Lecture 17

October 7th, 2011

Today's Outline:

I. Last Lecture

II. Quantitative Treatment of tides

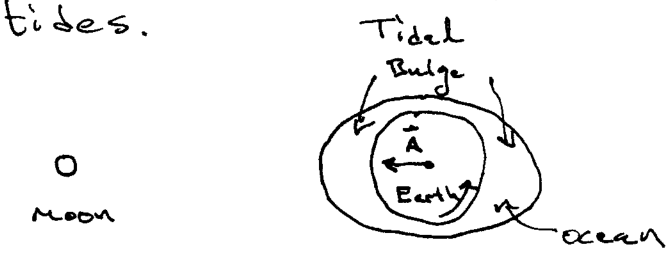
I. Last Lecture

(Completed discussion of central forces.)

In a linearly accelerated frame we found the modified Newton's law (acc. $\vec{A}$)

$$m\ddot{\vec{r}} = \vec{F} - m\vec{A}$$

We began discussion of tides: the earth is an accelerated frame (due to moon's pull) and this combined with gravitational pull of the moon is the primary cause of the tides.

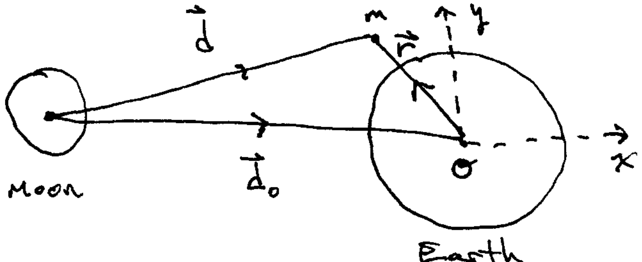


The acceleration $\vec{A}$'s magnitude and direction are determined by treating the Earth and moon as point masses concentrated at their respective centers

The gravitational pull of the moon is greater on the side closer to the moon and weaker on the opposite side. This, ~~is~~ along w/ Earth's accel, is why there are two high tides each day.

II Quantitative Treatment of Tides

leave up



The forces on m: (1) Earth's gravity: $m\vec{g}$

(2) Moon's gravity: $-GM_m m\,\hat{d}/d^2$ (M_m = moon's mass)

(3) Net non-gravitational Force: $\vec{F}_{ng}$

And the Earth is a noninertial frame with acceleration,

$$\vec{A} = -\frac{GM_m}{d_0^2}\hat{d}_0$$

We can use our noninertial frame law

$$m\ddot{\vec{r}} = \vec{F} - m\vec{A}$$

To find the height of the tides we'll leverage a nice argument: the tidal bulge is an equipotential surface.

Mathematical prerequisite: ~~A level~~ ~~set of f is a set on which f takes a constant value~~

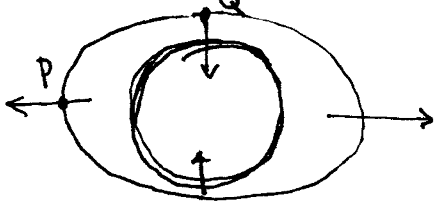
Level set of f: $f = \text{const.}$

If $g = g(x)$ then $dg = \frac{dg}{dx}dx$,

$$\Rightarrow m\ddot{\vec{r}} = \left(m\vec{g} - GM_m m\frac{\hat{d}}{d^2} + \vec{F}_{ng}\right) + GM_m m\frac{\hat{d}_0}{d_0^2}$$

$$\Rightarrow m\ddot{\vec{r}} = m\vec{g} \underbrace{- GM_m m\left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2}\right)}_{\equiv \vec{F}_{tid}} + \vec{F}_{ng}$$

The direction of $\vec{F}_{tid}$ is interplay of $\hat{d}_0$ and $\hat{d}$, d^2 and d_0^2:



while if $f = f(x,y,z)$ then

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$
$$= \vec{\nabla} f \cdot d\vec{r}$$

