Lecture 25

October 26th, 2011

Today's outline:

I. Last Lecture

II. Demo of precession

III. Euler's Equations

IV Intermediate Axis Theorem

I Last Lecture

We found the principle moments and axes of a cube.

We also found that a symmetric top or gyro responds to a weak torque by precessing:

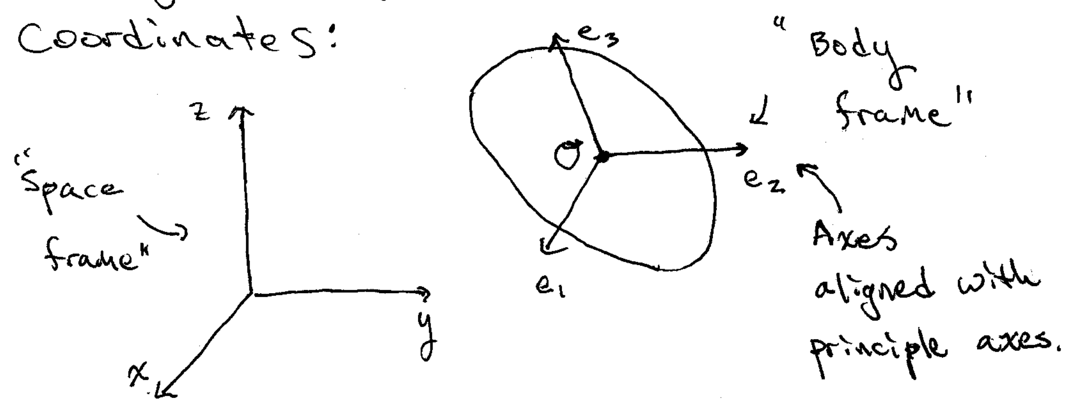
$$\dot{\vec{e}}_3 = \vec{\Omega} \times \vec{e}_3$$

with $\vec{\Omega} = \frac{MgR}{\lambda_3 \omega} \hat{z}$.

III Euler's Equations

Today we will derive Euler's Equations which are the analog of Newton's $m\vec{a} = \vec{F}$ for the rotational motion of a rigid body.

Coordinates:



The space frame is inertial, non-rotating and fixed in space. The body frame is noninertial, rotates with the body and fixed in the body. The picture is <u>deceptive</u> in one respect: we will generally choose O, the origin, at the CM or the fixed pivot of the body for <u>both</u> coordinate systems.

In the body frame $\vec{\omega} = \omega_1 \vec{e}_1 + \omega_2 \vec{e}_2 + \omega_3 \vec{e}_3$ and

$$\vec{L} = (\lambda_1 \omega_1, \lambda_2 \omega_2, \lambda_3 \omega_3) \quad [\text{body frame}]$$

Now if there's a torque $\vec{\Gamma}$ then

$$\left(\frac{d\vec{L}}{dt}\right)_{space} = \vec{\Gamma}$$

But then

$$\left(\frac{d\vec{L}}{dt}\right)_{space} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L}$$

And recall our convention that $(d\vec{L}/dt)_{body} \equiv \dot{\vec{L}}$, then

$$\dot{\vec{L}} + \vec{\omega} \times \vec{L} = \vec{\Gamma} \qquad [\text{body}]$$

These are <u>Euler's Equations</u>.

In components

$$\vec{\omega} \times \vec{L} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \lambda_1 \omega_1 & \lambda_2 \omega_2 & \lambda_3 \omega_3 \end{vmatrix} = \big(\omega_2 \omega_3 (\lambda_3 - \lambda_2), \; \omega_3 \omega_1 (\lambda_1 - \lambda_3), \; \omega_1 \omega_2 (\lambda_2 - \lambda_1)\big)$$

or

$$\lambda_1 \dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2 \omega_3 = \Gamma_1$$
$$\lambda_2 \dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_3 \omega_1 = \Gamma_2$$
$$\lambda_3 \dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1 \omega_2 = \Gamma_3$$

Euler's Eq.s [body]

<u>Example</u>: Spinning top from last time. The torque $\vec{\Gamma}$ was perp. to $\vec{e}_3$, so that $\Gamma_3 = 0$. Also $\lambda_1 = \lambda$, so:

$$\lambda_3 \dot{\omega}_3 = 0$$

in agreement with our earlier claims.

