Today's Outline

Lecture 15

Oct. ~~2nd~~ 3rd, 2011

I Relation between Energy & Eccentricity

II Unbound orbits

III Summary of Kepler orbits

IV Orbital transfer

Put effective potential and ellipsoidal orbit on upper board.

How does the geometry of the orbit we studied last lecture connect with the energetics:

$$E = U_{eff}(r_{min}) = -\frac{\gamma}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2}$$
$$= \frac{1}{2r_{min}}\left(\frac{\ell^2}{\mu r_{min}} - 2\gamma\right)$$

Last time we found $r = \frac{c}{1+\epsilon\cos\phi}$, so,

$$r_{min} = \frac{c}{1+\epsilon} = \frac{\ell^2}{\gamma\mu(1+\epsilon)}$$

Putting this into the equation for E we have,

$$E = \frac{\gamma\mu(1+\epsilon)}{2\ell^2}\left(\gamma(1+\epsilon) - 2\gamma\right)$$
$$= \frac{\gamma^2\mu}{2\ell^2}(1+\epsilon)(1+\epsilon-2)$$
$$= \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2-1)$$

Note that the prefactor $\gamma^2\mu/2\ell^2$ is positive. So this formula explicitly exhibits that for $\epsilon < 1$, the energy $E<0$ and the orbit is bound. While for $\epsilon \geq 1$, $E \geq 0$ and the orbit is unbound.

II. Assume $\epsilon \geq 1$ then ϕ_{max} is determined by

$$1+\epsilon\cos\phi_{max} = 0 \Rightarrow \epsilon\cos\phi_{max} = -1$$

and $r(\phi) \to \infty$ as $\phi \to \pm\phi_{max}$. These orbits are generally hyperbolae (with a parabolic orbit when $\epsilon = 1$). The demonstration of these claims, namely that

$$\frac{(x-\delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$$

is very similar to your HW problem (Taylor 8.16) and I've spared you the demonstration but check it if you're up for it.

III. Summary

Orbit Eq: $r(\phi) = \dfrac{c}{1+\epsilon\cos\phi}$

w/ $c = \dfrac{\ell^2}{\gamma\mu}$ $\quad \gamma = G m_1 m_2$

$$\mu = \frac{m_1 m_2}{(m_1+m_2)}$$

Energy/eccentricity: $E = \dfrac{\gamma^2\mu}{2\ell^2}(\epsilon^2-1)$

$= \gamma/2c\,(\epsilon^2-1)$

Eccen.	Energy	Orbit
$\epsilon = 0$	$E = -\frac{\gamma^2\mu}{2\ell^2} < 0$	circle
$0 < \epsilon < 1$	$E < 0$	ellipse
$\epsilon = 1$	$E = 0$	parabola
$\epsilon > 1$	$E > 0$	hyperbola

IV Orbital Transfer

A common problem for satellite engineers is the transfer of a satellite from one orbit to another.

The same analysis as before yields

$$r(\phi) = \frac{c}{1+\epsilon\cos(\phi-\delta)}$$

although closest approach to earth is called ~~a~~ perigee and the most distant pt of the orbit is apogee.

We retain δ now because we can't, in general, align the x-axis with both perigees in an orbit transfer.

The idea is that the satellite rockets give a brief strong impulse to the satellite that changes its orbit. Call this a thrust.

We assume we know the change in velocity due to this thrust from which we can find $E_1 \to E_2$ and $\ell_1 \to \ell_2$.

From l_1, l_2 we find

$$c_1 = l_1^2/\gamma\mu \quad c_2 = l_2^2/\gamma\mu$$

and from $E = \gamma/2c\,(\epsilon^2-1)$ we find ϵ_1 and ϵ_2. Finally, assuming the thrust occured at precisely ϕ_0 (an approximation) we equate the two orbit equations at this point to find δ_2 from δ_1,

$$\frac{c_1}{1+\epsilon_1\cos(\phi_0-\delta_1)} = \frac{c_2}{1+\epsilon_2\cos(\phi_0-\delta_2)}$$

Let λ, the "thrust factor", be s.t.

$$v_2 = \lambda v_1$$

($\lambda > 1$ speed up ; $\lambda < 1$ slow down).

At perigee $v = r\dot{\phi}$ and $l = \mu r v$ then, assuming the thrust doesn't significantly change the satellite mass,

$$l_2 = \lambda l_1$$

and

$$c_2 = \lambda^2 c_1$$

That's the whole story but there's lots of algebra and instead all the problems tend to focus on simpler versions.

<u>Tangential thrust at perigee</u>:

Choose thrust s.t. $\phi = 0 \Rightarrow \phi_0 = 0$

$\delta_1 = 0$, $\delta_2 = 0$:

$$c_1/(1+\epsilon_1) = \frac{c_2}{(1+\epsilon_2)}$$

$$\Rightarrow \frac{c_1}{1+\epsilon_1} = \frac{\lambda^2 c_1}{(1+\epsilon_2)}$$

$$\Rightarrow \epsilon_2 = \lambda^2(1+\epsilon_1) - 1$$
$$= \lambda^2\epsilon_1 + (\lambda^2-1)$$

$\lambda > 1 \Rightarrow \epsilon_2 > \epsilon_1$, orbit is more eccentric until escape.

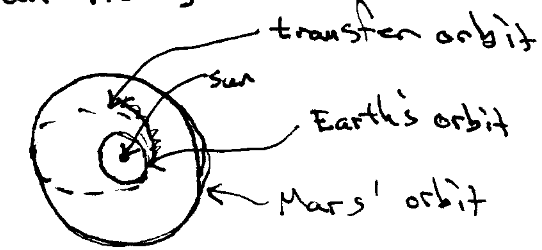
$\lambda < 1 \Rightarrow \epsilon_2 < \epsilon_1$ less eccentric $\rightarrow$ circular $\rightarrow$ perigee and apogee switch!

Fun example: Most efficient way to get to Mars: "the Hohmann transfer".

Both the Earth and Mars have roughly circular orbits:

	Eccentricity	Orbit radius
Earth	$\epsilon_e = 0.0167$	$R_e = 1.5 \times 10^{11}\,M$
Mars	$\epsilon_M = 0.0933$	$R_M = 2.3 \times 10^{11}\,M$

The Hohmann transfer involves two thrusts, the first takes you from the circular (earth) orbit to an elliptical orbit that intersects the desired orbit (Mars in our case) and the second which brings you onto the target orbit. (Note: I'm speaking about orbits about the sun now)



Take $\epsilon_e \approx \epsilon_M \approx 0$. Denote the transfer orbit with t subscripts. Then

$$c_1 = c_e = R_e$$

and $c_t = \lambda^2 R_e$ $\quad \epsilon_t = \lambda^2(0) + (\lambda^2 - 1)$
$$= (\lambda^2 - 1)$$

Now R_M must be ~~apogee~~ aphelion of transfer so,

~~apogee~~ aphelion of transfer orbit

$$R_M = \frac{c_t}{1-\epsilon_t} = \frac{\lambda^2 R_e}{1-(\lambda^2-1)} = \frac{\lambda^2 R_e}{2-\lambda^2}$$

$$\Rightarrow 2R_M - \lambda^2 R_M = \lambda^2 R_e \Rightarrow \lambda^2 (R_e + R_M) = 2R_M$$

$$\Rightarrow \lambda = \sqrt{\frac{2R_M}{R_e + R_M}} = \sqrt{\frac{4.6}{3.8}} = 1.10$$

Need a 10% increase in speed!

I leave to you to find λ_2, the thrust factor needed to go into Mars' orbit.

Say a few words about the exam.