pi Q} \right)^{\frac{1}{2}} \cos(t_1 + \eta_1 - \tfrac{1}{4}\pi). \tag{G.6}$$

Similarly, continuity through z_2 can be achieved by means of

$$z_1 < z \leqslant z_2 \begin{cases} g_2'(z) = \left(\dfrac{4t_2}{3Q} \right)^{\frac{1}{2}} \{\cos(\tfrac{1}{3}\pi + \eta_2) J_{\frac{1}{3}}(t_2) + \cos(\tfrac{1}{3}\pi - \eta_2) J_{-\frac{1}{3}}(t_2)\}, & \text{(G.7)} \\ t_2(z) = \displaystyle\int_{z}^{z_2} Q(\xi)\, d\xi, \quad Q(\xi) = +[Q^2(\xi)]^{\frac{1}{2}}, & \text{(G.8)} \end{cases}$$

$$z_2 < z, \quad g_3(z) = \frac{2}{\pi} \left(\frac{|\mathrm{Im}\, t_2|}{|Q|} \right)^{\frac{1}{2}} \times$$
$$\times \{\pi \sin \eta_2 I_{\frac{1}{3}}(|\mathrm{Im}\, t_2|) + \cos(\tfrac{1}{3}\pi - \eta_2) K_{\frac{1}{3}}(|\mathrm{Im}\, t_2|)\}, \tag{G.9}$$

choosing for

$$z_2 < z: \quad Q(z) = \mathrm{i}\,|Q(z)|, \quad |\mathrm{Im}\, t_2| = \int_{z_2}^{z} |Q(\xi)|\, \mathrm{d}\xi\,;$$

for large $|z - z_2|$ these solutions behave according to (G.6), (G.5) with the obvious replacements: $t_1 \to t_2$, $\eta_1 \to \eta_2$. We note incidentally that, for instance $g_2(z)$ can be written in terms of Airy functions[25]):

$$g_2(z) = \left(\frac{3t_1}{Qy}\right)^{\frac{1}{2}} \{\cos\eta_1\, \mathrm{Ai}(-y) + \sin\eta_1\, \mathrm{Bi}(-y)\} \tag{G.10}$$

$$\mathrm{Ai}(-y) = \tfrac{1}{3} y^{\frac{1}{2}} [J_{-\frac{1}{3}}(t_1) + J_{\frac{1}{3}}(t_1)], \quad \mathrm{Bi}(-y) = (\tfrac{1}{3} y)^{\frac{1}{2}} [J_{-\frac{1}{3}}(t_1) - J_{\frac{1}{3}}(t_1)], \tag{G.11}$$

where $y = (\frac{3}{2} t_1)^{\frac{2}{3}}$. We see from (G.6) that the consistency condition $g_2(z) = g_2'(z)$ can be fulfilled for $z_1 \ll z \ll z_2$ if

$$\int_{z_1}^{z_2} Q(\xi)\, \mathrm{d}\xi = -\eta_1 - \eta_2 + (2N + \tfrac{1}{2})\pi, \quad N \gg 1 \tag{G.12}$$

which provides a relation between η_1 and η_2.

We know (Brussaard and Tolhoek[16])) that, for large f, $d^{(f)}_{\delta\delta'}(\theta)$ satisfies

$$\left\{\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)} \sin^2\theta \frac{\mathrm{d}}{\mathrm{d}(\cos\theta)} + (f + \tfrac{1}{2})^2 \sin^2\theta\, [(1-\mu^2) \times (1-\nu^2) - (\cos\theta - \mu\nu)^2]\right\} d^{(f)}_{\delta\delta'}(\theta) = 0, \tag{G.13}$$

where $(f+\frac{1}{2})\,\mu = \delta'$, $(f+\frac{1}{2})\,\nu = \delta$; we shall consider only the domain $|\cos\theta| \leqslant 1$. If we put $\cos\theta = \mathrm{tgh}\, z$, $d^{(f)}_{\delta\delta'}(\theta) = g(z)$, then we obtain (G.1) with

$$Q^2(z) = (f + \tfrac{1}{2})^2\, [(1-\mu^2)(1-\nu^2) - (\mathrm{tgh}\, z - \mu\nu)^2], \tag{G.14}$$

and transition points

$$\cos\theta_{1,2} \equiv \mathrm{tgh}\, z_{1,2} = \mu\nu \mp \sqrt{(1-\mu^2)(1-\nu^2)}; \tag{G.15}$$

when $\mu = \nu$, z_2 corresponds to $\cos\theta_2 = 1$; therefore we shall limit ourselves to $\delta \neq \delta'$.

