Lecture 24

October 24th, 2011

Today's Outline:

I Last Lecture

II Cube Example again

III Precession due to weak torque

I. Last Lecture

We showed that a rigid body has three perpendicular principle axes about any origin $\mathcal{O}$.

Principle axis condition:

$$\vec{L} = \lambda \vec{\omega}$$

In general,

$$\vec{L} = \overleftrightarrow{I} \vec{\omega}$$

$$\Rightarrow \vec{L} = \overleftrightarrow{I} \vec{\omega} = \lambda \vec{\omega}.$$

To find eigenvalues (principle moments) of $\overleftrightarrow{I}$ solve:

$$\det(\overleftrightarrow{I} - \lambda \overleftrightarrow{1}) = 0$$

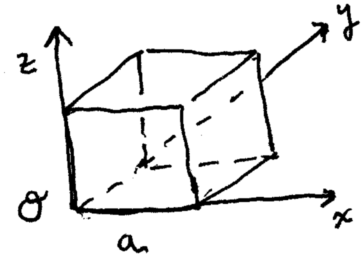
To find eigenvectors (principle axes) solve

$$\overleftrightarrow{I} \vec{\omega} = \lambda \vec{\omega}$$

now with given eigen value.

for $\vec{\omega}$.

II Cube Example again:



Recall that about $\mathcal{O}$ we have

$$\overleftrightarrow{I} = \underbrace{\frac{Ma^2}{12}}_{\equiv I_0} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$$

Then the secular equation is

$$\det(\overleftrightarrow{I} - \lambda \overleftrightarrow{1}) = \det \begin{pmatrix} 8I_0 - \lambda & -3I_0 & -3I_0 \\ -3I_0 & 8I_0 - \lambda & -3I_0 \\ -3I_0 & -3I_0 & 8I_0 - \lambda \end{pmatrix}$$

$$= 0$$

I'll skip the algebra of calculating this determinant, you find,

$$(2I_0 - \lambda)(11I_0 - \lambda)^2 = 0.$$

Then

$$\lambda_1 = 2I_0, \quad \lambda_2 = \lambda_3 = 11I_0$$

Now, for the eigenvectors; put λ_1 into

$$(\overleftrightarrow{I} - \lambda_1 \overleftrightarrow{1})\vec{\omega} = 0$$

to find

$$\begin{pmatrix} 6I_0 & -3I_0 & -3I_0 \\ -3I_0 & 6I_0 & -3I_0 \\ -3I_0 & -3I_0 & 6I_0 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

$$\Rightarrow \quad \begin{aligned} 2\omega_x - \omega_y - \omega_z &= 0 \quad ① \\ -\omega_x + 2\omega_y - \omega_z &= 0 \quad ② \\ -\omega_x - \omega_y + 2\omega_z &= 0 \quad ③ \end{aligned}$$

$① - ② \Rightarrow 3\omega_x - 3\omega_y = 0 \Rightarrow \omega_x = \omega_y$

$① \Rightarrow \omega_x - \omega_z = 0 \Rightarrow \omega_x = \omega_z$

So our first eigenvector (after normalizing) is

$$\vec{e}_1 = \frac{1}{\sqrt{3}}(1, 1, 1)$$

For λ_2 and λ_3 we have

$$-3I_0 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

Then

$$\omega_x + \omega_y + \omega_z = 0$$

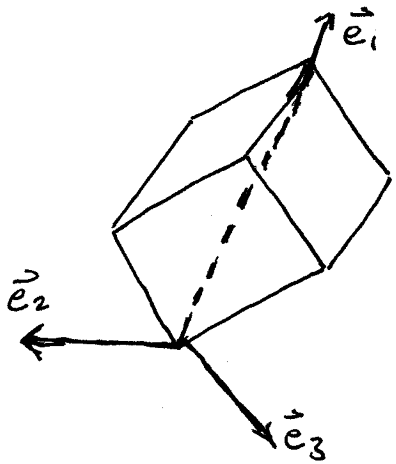
is the only equation. This does not uniquely determine $\vec{\omega}$ (one condition + normalization as a second condition is still only two conditions on three variables, $\omega_x, \omega_y, \omega_z$).