<u>Example</u>: $\vec{\Gamma} = 0$, ~~then~~ and $\lambda_1 \neq \lambda_2 \neq \lambda_3$. Then

$$\lambda_1 \dot{\omega}_1 = (\lambda_2 - \lambda_3)\omega_2 \omega_3$$
$$\lambda_2 \dot{\omega}_2 = (\lambda_3 - \lambda_1)\omega_3 \omega_1$$
$$\lambda_3 \dot{\omega}_3 = (\lambda_1 - \lambda_2)\omega_1 \omega_2$$

Suppose at $t = 0$, $\omega_1 = \omega_2 = 0$ $\omega_3 \neq 0$ then $\dot{\omega}_i = 0$ in the body frame and ω_i = const. But for $\vec{\omega} = \omega_3 \vec{e}_3$

$$\vec{L} = \lambda_3 \vec{\omega}$$

and $\vec{L}$ is constant (no torques) in any inertial frame, so:

Initial spinning about principle axis

$\Rightarrow$ Constant spinning about that axis (direction and rate).

Suppose at $t=0$ any two of $\omega_1, \omega_2, \omega_3$ are nonzero. Then some $\dot{\omega}_i \neq 0$ (e.g. $\omega_1 \neq 0$, $\omega_2 \neq 0$, $\dot{\omega}_3 \neq 0$) $\Rightarrow$ $\vec{\omega}$ is not constant.

Not initially spinning about principle axis

$\Rightarrow$ $\vec{\omega}$ is not constant, but evolves.

IV Int. Axis Theorem

[A.K.A Tennis Racket Theorem]

If the only way to get constant rotation is about a principle axis then how <u>stable</u> are these rotations?

The answer is a wonderful mixture of tools from this course.

Suppose initially $\omega_1 = \omega_2 = 0$, $\omega_3 \neq 0$ and at $t=0$ we give the system a small kick s.t. ω_1 and ω_2 are nonzero. Will their values continue to increase or do something else? (stability question rephrased)

First, if ω_1 and ω_2 are small then

$$\dot{\omega}_3 = \frac{1}{\lambda_3}(\lambda_1 - \lambda_2)\underbrace{\omega_1\omega_2}_{\text{very small}} \approx 0$$

and so

$$\lambda_1 \dot{\omega}_1 = \underbrace{[(\lambda_2 - \lambda_3)\omega_3]}_{\text{roughly constant}}\omega_2$$

$$\lambda_2 \dot{\omega}_2 = [(\lambda_3 - \lambda_1)\omega_3]\omega_1$$

Now

$$\lambda_1 \ddot{\omega}_1 \approx [(\lambda_2 - \lambda_3)\omega_3]\dot{\omega}_2$$

$$\Rightarrow \ddot{\omega}_1 \approx -\left[\frac{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)}{\lambda_1\lambda_2}\omega_3^2\right]\omega_1$$

Thus ω_1 oscillates if [...] is positive and exponentially grows if [...] is negative. Thus if λ_3 is the largest or smallest principle moment the rotation is stable!

While if λ_3 is intermediate between λ_1 and λ_2 then the rotation is unstable and oscillations about the other principle axes will develop!

We've been assuming $\lambda_1 \neq \lambda_2 \neq \lambda_3$, check out $\lambda_1 = \lambda_2 \neq \lambda_3$ in your book — precession arises even without torques!

Comment on the limitations of Euler's equations:

- Finding the torque in the body frame $(\Gamma_1, \Gamma_2, \Gamma_3)$ is in general difficult (Have to solve for the motion and its cause simultaneously).
- Like Newton's law's limitations the limitations of Euler's equations can be overcome by going to the Lagrangian Formalism (see textbook for intro). For example you get the very cool (and aesthetic) phenomenon of nutation. We'll forgoe this material in favor of special topics but do take a look.