In order to apply (G.2)–(G.9), we must provide t_1, t_2, η_1 and η_2 explicitly. From (G.4), (G.14) and (F.4) we have

$$\int^{z} (f + \tfrac{1}{2})\, [1 - \mu^2 - \nu^2 + 2\mu\nu\, \mathrm{tgh}\, z - (\mathrm{tgh}\, z)^2]^{\frac{1}{2}}\, \mathrm{d}z = (f + \tfrac{1}{2})\, [F(z) - \mu A(z) - \nu B(z)], \tag{G.16}$$

where $\pi - F$, A, B are the *internal* dihedral angles between the planes belonging respectively to f, $a+\delta'$, $b+\delta$ (fig. 4 with $\mu = \cos\alpha$, $\nu = \cos\beta$). We notice from (G.15) that z_1 corresponds to $\theta_1 = \alpha + \beta$, $A = B = F = 0$. On the other hand z_2 is associated to the case $\theta_2 = |\alpha - \beta|$; we have for $\alpha > \beta$: $B = F = \pi$, $A = 0$ and for $\alpha < \beta$: $A = F = \pi$, $B = 0$. Therefore[34]):

$$t_1(\theta) \equiv t = (f + \tfrac{1}{2})(F - A\mu - B\nu),$$
$$F = \arccos\left(\frac{\mu\nu - \cos\theta}{|\sin\alpha\sin\beta|}\right), \quad A = \arccos\left(\frac{\nu - \mu\cos\theta}{|\sin\alpha\sin\theta|}\right), \tag{G.17}$$
$$B = \arccos\left(\frac{\mu - \nu\cos\theta}{|\sin\beta\sin\theta|}\right),$$

and similarly for t_2. Furthermore (G.12) yields now:

$$-\eta_1 - \eta_2 + (2M + \tfrac{1}{2})\pi = \int_{z_1}^{z_2} Q(\xi)\, \mathrm{d}\xi = \begin{cases} (f + \frac{1}{2})(\pi - \pi\nu), & \alpha > \beta, \quad (\delta > \delta'), \\ (f + \frac{1}{2})(\pi - \pi\mu), & \alpha < \beta, \quad (\delta < \delta'). \end{cases} \tag{G.18}$$

From the asymptotic behaviour of Jacobi polynomials[19]) and relation (A.18) it is easy to obtain:

$$d^{(f)}_{\delta\delta'}(\theta) \simeq \left[\frac{2}{\pi(f + \frac{1}{2})\sin\theta}\right]^{\frac{1}{2}} \cos\{(f + \tfrac{1}{2})\,\theta + \tfrac{1}{2}\pi(\delta - \delta') - \tfrac{1}{4}\pi\}, \tag{G.19}$$

valid when f is large and $|\delta|, |\delta'| \ll f$; performing this limit in (G.17), we have: $A \simeq B \simeq \frac{1}{2}\pi$, $F \simeq \pi - \theta$ and consequently $t_1(\theta) \simeq (f + \frac{1}{2})(\pi - \theta) - \frac{1}{2}\pi(\delta + \delta')$. Using this result and identifying the arguments of the cosines in (G.6) and (G.19), we obtain

$$\eta_1 = \pi(\delta' - f). \tag{G.20}$$

Therefore from (G.18)

$$\eta_2 = \begin{cases} \pi(\delta - \delta') & \delta > \delta', \\ 0 & \delta < \delta'; \end{cases} \tag{G.21}$$

obviously η_1 and η_2 are determined modulo $2N\pi$. It must be stressed that only for "physical" values of δ, δ' the phases η_1, η_2 (except when $\delta < \delta'$) are integer multiples of π; in this case the exponentially increasing term in (G.5) is ruled out.

