Lecture 20

October 14th, 2011

Today's Outline

I. Last Lecture

II. Free-fall with $\vec{F}_{cor}$

III. Foucault's pendulum

I. Last Lecture

We carefully worked out:

- the direction of the Centrifugal force
- the impact of $\vec{F}_{cf}$ on $\vec{g}$.
- The direction of the coriolis force (in both the Northern and the Southern hemispheres).

II Free-fall again

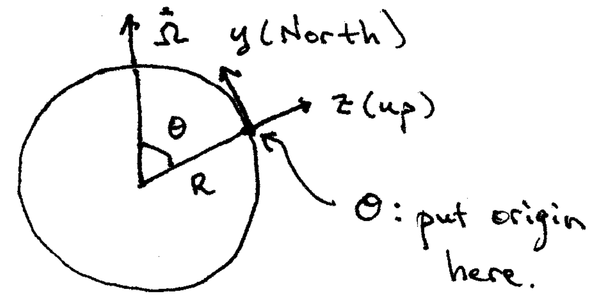
$$m\ddot{\vec{r}} = m\vec{g}_0 + \vec{F}_{cf} + \vec{F}_{cor}$$

$$= m\vec{g} + 2m\dot{\vec{r}} \times \vec{\Omega}$$

$$\Rightarrow \ddot{\vec{r}} = \vec{g} + 2\dot{\vec{r}} \times \vec{\Omega}$$

Notice that this equation only depends on $\ddot{\vec{r}}$ and $\dot{\vec{r}}$ $\Rightarrow$ we can arbitrarily shift our origin (see figure):

Cross-section of Earth:



In these coord.s

$$\dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z}) \quad \text{and}$$

$$\vec{\Omega} = (0, \Omega \sin\theta, \Omega\cos\theta), \text{ so that}$$

$$\dot{\vec{r}} \times \vec{\Omega} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & \Omega s\theta & \Omega c\theta \end{vmatrix} = (\dot{y}\Omega c\theta - \dot{z}\Omega s\theta, \ -\dot{x}\Omega c\theta, \ \dot{x}\Omega s\theta)$$

The E.O.M. is then

$$\ddot{x} = 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta)$$

$$\ddot{y} = -2\Omega\dot{x}\cos\theta$$

$$\ddot{z} = -g + 2\Omega\dot{x}\sin\theta$$

which is a set of coupled differential equations. Not easy to solve, so we will make a series of approximations:

First $\Omega \ll 1$, so let's take $\Omega \approx 0$

$$\Rightarrow \ddot{x} = 0 \quad \ddot{y} = 0 \quad \text{and} \quad \ddot{z} = -g$$

with solution,

$$x = 0 \quad y = 0 \quad z = h - \tfrac{1}{2}gt^2$$

assuming $z(0) = h$, $\dot{z}(0) = 0$. This is called the zeroth ~~get next~~ order approximation because it only has $(\Omega)^0 = 1$ in it. To get the first order we put the zeroth order R.H.S of the back into the E.O.M. to find

$$\ddot{x} = 2\Omega g t\sin\theta, \quad \ddot{y} = 0, \quad \ddot{z} = -g$$

$$\Rightarrow \dot{x}(t) = \Omega g t^2\sin\theta + \cancel{v_0}^{\,0} \Rightarrow x(t) = \tfrac{1}{3}\Omega g t^3\sin\theta + \cancel{x_0}^{\,0}$$

First order is then

$$x(t) = \tfrac{1}{3}\Omega g t^3\sin\theta \quad y = 0 \quad z = h - \tfrac{1}{2}gt^2$$

This reasonably has only $\Omega^0 = 1$ and $\Omega^1 = \Omega$ in it. We could continue in this manner to get as many powers of Ω as we wanted. How big is this effect?

Drop a pebble down a 100 meter mine shaft at the equator and we have

$$z = h - \tfrac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2h}{g}}$$

while

$$x = \tfrac{1}{3}\Omega g\left(\frac{2h}{g}\right)^{3/2} \approx 2.2\,\text{cm},$$

generally a small effect.

This example illustrates:

