Lecture 23
October 21st, 2011

Today's Outline:

I Last Lecture

II Example: Cube

III Class vote
(Remind Midterm 2 in one week)

IV Principal axes

V Calculating Principle Axes

I Last Lecture

We found that in general $\vec{\omega}$ and $\vec{L}$ are not even parallel. All three components of $\vec{\omega}$ contribute to each component of $\vec{L}$. This is summarized by

$$L_i = I_{ij}\,\omega_j$$

← Einstein summation on j

or

$$\vec{L} = \overleftrightarrow{I}\,\vec{\omega}$$

The inertia tensor is given by

$$I_{11} = I_{xx} = \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2)$$

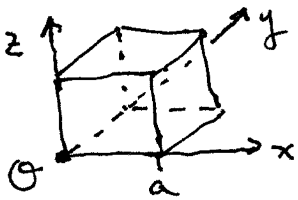
and similarly for I_{22}, I_{33}. While

$$I_{12} = I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha$$

and similarly for I_{ij} $(i \neq j)$.

II Example: Cube: Continuous

$$\rho = \frac{M}{a^3}$$



$$I_{xx} = \int_0^a\int_0^a\int_0^a dx\,dy\,dz\ \rho\,(y^2+z^2)$$

$$= \rho\left(a^2\left(\frac{y^3}{3}\Big|_0^a\right) + a^2\left(\frac{z^3}{3}\Big|_0^a\right)\right) = \frac{2}{3}\rho a^5 = \frac{2}{3}Ma^2.$$

$$I_{xy} = -\int_0^a\int_0^a\int_0^a \rho\, xy\,dx\,dy\,dz$$

$$= -\rho a\left(\left(\frac{x^2}{2}\Big|_0^a\right)\left(\frac{y^2}{2}\Big|_0^a\right)\right) = -\frac{1}{4}\rho a^5 = -\frac{1}{4}Ma^2.$$

By Symmetry,

$$\overleftrightarrow{I} = Ma^2\begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3}\end{pmatrix}.$$

[about corner]

III class vote:

Option 1: 3 Lect.s Ham. Theory, 3 Lect.s chaos & 3 Lect.s Cont. Mech.

Total: 34

Option 2: 4.5 Lect.s Ham. Theory & 4.5 Lect.s chaos

Total: 3

IV Principal Axes

If in general $\vec{L}$ is not parallel to $\vec{\omega}$, under what circumstances are they parallel? As an equation the condition is

$$\vec{L} = \lambda \vec{\omega}.$$

If this is satisfied we call the common axis of $\vec{L}$ and $\vec{\omega}$ a principle axis.

We know that

$$\vec{L} = \overleftrightarrow{I}\vec{\omega} = (I_{xz}\omega,\ I_{yz}\omega,\ I_{zz}\omega)$$

If we require $\vec{L} = \lambda\vec{\omega}$ then it must be that

$$I_{xz} = I_{yz} = 0$$

Thus for ~~z~~ the z-axis to be a princ. axis our Inertia tensor must have the form

$$\overleftrightarrow{I} = \cancel{\begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & I_{yz} \\ 0 & I_{yz} & I_{zz} \end{pmatrix}} = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{yx} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

Let's build some intuition by reverse ~~working~~ engineering ~~backwards~~. Assume the z-axis is a principle axis and take $\vec{\omega}$ to be about it

$$\vec{\omega} = (0, 0, \omega)$$

and $\lambda = I_{zz}$. Clearly if

called principle moments

$$\overleftrightarrow{I} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

then the x-, y and z-axes are all principle axes. In this case $\vec{\omega}$ and $\vec{L}$ are parallel for rotations about the coordinate axes and only for such rotations. (unless some principle moments are equal).

As is clear from the example of II $\overleftrightarrow{I}$ is a function ~~property~~ of both the body's mass distribution <u>and</u> the coordinates that we choose. Thus there's potential for us to simplify $\overleftrightarrow{I}$ by rotating our coordinates about O (note, we generally want to leave O fixed as it's natural to choose it to be the fixed pt of Euler's theorem).

<u>Mathematical Interlude</u>: Any symmetric real matrix $\overleftrightarrow{A}$ can be diagonalized by a rotation

$$\overleftrightarrow{A}' = \overleftrightarrow{R}\overleftrightarrow{A}\tilde{\overleftrightarrow{R}}$$

Diagonal form of $\overleftrightarrow{A}$.

See Taylor's Appendix for the proof.

This means that:

<u>Existance of Principal Axes</u>

For any rigid body and any pt O there are three perpendicular principal axes through O.

(Note: principle axes are fixed in the body — if we commit to these coordinates we're committing to working in a rotating frame)

V Calculating Principle Axes

$$\vec{L} = \overleftrightarrow{I}\vec{\omega}$$

Principle Axis: $\vec{L} = \lambda\vec{\omega}$

Combine to find

$$\overleftrightarrow{I}\vec{\omega} = \lambda\vec{\omega}$$

an eigenvalue / eigenvector equation! Rewrite,

$$(\overleftrightarrow{I} - \lambda\overleftrightarrow{1})\vec{\omega} = 0$$

↑ identity matrix $\overleftrightarrow{1}\vec{a} = \vec{a}$

For this to have non-trivial solutions we require,

$$\det(\overleftrightarrow{I} - \lambda\overleftrightarrow{1}) = 0$$

We solve this "characteristic" (or "secular") equation for the eigenvalues λ. Then put these into

$$\overleftrightarrow{I}\vec{\omega} = \lambda\vec{\omega}$$

to find the eigenvectors $\vec{\omega}$.

Cube example again

$$\overleftrightarrow{I} = \underbrace{\frac{Ma^2}{12}}_{\equiv I_0}\begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$$