The overall normalization in (G.2), (G.3) and (G.9) has been chosen in agreement with (A.15).

Appendix H. The 1 + 1 + 2 case

In this last appendix we want to show that (3.6) is in agreement with our asymptotic formulae (1.4), (1.8). To this end we consider the limiting case in which f, δ, δ', though large, are still small with respect to a, b, c; therefore we may use in (3.6) our results of appendix G.

In the oscillatory region we deduce from (3.6), (G.6), (G.17), (G.20)

$$\begin{Bmatrix} c & a & b \\ f & b+\delta & a+\delta' \end{Bmatrix} \simeq \frac{(-1)^{a+b+c+f+\delta}}{[(2a+1)(2b+1)]^{\frac{1}{2}}} \left[\frac{2}{\pi(f+\frac{1}{2})}\right]^{\frac{1}{2}} \times$$
$$\times [(1-\mu^2)(1-\nu^2)-(\cos\theta-\mu\nu)^2]^{-\frac{1}{2}} \cos\{t+\pi(\delta'-f)-\tfrac{1}{4}\pi\}$$

and recalling (B.6):

$$\begin{Bmatrix} c & a & b \\ f & b+\delta & a+\delta' \end{Bmatrix} \simeq \frac{(-1)^{a+b+c+\delta+\delta'}}{[12\pi V]^{\frac{1}{2}}} \cos(t-\tfrac{1}{4}\pi). \tag{H.1}$$

On the other hand we note that as a, b, c increase in the tetrahedron of fig. 4, the two faces belonging to c become almost parallel; therefore: $\theta_c \simeq \pi$, $\theta_a \equiv A \simeq \pi - \theta_{a+\delta'}$, $\theta_b \equiv B \simeq \pi - \theta_{b+\delta}$ and from the definition (3.15b) we obtain

$$\begin{aligned} \Omega &= (a+b+c+\delta+\delta'-\tfrac{1}{4})\pi + (f+\tfrac{1}{2})F - \delta' A - \delta B = \\ &= t + (a+b+c+\delta+\delta'-\tfrac{1}{4})\pi, \end{aligned} \tag{H.2}$$

which, when introduced into (1.4a) leads to (H.1).

Let us now consider the classically forbidden regions. First we notice that the transition points $z_{1,2}$ relative to $d^{(f)}_{\delta\delta'}(\theta)$ discussed in appendix G correspond to configurations in which the tetrahedron of fig. 4 becomes flat; this can be checked by means of (G.14) and (B.6). More precisely, since in z_1: $\theta=\alpha+\beta$, the tetrahedron enters the region B_3 with

$$z_1) \quad \Phi = (a+b+c+\delta+\delta')\pi, \tag{H.3}$$

while in z_2, $\theta=|\alpha-\beta|$ and it enters B_4 in either one of the two ways

$$z_2) \quad \Phi = \begin{cases} (a+b+c+f+\delta')\pi & \delta > \delta', \\ (a+b+c+f+\delta)\pi & \delta < \delta'. \end{cases} \tag{H.4}$$

On the other hand, from (G.5) and (3.6) we have for the forbidden region relative to z_1

$$\begin{Bmatrix} c & a & b \\ f & b+\delta & a+\delta' \end{Bmatrix} \simeq \frac{(-1)^{a+b+c+f+\delta}}{2[12\pi|V|]^{\frac{1}{2}}} \times$$
$$\times \{2\sin\pi(\delta'-f)\,e^{|\mathrm{Im}\,t|} + \cos\pi(\delta'-f)\,e^{-|\mathrm{Im}\,t|}\}. \tag{H.5}$$

Noticing that $|\mathrm{Im}\,t| = |\mathrm{Im}\,\Omega|$, we see from (5.8), (H.2), (H.3) and (H.5) that for physical values of the angular momenta (H.5) and (5.8) become identical even in sign. The same holds also for the forbidden region relative to z_2; in fact we have from (G.21):

$$(-1)^{\eta_2/\pi+a+b+c+f+\delta} = \begin{cases} (-1)^{a+b+c+f+\delta'} & \delta > \delta' \\ (-1)^{a+b+c+f+\delta} & \delta < \delta' \end{cases}$$

in agreement with (H.4).