Now consider a level set

$$f(x,y,z) = c \quad (\text{a const.})$$

If we choose a displacement within this surface then $d\vec{r}$ is tangent to $f = c$ and, certainly, $df = 0$. Then,

$$df = 0 = \vec{\nabla} f \cdot d\vec{r}$$

This means that $\vec{\nabla} f$ is normal (i.e. perpendicular) to $f=c$. Example:

$$f(x,y,z) = x^2+y^2+z^2, \quad \vec{\nabla} f = 2(x,y,z) \propto \hat{r} \quad \checkmark.$$

An equipotential is just a level set of the potential energy function. We're going to argue that $\vec{\nabla} U$ is normal to the tidal bulge and hence that the tidal bulge is an equipotential. (Note: reversal of mathematical argument just given.)

Consider a water droplet, it is subject to (in our Earth frame): $m\vec{g}$, $\vec{F}_{tid}$, and $\vec{F}_p$, the pressure of the surrounding water. Water can't exert a shear force $\Rightarrow$ $\vec{F}_p$ is normal to surface of water. But our droplet is in equilibrium and so

$$m\vec{g} + \vec{F}_{tid}$$

is also normal to the water surface.

Write in terms of potentials:

$$m\vec{g} = -\vec{\nabla} \underset{\substack{\uparrow \\ \text{potential of} \\ \text{Earth's grav. pull}}}{U_{eg}} \qquad \vec{F}_{tid} = -\vec{\nabla} U_{tid}$$

$$U_{tid} = -GM_m m \left(\frac{1}{d} + \frac{x}{d_0^2} \right)$$

and

$$U = U_{eg} + U_{tid}$$

Then $m\vec{g} + \vec{F}_{tid} = -\vec{\nabla} U$ is normal to the surface of the water and indeed it must be an equipotential.

This means that

$$U(Q) = U(P)$$

$$\Rightarrow \quad U_{tid}(Q) - U_{tid}(P) = U_{eg}(P) - U_{eg}(Q)$$

$$= mgh$$

where h is the height of the tidal bulge. What about $U_{tid}(Q)$ and $U_{tid}(P)$?

For Q: $d = \sqrt{d_o^2 + r^2}$ and $r \approx R_e$, the radius of the Earth. So

$$U_{tid}(Q) = -GM_m M \left(\frac{1}{\sqrt{d_o^2 + R_e^2}} + 0 \right)$$

$$= -\frac{GM_m M}{d_o} \left(1 + R_e^2 / d_o^2 \right)^{-1/2}$$

$$\approx -\frac{GM_m M}{d_o} \left(1 - \frac{R_e^2}{2 d_o^2} \right)$$

For P: $d \approx d_o - R_e$ $x \approx -R_e$

and

$$U_{tid}(P) = -GM_m M \left(\frac{1}{d_o - R_e} - \frac{R_e}{d_o^2} \right)$$

$$= -\frac{GM_m M}{d_o} \left(\frac{1}{1 - R_e/d_o} - \frac{R_e}{d_o} \right)$$

$$\approx -\frac{GM_m M}{d_o} \left(1 + \frac{R_e}{d_o} + \frac{R_e^2}{d_o^2} - \frac{R_e}{d_o} \right)$$

$$= -\frac{GM_m M}{d_o} \left(1 + \frac{R_e^2}{d_o^2} \right)$$

Putting it together

$$-\frac{GM_m M}{d_o} \left(1 - \frac{R_e^2}{2 d_o^2} \right) + \frac{GM_m M}{d_o} \left(1 + \frac{R_e^2}{d_o^2} \right)$$

$$= \frac{3 G M_m m R_e^2}{2 d_o^3} = m g h$$

Now, $g = G M_e / R_e^2$ and so

$$\frac{3 \cancel{G} M_m \cancel{m} R_e^2}{2 d_o^3} = \frac{G M_e}{R_e^2} h$$

$$\Rightarrow h = \frac{3 M_m R_e^4}{2 M_e d_o^3}$$

$\Rightarrow$ $h = 54$ cm [moon alone]

Sun also contributes, exact same analysis with sun's mass and distances yields

$h = 25$ cm [sun alone].