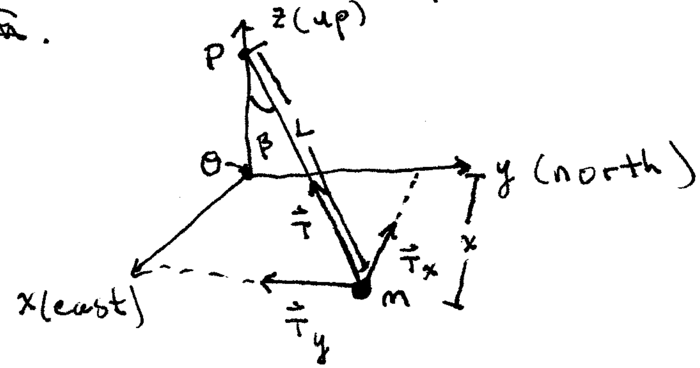
- how to calculate cross-products (a reminder)

and

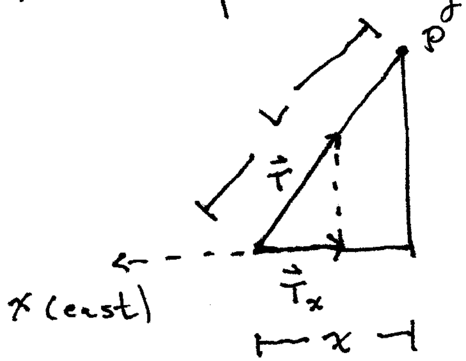
- One approach to solving coupled differential equations.

III Foucault's pendulum

Foucault's pendulum is a spherical pendulum (like your HW problem) with a massive bob and a long wire. The pendulum is suspended from a pivot P. ~~fixed to the Earth~~.



Now lets look at our figure in the plane spanned by $\vec{T}_x$ and $\vec{T}$:



By similar triangles we have

$$\frac{T_x}{T} = -\frac{x}{L}$$

Minus sign means T_x points in neg. x - direction.

The E.O.M. for the pendulum is

$$m\ddot{\vec{r}} = \vec{T} + m\vec{g}_o + \vec{F}_{cf} + 2m\dot{\vec{r}}\times\vec{\Omega}$$

$$= \vec{T} + m\vec{g} + 2m\dot{\vec{r}}\times\vec{\Omega}$$

Consider case where β is small so that

$$T_z = T\cos\beta \approx T$$

For small β we also have $\ddot{z}$ and $\dot{z}$ small and the z-component of the E.O.M. becomes,

$$0 = T_z - mg \Rightarrow T_z \approx T \approx mg.$$

Similarly $T_y = -T\,y/L = -mgy/L$

Putting it together our E.O.M are

$$\ddot{x} = -gx/L + 2\dot{y}\,\Omega\cos\theta$$

$$\ddot{y} = -gy/L - 2\dot{x}\,\Omega\cos\theta$$

Noting that $g/L = \omega_o$ and $\Omega\cos\theta = \Omega_z$ we have

$$\ddot{x} - 2\Omega_z\dot{y} + \omega_o^2 x = 0$$

$$\ddot{y} + 2\Omega_z\dot{x} + \omega_o^2 y = 0$$

Another set of coupled equations!

These are almost harmonic oscillator equations. We'll use another new technique to solve them. Let

$$\eta = x + iy$$

and multiply the $\ddot{y}$ equation by $i = \sqrt{-1}$ and add it to the $\ddot{x}$ equation, to find

$$\ddot{\eta} + 2i\Omega_z \dot{\eta} + \omega_0^2 \eta = 0$$

This is a 2^{nd} order, linear, homogeneous diff. eq. We can go back to our

Standard guess:

$$\eta(t) = e^{\lambda t}$$

(Note: notation differs from book but agrees with lecture 1.)

$$\Rightarrow \lambda^2 + 2i\Omega_z \lambda + \omega_0^2 = 0$$

$$\Rightarrow \lambda = \frac{-2i\Omega_z \pm \sqrt{-4\Omega_z^2 - 4\omega_0^2}}{2}$$

$$= -i\left(\Omega_z \mp \sqrt{\Omega_z^2 + \omega_0^2}\right)$$

(so book's α = our $i\lambda$).

Recall $\Omega_z \ll \omega_0$ and so

$$\lambda \approx -i\left(\Omega_z \mp \omega_0\right)$$

Our general solution is then

$$\eta(t) = c_1 e^{-i(\Omega_z - \omega_0)t} + c_2 e^{-i(\Omega_z + \omega_0)t}$$

$$= e^{-i\Omega_z t}\left[c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t}\right]$$

To set c_1 and c_2 we need initial conditions, let's choose: $x(0) = A$ $y(0) = 0$

$v_{x_0} = v_{y_0} = 0$ then $\eta(0) = A$ and $\dot{\eta}(0) = 0$ but

$$\eta(0) = c_1 + c_2$$

$$\dot{\eta}(0) = -i(\Omega_z - \omega_0)c_1 - i(\Omega_z + \omega_0)c_2$$

$$\approx i\omega_0 c_1 - i\omega_0 c_2$$

Then, $c_1 + c_2 = A$ $c_1 - c_2 = 0$

$$\Rightarrow c_1 = c_2 = A/2.$$

and

$$\eta(t) = x + iy = \underbrace{A\, \overbrace{e^{-i\Omega_z t}}^{\text{rotates plane of oscillation}} \cos(\omega_0 t)}_{\text{usual Pendulum motion}}$$