$$\vec{P} = M\dot{\vec{R}} \qquad \left(\begin{array}{l}\vec{R} = \text{CM position} \\ M = \text{total mass}\end{array}\right)$$

$$\dot{\vec{P}} = \vec{F}^{ext} = M\ddot{\vec{R}}$$

$$\vec{L} = \vec{R} \times \vec{P} + \sum_\alpha \vec{r}_\alpha{}' \times m_\alpha \dot{\vec{r}}_\alpha{}'$$

position relative to CM $(\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}')$

or

$$\vec{L} = \vec{L}(\text{motion of CM}) + \vec{L}\left(\begin{array}{c}\text{motion} \\ \text{relative to CM}\end{array}\right).$$

$$\tfrac{1}{2} M \dot{\vec{R}}^2 + \tfrac{1}{2} m_\alpha \dot{\vec{r}}_\alpha{}'^2$$

← Einstein summation

or

$$T = T(\text{motion of CM}) + T\left(\begin{array}{c}\text{motion} \\ \text{relative to CM}\end{array}\right)$$

or for a rigid body

$$T = T(\text{motion of CM}) + T\left(\begin{array}{c}\text{rotation} \\ \text{about } \cancel{\text{relative to}} \text{ CM}\end{array}\right)$$

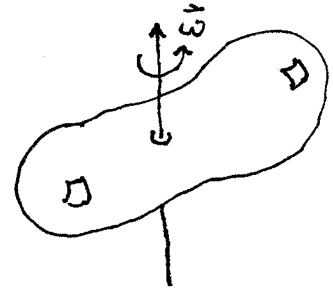
and

$$U = U^{ext} + U^{int}$$

↖ for rigid body w/ conservative internal forces, a constant that can be dropped.

III Why must we distinguish $\vec{\omega}$ and $\vec{L}$?

<u>Example</u>: Fixed axis of rotation.



"speared potato"

Choose fixed axis to be z axis:

$$\vec{\omega} = (0, 0, \omega)$$

We want to calculate

$$\vec{L} = \sum \vec{l}_\alpha = \sum \vec{r}_\alpha \times m_\alpha \vec{v}_\alpha$$

Well,

$$\vec{r}_\alpha = (x_\alpha, y_\alpha, z_\alpha)$$

and $\vec{v}_\alpha = (\vec{\omega} \times \vec{r}_\alpha)$ (recall $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$)

$$\vec{v}_\alpha = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x_\alpha & y_\alpha & z_\alpha \end{vmatrix} = (-y_\alpha \omega, x_\alpha \omega, 0)$$

$$l_\alpha = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_\alpha & y_\alpha & z_\alpha \\ -m_\alpha y_\alpha \omega & m_\alpha x_\alpha \omega & 0 \end{vmatrix} = m_\alpha \omega (-x_\alpha z_\alpha, -y_\alpha z_\alpha, x_\alpha^2 + y_\alpha^2) \text{ (no sum)}$$

Then

$$L_z = \sum m_\alpha (x_\alpha^2 + y_\alpha^2)\,\omega$$

$$= m_\alpha \rho_\alpha^2 \omega = I_z \omega$$

with $I_z = m_\alpha \rho_\alpha^2$. This may look familiar. But

$$L_x = -\sum \underbrace{m_\alpha x_\alpha z_\alpha}\,\omega$$

$$L_y = -\sum \underbrace{m_\alpha y_\alpha z_\alpha}\,\omega$$

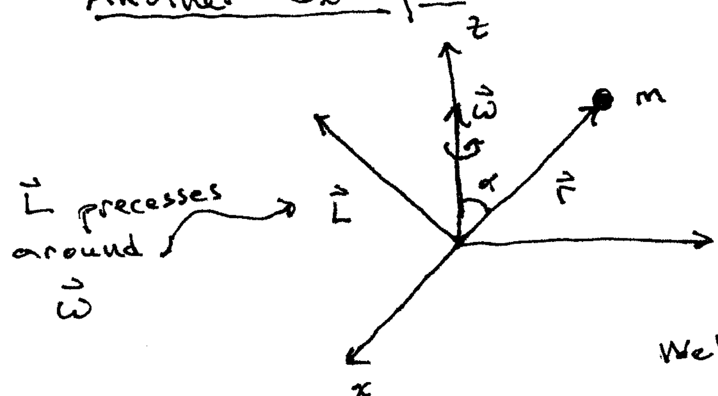
are new. These prefactors are called the "products of inertia"

