Today's Outline

I. Last Lecture

II. ~~Wrap up~~ Recall Standard guess
(2D Oscillations moved to section)

III Damped Oscillations

IV Damped oscillation Regions

II. SHO

$$m\ddot{x} = -kx$$

$$\dot{x} = re^{rt} \qquad \ddot{x} = r^2 e^{rt}$$

so,

$$r^2 m \cancel{e^{rt}} = -k \cancel{e^{rt}}$$

$$\Rightarrow \quad r = \pm i\sqrt{\frac{k}{m}} \equiv \pm i\omega_0$$

Solution is a linear combination (superposition):

I. Oscillations around stable equilibria are ubiquitous.

$$k = U''(x_{eq})$$

and we began to investigate the standard guess

$$x(t) = e^{rt}$$

← switch to book's notation (I used λ).

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

Euler's exquisite Creation:

$$e^{i\pi} + 1 = 0$$

Generally,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Amazing secret:

$$e^{i\theta} = c\theta + is\theta$$

Use this to derive all of trigonometry!

Example:

$$e^{i2\theta} = C2\theta + iS2\theta$$

$$= (e^{i\theta})^2 = (C\theta + iS\theta)^2$$

$$= C^2\theta - S^2\theta + 2iC\theta S\theta$$

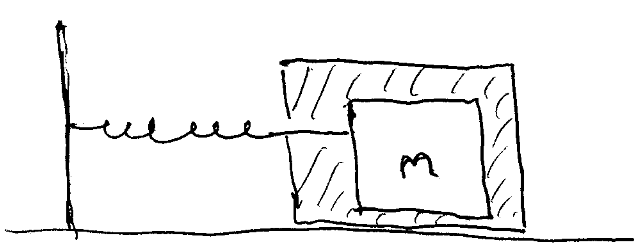
Real parts equal implies

$$C2\theta = C^2\theta - S^2\theta = 2C^2\theta - 1 = \cdots$$

III Damped Harmonic Oscillations

Hooke's law: $F_{spring} = -kx$

Immerse the mass in a tank of viscous medium.



Challenge: For the duration of this course never look up a trig formula.

Challenge 2: Figure out four distinct ways to write the general solution to

$$m\ddot{x} = -kx$$

using complex numbers (solution in your text)

Assume force is proportional to velocity and opposite in direction

$$f_{viscous} = -bv$$

(e.g. neglecting turbulence)

Newton's ~~second~~ 2nd law:

$$F_{net} = F_{spring} + f_{viscous} = ma = m\ddot{x}.$$

So,

$$m\ddot{x} = -kx - bv$$

or

$$m\ddot{x} + b\dot{x} + kx = 0$$

or

$$\ddot{x} + \frac{b}{m}\dot{x} + \omega_0^2 x = 0,$$

recall $\omega_0 = \sqrt{k/m}$.

Introduce shorthand $\beta = \frac{b}{2m}$,

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

Solve for $x(t)$? Use standard guess, $x = e^{rt}$.

$$\dot{x} = r e^{rt} \qquad \ddot{x} = r^2 e^{rt}$$

$$r^2 e^{rt} + 2\beta r e^{rt} + \omega_0^2 e^{rt} = 0$$

$$\Rightarrow r^2 + 2\beta r + \omega_0^2 = 0$$

$$\Rightarrow r = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega_0^2}}{2}$$

$$= -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

Call these

$$r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2} \qquad r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$$

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Important differences in this solution depending on regime of the parameters β, ω_0.

IV. Regimes

Three Regimes:

(1) Overdamped: $\beta > \omega_0$.

Then $B \equiv \sqrt{\beta^2 - \omega_0^2}$ is real and the general solution is

$$x(t) = C_1 e^{(-\beta + B)t} + C_2 e^{(-\beta - B)t}.$$

Both terms are declining exponentials (note: $\beta > B > 0$).