References and footnotes

1) G. Racah, Phys. Rev. **61**, 186 (1942); **62**, 438 (1942); **63**, 367 (1943); **76**, 1352 (1949).
2) See the exhaustive bibliography in L. C. Biedenharn and H. Van Dam, *Quantum Theory of Angular Momentum* (Academic Press, New York, 1965).
3) L. C. Biedenharn, private communication to one of us (T.R.), 1958.
4) E. P. Wigner, Am. J. Math. **63**, 57 (1941); On the Matrices which Reduce the Kronecker Products of Representations of S. R. Groups (unpublished, 1940); hectographed paper (Princeton, 1951) (see ref. 2).
5) W. T. Sharp, Some Formal Properties of the 12*j*-symbol, Chalk River Report, TPJ-81, Ontario, 1955.
6) A. P. Yutsis, I. B. Levinson and V. V. Vanagas, *The Mathematical Apparatus of the Theory of Angular Momentum* (translated from Russian) (Jerusalem, 1962); A. P. Yutsis and A. A. Bandzaitis, *Teoria Momenta Kolichestva Dvizheniya Kvantovoi Mekhanike* (Vilnius, 1965) (in Russian).
7) In addition to the three-dimensional representation which will be used in this paper, we recall the graphical techniques developed by A. Yutsis and coworkers[6]) as well as its dual, exemplified by U. Fano and G. Racah, *Irreducible Tensorial Sets* (Academic Press, New York, 1959) appendices.
8) We introduce $a+\frac{1}{2}$ rather than a... as length of the edge corresponding to a... because, for high quantum numbers, the length $[a(a+1)]^{\frac{1}{2}}$ of the angular momentum vector is closer to $a+\frac{1}{2}$ in the semiclassical limit; it turns out that any other choice is inconsistent with numerical results and with existing formulae.
9) N. F. Tartaglia, *General trattato de numeri et misure* (Venezia, 1560).
10) See the footnote 142 of ref. 22.
11) A. Cayley, Cambridge Math. J. **2**, 267 (1841).
12) S. Adler has kindly pointed out to us that:

$$2^9(3!)^2V^2 \equiv -(2\alpha+1)(2\gamma+1)(2\varepsilon+1) - 2[2q_4+1]_2[2q_2+1]_2[2q_1+1]_2+1, \text{ mod. } 8$$

where q_1, q_2, q_4 are defined above and $[A]_2=0$, 1 if A is even, odd;

$$\alpha=[2a+1]_2[2q_4+1]_2, \quad \gamma=[2c+1]_2[2q_2+1]_2, \quad \varepsilon=[2e+1]_2[2q_1+1]_2.$$

It follows that if all the q's are integer and $\alpha, \gamma, \varepsilon = 0$, or 1:

$$2^9(3!)^2V^2 \equiv -(2\alpha+1)(2\gamma+1)(2\varepsilon+1)-1 \equiv -2, -4 \text{ mod. } 8;$$

therefore $V^2 \neq 0$.

