# Lecture 14

Sept. 30th, 2011

## Today's Outline

I. Last time

II. Continue Qualitative Analysis

III Solving the radial E.O.M.

IV Bounded Kepler Orbits

## I. Last time

We

- Reduced central force motion to an equivalent 1 D.O.F. problem
- Found the radial E.O.M.:

Kepler problem

$$\mu\ddot{r} = \frac{l^2}{\mu r^3} - \frac{dU}{dr} = \frac{l^2}{\mu r^3} - \frac{Gm_1m_2}{r^2}$$

- Found the total energy

$$E = K.E. + P.E.$$
$$= \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\mu r^2\dot{\phi}^2 + U(r)$$
$$= \tfrac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r)$$
$$= \tfrac{1}{2}\mu\dot{r}^2 + U_{eff}(r).$$

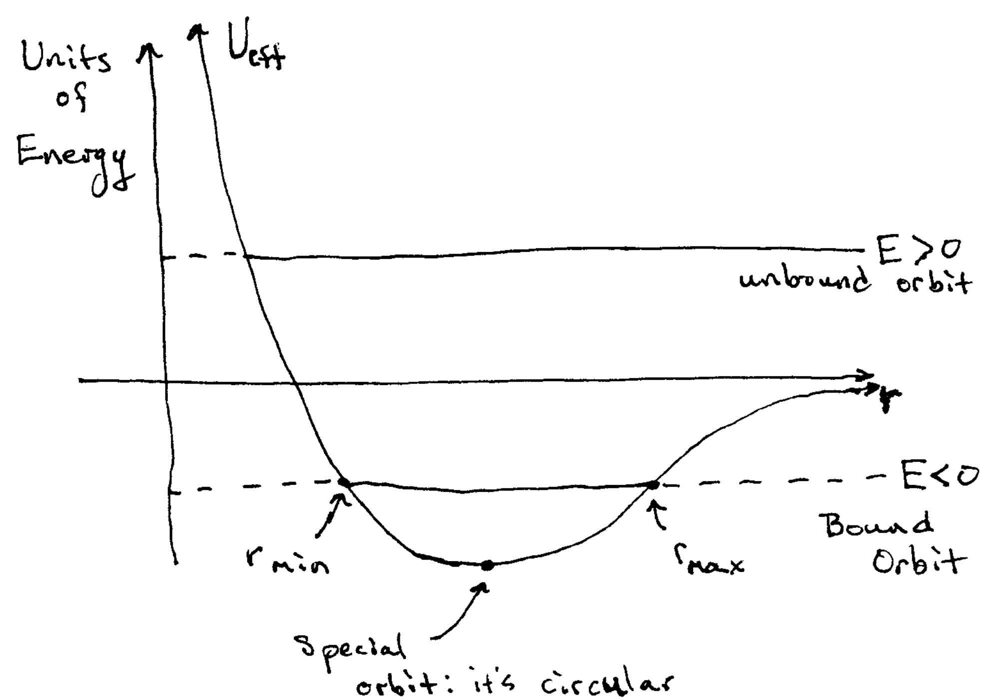
## II. Continue Qualitative Analysis

The effective potential for the Kepler problem is

dominates at large r

$$U_{eff} = -\frac{Gm_1m_2}{r} + \frac{l^2}{2\mu r^2}$$

dominates at small r



III Solving the radial E.O.M.

At this point we need to invoke two pieces of mathematical cleverness to solve our E.O.M.

The first one is somewhat intuitive, we are going to change the parametrization from $r(t)$ to $r(\phi)$ — this will tell us the geometric shape

We introduce a new variable,

$$u \equiv 1/r \Rightarrow r = 1/u.$$

Recall
Our radial eq. ~~is then~~

$$\mu \ddot{r} = \frac{l^2}{\mu r^3} - \frac{G m_1 m_2}{r^2}$$

We need:

$$\dot{r} = \frac{l}{\mu r^2}\frac{dr}{d\phi} = \frac{l u^2}{\mu}\frac{d}{d\phi}\left(\frac{1}{u}\right) = \frac{l u^2}{\mu}\left(-\frac{1}{u^2}\right)\frac{du}{d\phi}$$

$$= -\frac{l}{\mu}\frac{du}{d\phi}$$

of the orbit rather than $p_2/5$ giving us ~~a par~~ its time parametrization.

In practice this means we need to use the chain rule to replace $d/dt$ with $d/d\phi$,

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi} = \frac{l}{\mu r^2}\frac{d}{d\phi}.$$

The second bit of cleverness is not obvious, we'll see why it's nice in the end.

Also,

$$\ddot{r} = \frac{d}{dt}\left(-\frac{l}{\mu}\frac{du}{d\phi}\right) = -\frac{l^2}{\mu^2 r^2}\frac{d^2u}{d\phi^2}$$

$$= -\frac{l^2 u^2}{\mu^2}\frac{d^2u}{d\phi^2}$$

Putting this into the E.O.M. gives

$$-\mu\frac{l^2u^2}{\mu^2}\frac{d^2u}{d\phi^2} = \frac{l^2u^3}{\mu} - G m_1 m_2 u^2$$

$$\Rightarrow \frac{d^2u}{d\phi^2} = -u + \frac{G m_1 m_2 \mu}{l^2}$$

This equation we have studied! It's an harmonic oscillator with a (constant) forcing term. Introduce two shorthands

$$u' \equiv \frac{du}{d\phi} \quad \text{and let} \quad \gamma \equiv G m_1 m_2$$

then
$$u'' = -u + \frac{\gamma\mu}{l^2}.$$

General solution to an inhomogeneous equation = gen. sol. of homog. + part. sol.

