Today's Outline

Lecture 13

Sept. 28th, 2011

I. Last time

II. Complete Reduction

III. Qualitative Analysis

IV. Radial E.O.M. for the Kepler problem

swap order

I. Last time

- Introduced coordinates

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \qquad \left(\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r} \right.$$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M} \qquad \left. \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r} \right)$$

for relative position and CM position.

- Transformed to CM frame where $\dot{\vec{R}} = 0$ and chose $\vec{R} = 0$ as origin.

- Found the (conservative) central force Lagrangian in this frame:

$$\mathcal{L} = \mathcal{L}_{rel} = \frac{1}{2}\frac{m_1 m_2}{M}\dot{\vec{r}}^2 - U(r)$$

Note that $m_1 m_2 / M$ has units of mass and introduce shorthand

$$\mu \equiv \frac{m_1 m_2}{M} = \frac{m_1 m_2}{m_1 + m_2}$$

the "reduced" mass. Then

$$\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$$

{Say: like the Lagrangian of a single particle of mass μ}

II. Recall examples:

$$U = -\frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|} \qquad \text{gravitation}$$

$$U = \frac{k q Q}{|\vec{r}_1 - \vec{r}_2|} \qquad \text{Coulomb } \cancel{\text{force}} \text{ potential}$$

Once again we note a symmetry: both these potentials are invariant under overall rotations! Denote a rotation matrix by $\overleftrightarrow{R}$ then

$$|\overleftrightarrow{R}\vec{r}_1 - \overleftrightarrow{R}\vec{r}_2| = |\overleftrightarrow{R}(\vec{r}_1 - \vec{r}_2)| = |\vec{r}_1 - \vec{r}_2|$$

because rotations don't change the length of a vector.

Then, Noether's theorem implies that the total angular momentum is conserved!

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$

In the CM frame $\dot{\vec{R}} = 0$ and $\vec{R} = 0$, so,

$$\vec{r}_1 = \frac{m_2}{M}\vec{r} \qquad \text{and} \qquad \vec{r}_2 = -\frac{m_1}{M}\vec{r}$$

and

$$\vec{L} = \frac{m_2}{M}\vec{r} \times \frac{m_1 m_2}{M}\dot{\vec{r}} + \left(-\frac{m_1}{M}\right)\vec{r} \times m_2\left(-\frac{m_1}{M}\right)\dot{\vec{r}}$$

$$= \frac{m_1 m_2}{M^2}\left(m_2 \vec{r} \times \dot{\vec{r}} + m_1 \vec{r} \times \dot{\vec{r}}\right) = \vec{r} \times \mu \dot{\vec{r}}$$

By convention we choose the z-axis to be parallel to $\vec{L}$. Then all of the motion lies in the xy-plane (viz. $\vec{r}$ and $\dot{\vec{r}}$ lie in plane perp. to $\vec{L}$).

We have reduced the problem to one in 2 D.O.F. at this point.

Now choose polar coordinates in the xy-plane (r, ϕ). Our Lagrangian is,

$$\mathcal{L} = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r).$$

The vector $\vec{L}$ is constant, which means both its magnitude and its direction are constant.

Because $\vec{L} = \vec{r} \times \mu \dot{\vec{r}}$ it must be perpendicular to both $\vec{r}$ and $\dot{\vec{r}}$. As long as $\vec{L} \neq 0$, the two vectors $\vec{r}$ and $\dot{\vec{r}}$ span a plane.

Convention: We are free to orient our axes w/in the CM frame however we like.

We notice that ϕ is ignorable

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \text{const.}$$

$$\Rightarrow \quad \mu r^2 \dot{\phi} = \text{const} \equiv \ell \quad (\text{Eq. } 1)$$

This is just the z-component of angular momentum that we knew we had around (recall $\mu r^2 = I$ for pt. particle and $\dot{\phi} = \omega =$ angular speed, $L_z = I\omega$). Initial conditions set the value of ℓ and so we can use (Eq. 1) to get rid of ϕ,

$$(\text{Eq. } 1) \Rightarrow \dot{\phi} = \frac{l}{\mu r^2},$$

once we're working with the E.O.M.

This will complete our reduction, all that will remain is 1 D.O.F., r.

Reduction has been a major success for central forces; we've achieved the best possible outcome

$$6 \text{ D.O.F.} \longrightarrow 1 \text{ D.O.F.}$$

III Radial E.O.M. for the Kepler problem.

We have,

$$U = -\frac{G m_1 m_2}{r}$$

and so,

$$\mathcal{L} = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\phi}^2 + \frac{G m_1 m_2}{r}.$$

Then,

$$\frac{\partial \mathcal{L}}{\partial r} = \mu r \dot{\phi}^2 - \frac{G m_1 m_2}{r^2}$$

and

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{d}{dt}(\mu \dot{r}) = \mu \ddot{r}.$$

Now we proceed to solving the 1 E.O.M for r. There is one caveat; we're no longer working with the physically intuitive variables $\vec{r}_1$ and $\vec{r}_2$. This means that we have to decode what we learn along the way.

Let's specialize to the Kepler problem (see your book for general case).

So,

$$\mu \ddot{r} = \mu r \dot{\phi}^2 - \frac{G m_1 m_2}{r^2}$$

Now let's use l from above,

$$\mu \ddot{r} = \mu r \cdot \frac{l^2}{\mu^2 r^4} - \frac{G m_1 m_2}{r^2}$$

$$= \frac{l^2}{\mu r^3} - \frac{G m_1 m_2}{r^2}$$

In this form it's tempting to think of the first term on the R.H.S. as another radial force, the "centrifugal force",

$$F_{cf} = \frac{l^2}{\mu r^3}$$

Because this term actually originated with the kinetic energy we call it a "fictitious" force (note it's just $\mu r\dot{\phi}^2 = \frac{\mu v_\phi^2}{r}$, with $v_\phi = r\dot{\phi}$, from 7A).

We can also derive this force from a potential,

$$U_{cf} = \frac{\ell^2}{2\mu r^2},$$

will do the trick: $F_{cf} = -\frac{d}{dr}U_{cf}$.

Thus

$$\mu\ddot{r} = -\frac{d}{dr}\left(U_{cf} + U\right)$$

and we call the combination

$$U_{cf} + U \equiv U_{eff}$$

the "effective potential"

The introduction of U_{cf} is, of course, useful for any central force problem.

IV Qualitative Analysis

There's one more conservation law hiding in the woodwork. What is it? What's the remaining symmetry? Answers: The symmetry is time translation invariance and it corresponds to conservation of energy (note that $\mathcal{L}$ doesn't explicitly depend on t):

$$E = K.E. + P.E.$$

$$= \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\mu r^2\dot{\phi}^2 + U(r)$$

$$= \tfrac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r)$$

$$= \tfrac{1}{2}\mu\dot{r}^2 + U_{eff}(r).$$

On Friday, we'll use this conservation law to extract as much qualitative information about the motion as possible.