[Aside: For exponentials the "characteristic time", τ, is the time it takes to get to $1/e$ of its original value]

In the overdamped case there are two characteristic times

$$\tau_1 = \frac{1}{-\beta + B} \; ; \quad \tau_2 = \frac{1}{\beta + B}.$$

In fact, for <u>Huge</u> damping $\beta \approx B$ and $\tau_1 \to \infty$

(2) <u>Underdamped</u>: $\beta < \omega_0$. Then

$$\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2} \quad \text{is real}$$

$$r_{1,2} = -\beta \pm i\omega_1$$

and the general solution is

$$x(t) = C_1 e^{(-\beta + i\omega_1)t} + C_2 e^{(-\beta - i\omega_1)t}$$

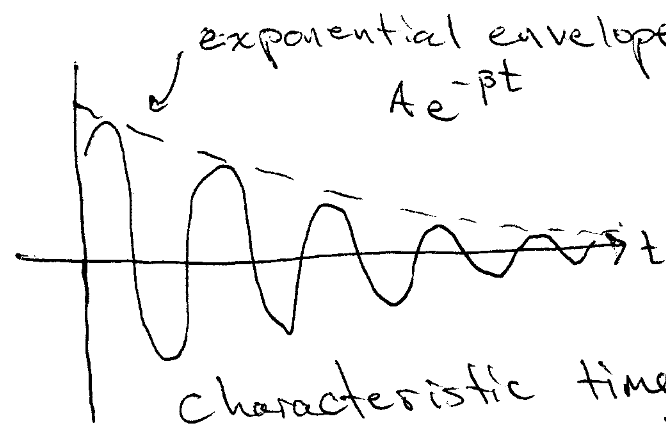
$$= e^{-\beta}\left(C_1 [\cos\omega_1 t + i \sin\omega_1 t] + C_2 [\cos\omega_1 t - i \sin\omega_1 t] \right)$$

$$= e^{-bt}\left(D_1 \cos\omega_1 t + D_2 \sin\omega_1 t \right)$$

with $D_1 = C_1 + C_2$, $D_2 = i(C_1 - C_2)$.

$$\boxed{x(t) = A e^{-\beta t} \cos(\omega_1 t - \delta)}$$

Sinusoidal oscillations with exponentially decreasing amplitude $A e^{-\beta t}$.



Characteristic time for damping $\tau = 1/\beta$.

Two times at work here: damping $\tau = 1/\beta$; and the period of the oscillations

$$T = \frac{2\pi}{\omega_1}$$

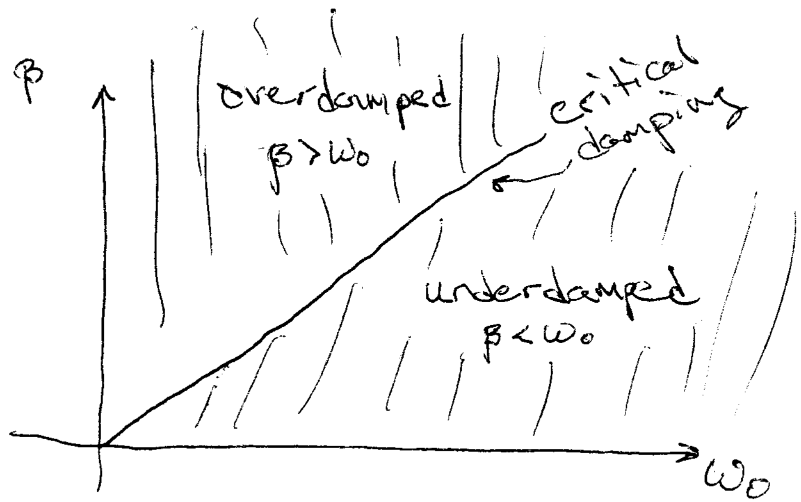
Question: How many oscillations in one characteristic time?

$$\# = \frac{\tau}{T} = \frac{1}{\beta}\cdot\frac{\omega_1}{2\pi} = \frac{1}{\pi}\left(\frac{\omega_1}{2\beta}\right)$$

Quality factor $\equiv Q \equiv \omega_1/2\beta$

"How many times does it ring?"

IF there's time:



(3) **Critical Damping:**

$\beta = \omega_0 \Rightarrow r_1 = r_2$

General solution is now,

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t}$$

$$= e^{-\beta t}(C_1 + C_2 t)$$

again $\tau = 1/\beta$

The derivatives in our damped oscillator EOM can be collected into a **differential operator:**

$$D \equiv \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$$

$$Dx = \ddot{x} + 2\beta\dot{x} + \omega_0^2 x$$

This operator is linear

$D(ax) = aDx \qquad D(x_1 + x_2) = Dx_1 + Dx_2$ ← superposition

Linear operators with constant coefficients are the ones for which the standard guess works. Show it!