Guess a particular solution 

$$u_p = \text{const.} = K \quad \text{call it}$$

Then $u_p' = 0$, $u_p'' = 0$ and

$$0 = -K + \frac{\gamma\mu}{l^2} \Rightarrow K = \frac{\gamma\mu}{l^2}$$

The general solution is,

$$u(\phi) = A\cos(\phi - \delta) + \frac{\gamma\mu}{l^2}$$

and by choosing the origin of $\phi$ properly we can get rid of $\delta$,

$$u(\phi) = A\cos(\phi) + \frac{\gamma\mu}{l^2}$$

A bit more notation:

let $$c \equiv \frac{1}{K} = \frac{l^2}{\gamma\mu} \qquad [c] = \text{length}$$

and $$\epsilon \equiv \frac{A l^2}{\gamma\mu} = A \cdot c \qquad \text{Unitless "eccentricity"}$$

$$\Rightarrow u(\phi) = \frac{1}{r(\phi)} = \frac{\epsilon}{c}\cos(\phi) + \frac{1}{c}$$
$$= \frac{1}{c}(1 + \epsilon\cos\phi)$$

or

$$r(\phi) = \frac{c}{1 + \epsilon\cos\phi}$$

IV Bounded Kepler orbits

Note that if $\epsilon < 1$ then $r$ is bounded for all $\phi$ but if $\epsilon \geq 1$ then at some $\phi$ $r$ runs off to infinity. We will find that this is the difference btwn the bound and unbound orbits.

Let's focus on bound orbits first, $\epsilon < 1$. The bounds on $r$ are at $\phi = 0$ and $\phi = \pi$

We have,

$\phi = 0$

$$r_{min} = \frac{c}{1+\epsilon}$$

"perihelion"

$\phi = \pi$

$$r_{max} = \frac{c}{1-\epsilon}$$

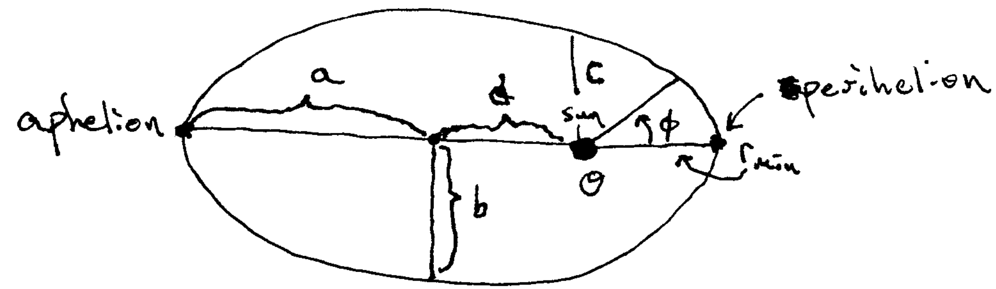
"aphelion"

On the homework you will show

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \left(\begin{array}{c}\text{Equation} \\ \text{of an} \\ \text{ellipse}\end{array}\right)$$

for this orbit. The various constants are

$$a = \frac{c}{1-\epsilon^2} \qquad b = \frac{c}{\sqrt{1-\epsilon^2}} \qquad d = a\epsilon$$




$a + d = r_{max}$. Note that our origin is at one of the foci, this is the origin of the "$x+d$" above.

Indeed,

$$\frac{b}{a} = \frac{c}{\sqrt{1-\epsilon^2}} \; \frac{1-\epsilon^2}{c} = \sqrt{1-\epsilon^2}$$

and this verifies that $\epsilon$ is the eccentricity. Note the limits

$\epsilon \to 0 \Rightarrow b/a = 1 \Rightarrow$ a circle

and $\epsilon \to 1 \Rightarrow b/a \to 0$ a highly stretched ellipse. From this it follows that $d = a\epsilon$ is indeed the distance to a focus.

How does this geometry relate to the physics?

$$E = U_{eff}(r_{min}) = -\frac{\gamma}{r_{min}} + \frac{l^2}{2\mu r_{min}^2}$$

$$= \frac{1}{2r_{min}}\left(\frac{l^2}{\mu r_{min}} - 2\gamma\right)$$

But

$$r_{min} = \frac{c}{1+\epsilon} = \frac{l^2}{\gamma\mu(1+\epsilon)}$$

Putting this into the equation for E we have,

$$E = \frac{\gamma\mu(1+\epsilon)}{2\ell^2}\left(\gamma(1+\epsilon) - 2\gamma\right)$$

$$= \frac{\gamma\mu}{2\ell^2}(1+\epsilon)(\gamma\epsilon - \gamma)$$

$$= \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2 - 1)$$

$$= \frac{\gamma}{2c}(\epsilon^2 - 1)$$

So indeed for $\epsilon < 1$ $E < 0$ and the orbit is bounded. While for $\epsilon > 1$ $E > 0$ and the orbit is unbounded.