Instead it indicates

$$\vec{\omega} \cdot \vec{e}_1 = 0$$

So $\vec{e}_2$ and $\vec{e}_3$ are <u>any</u> two orthogonal vectors in the plane orthogonal to $\vec{e}_1$. (Note: This degeneracy is due to the symmetry of the cube.)

Pictorially,



From symmetry can guess the principle axes: $\vec{e}_3$ along top axis, $\vec{e}_1$ and $\vec{e}_2$ any two vectors orthogonal to $\vec{e}_3$ and each other.

For $\vec{\omega} = \omega \vec{e}_3$ we have

$$\vec{L} = \lambda_3 \vec{\omega} = \lambda_3 \omega \vec{e}_3 .$$

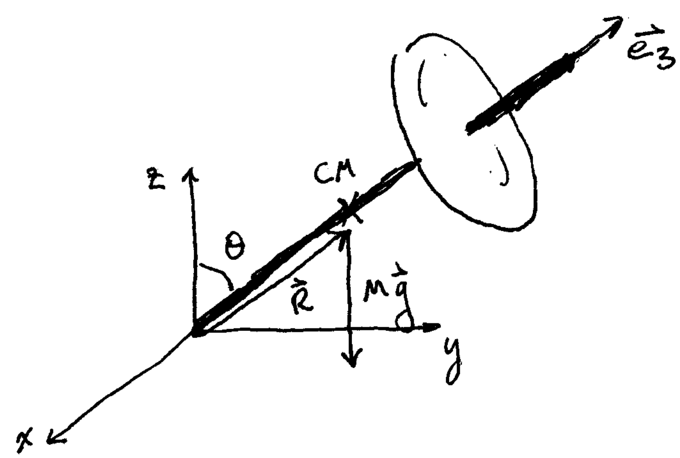
(Note:

$$I = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$$

in principle axis frame)

III Precession due to weak torque

Example: Top or gyroscope! Assume it is axis symmetric, simplified picture below.



If $g=0$ then there is no torque and this $\vec{L}$ is conserved. However for nonzero g we have a torque

$$\vec{\Gamma} = \vec{R} \times M\vec{g} = RMg \sin\theta \, \hat{\Gamma}$$

We'll assume this torque is small. Then

$$\vec{\Gamma} = \dot{\vec{L}}$$

implies that $\vec{L}$ begins to evolve and $\omega_1 = \vec{\omega} \cdot \vec{e}_1 \neq 0$ and $\omega_2 = \vec{\omega} \cdot \vec{e}_2 \neq 0$. For small torques these are small and we can take $\vec{L} \approx \lambda_3 \omega \vec{e}_3$ still.

Now, $\vec{\Gamma}$ is perpendicular to $\vec{e}_3$ and with $\vec{L} \approx \lambda_3 \omega \vec{e}_3$ it is also perp. to $\vec{L}$. <u>Hence $\vec{\Gamma}$ only changes the direction of $\vec{L}$</u> (with this approx.).

Specifically,

$$\dot{\vec{L}} = \vec{\Gamma}$$

$$\Rightarrow \quad \lambda_3 \omega \dot{\vec{e}}_3 = \vec{R} \times M\vec{g}$$

$$= (R\vec{e}_3) \times [Mg(-\hat{z})]$$

and

$$\dot{\vec{e}}_3 = \frac{MgR}{\lambda_3 \omega} \hat{z} \times \vec{e}_3$$

$$= \vec{\Omega} \times \vec{e}_3$$

with

$$\vec{\Omega} \equiv \frac{MgR}{\lambda_3 \omega} \hat{z}$$

But this is precisely our useful relation from Chapter 9 $\left(\frac{d\vec{e}}{dt} = \vec{\Omega} \times \vec{e}\right)$, and we know how to interpret it:

The axis of the top, $\vec{e}_3$, rotates with angular velocity $\vec{\Omega}$ about the z-axis.

<u>Conclusion</u>: The top precesses, that is it moves slowly around a cone with fixed opening angle θ at an angular frequency $\Omega = MgR / \lambda_3 \omega$

Fascinating example: The Earth spins about an axis inclined at 23° to the normal of the orbital plane. Because of the Earth's equatorial bulge the Sun and Moon exert weak torques on the Earth, and cause it to precess ~~this~~ (one complete turn in 26,000 years). This is called the <u>precession of the equinoxes</u>.