So, why must we distinguish $\vec{\omega}$ and $\vec{L}$? [The answer is even richer than why we distinguish $\vec{v}$ and $\vec{p}$!] Unlike $\vec{v}$ and $\vec{p}$ they don't just differ by a scalar ($\vec{p} = m\vec{v}$) but they can even point in different directions (!!!):

$\vec{\omega}$ is not always parallel to $\vec{L}$.

Why? $\vec{L}$ depends on how the mass is distrubuted around the rotation axis, that is, on the body's shape.

<u>Another example</u>:



We'll return to this example.

We've done two examples, now let's do the general case.

IV Inertia Tensor

General case has

$$\vec{\omega} = (\omega_x, \omega_y, \omega_z)$$

and

$$\vec{L} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha$$

$$= \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$$

Recall (or use supplement on indices to show) the

Thus, after putting in $\sum_\alpha m_\alpha \cdot$, we have

non-diag contains "products of inertia"

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Diag contains "moments of inertia"

Reading off of (Eq $*$) we have

$$I_{xx} = \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2) \quad \text{etc.}$$

and

$$I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha \quad \text{etc.}$$

bac - cab rule:

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

So that,

$$\vec{r} \times (\vec{\omega} \times \vec{r}) = \vec{\omega}(r^2) - \vec{r}(\vec{r} \cdot \vec{\omega})$$

$$= \big(\omega_x r^2 - x(x\omega_x + y\omega_y + z\omega_z),\ \omega_y r^2 - y(x\omega_x + y\omega_y + z\omega_z),\ \omega_z r^2 - z(x\omega_x + y\omega_y + z\omega_z)\big)$$

$$= \big(\omega_x(y^2+z^2) - xy\omega_y - xz\omega_z,\ \omega_y(x^2+z^2) - yx\omega_x - yz\omega_z,\ \omega_z(x^2+y^2) - zx\omega_x - zy\omega_y\big) \qquad (\text{Eq } *)$$

Shorthand notation

$$L_i = I_{ij}\,\omega_j$$

← Einstein summation on j index

or

$$\vec{L} = \overleftrightarrow{I}\,\vec{\omega}$$

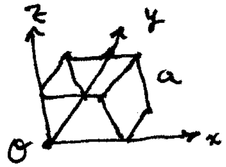
$\overleftrightarrow{I}$ is the moment of inertia or <u>inertia tensor</u>. For practical purposes its just a matrix. While we have it up, note that $I_{xy} = I_{yx}$ etc, or

$$I_{ij} = I_{ji}$$

or in Matrix notation

$$\overleftrightarrow{I} = \tilde{\overleftrightarrow{I}}$$ ← transpose = flip matrix along its main diagonal.

Example: Cube : continuous



$$\rho = M/a^3$$

$$I_{xx} = \int_0^a\int_0^a\int_0^a dx\,dy\,dz\;\rho\,(y^2+z^2)$$

$$= \rho\left(a^2\left(\frac{y^3}{3}\Big|_0^a\right) + a^2\left(\frac{z^3}{3}\Big|_0^a\right)\right)$$

$$= \frac{2}{3}\rho a^5 = \frac{2}{3} M a^2$$

$$I_{xy} = -\int_0^a\int_0^a\int_0^a \rho\, x\, y\; dx\,dy\,dz$$

$$= -\rho a\left(\left(\frac{x^2}{2}\Big|_0^a\right)\left(\frac{y^2}{2}\Big|_0^a\right)\right)$$

$$= -\frac{1}{4}\rho a^5 = -\frac{1}{4} M a^2$$

By symmetry we have

$$I = Ma^2 \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}.$$ [about Corner]

Next time: We'll see how

$$\overleftrightarrow{I} = \tilde{\overleftrightarrow{I}}$$

Can significantly simplify things.