13) E. P. Wigner, *Group Theory* (Academic Press, New York, 1959).
14) W. Pauli, in: *Handbuch der Physik*, Band 24/1 (Springer-Verlag, Berlin, 1933) p. 170.
15) T. Regge, Nuovo Cimento **10**, 296 (1958); **11**, 116 (1959).
16) G. Racah, Phys. Rev. **84**, 910 (1951); *Lectures on Group Theory* (Institute for Advanced Study, Princeton, 1951); L. C. Biedenharn, J. Math. Phys. **31**, 287 (1953); K. Alder, A. Bohr, T. Huus, B. Mottelson and A. Winther, Rev. Mod. Phys. **28**, 432 (1956); P. J. Brussaard and H. A. Tolhoek, Physica **23**, 955 (1957); A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, 1957).
17) L. C. Biedenharn, ref. 16, and J. P. Elliott, Proc. Royal Soc. (London) A **218**, 370 (1953).
18) This property stems from a particular case of a theorem by Schläfli on elliptic tetrahedra; see H. S. M. Coxeter, *Non-Euclidean Geometry* (Toronto, 1957).
19) G. Szegö, *Orthogonal Polynomials* (American Mathematical Society, New York, 1959).
20) A discussion of this theorem, due originally to G. Racah, can be found in the appendix to the book by U. Fano and G. Racah, ref. 7.
21) See for instance eq. (16.8) ref. 6.
22) E. Hammer, *Trigonometrie* (Stuttgart, 1923).
23) A. Erdélyi, *Asymptotic Expansions* (Dover Publications, New York, 1956).
24) We do not expect to be true in general that the 6*j*-symbol is a function of the variables V and $\Pi_{h=1}^{4} A_h$ only.
25) J. C. P. Miller, *The Airy Integral* (Cambridge, 1946).
26) R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
27) T. Regge, Nuovo Cimento **19**, 551 (1961).
28) J. A. Wheeler, Geometrodynamics and the Issue of the Final State (1963), in: *Relativity, Groups and Topology*, Eds. C. and B. DeWitt (Gordon and Breach, New York, 1964).
29) B. L. Van der Waerden, *Die gruppentheoretische Methode in der Quantenmechanik* (Springer-Verlag, Berlin, 1932).
30) A. Messiah, *Quantum Mechanics* (North-Holland Publ. Co., Amsterdam, 1964) appendix C.
31) R. F. Scott, *The Theory of Determinants* (Cambridge, 1904) chapter XVII.
32) T. Regge and G. Barucchi, Nuovo Cimento **34**, 106 (1964).
33) R. E. Langer, Phys. Rev. **51**, 669 (1937).
34) The discussion which is carried out here as well as in appendix H corresponds actually to $\delta > 0$, $\delta' > 0$; however, it can be easily extended to the other cases yielding slight modifications in t_1, t_2, η_1, η_2. For instance, when $\delta < 0$, $\delta' < 0$ one would find $\theta_1 = 2\pi - \alpha - \beta$, $t_1(\theta) = (f+\frac{1}{2})\,[F-(A-\pi)\,\mu - (B-\pi)\,\nu]$, $\eta_1 = -\pi(f+\delta)$, while η_2 would be the same as in (G.21) for "physical" values of δ, δ'.

Spectroscopic and group theoretical methods in physics

ON RACAH COEFFICIENTS AS COUPLING COEFFICIENTS FOR THE VECTOR SPACE OF WIGNER OPERATORS

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1. Introduction and summary

One of the most important contributions made by Giulio Racah to theoretical physics was the systematic development of the theory of tensor operators for angular momentum; the results of this development – called, in honor of its principal creators, the Racah–Wigner calculus –, and its application, are now part of the equipment of every working physicist[1]).

There are two quite distinct reasons that underlie the success of this work. The first is the fundamental nature of angular momentum in quantum mechanics, and this stems ultimately from symmetry (isotropy) of space-time (Poincaré group) which is the deepest presently known foundation for quantum mechanics. This may be termed the 'physical reason' for the importance of the Racah–Wigner calculus. The second reason is mathematical, and provides the ultimate source for the very existence of the Racah–Wigner calculus: the angular momentum group (SU_2) has the property of being *simply reducible*[2]). This is the property which guarantees that the Racah–Wigner calculus is uniquely defined by the group and contains no inherent ambiguities. The really essential part of Wigner's definition of simple reducibility is that in reducing the Kronecker product of two irreducible representations ('irreps'), a given irrep occurs either once or not at all. One is forced by the structure of quantum mechanics (tensor operators acting on states) to 'multiply' irreps; if this 'product' upon being reduced into elements (irreps) has a given irrep occurring more than once, an inherent ambiguity may occur which is not decideable within the original symmetry group.

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