Lecture 26

October 31st, 2011

Today's outline

I. Last words on rigid bodies

II Intro to coupled oscillations

I. Last words on rigid bodies

We derived Euler's Equations:

$$\begin{aligned} \lambda_1 \dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3 &= \Gamma_1 \\ \lambda_2 \dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_3\omega_1 &= \Gamma_2 \\ \lambda_3 \dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2 &= \Gamma_3 \end{aligned} \quad \begin{bmatrix}\text{body} \\ \text{frame}\end{bmatrix}$$

There is a big limitation to these equations:

- Finding the torque in the body frame $(\Gamma_1, \Gamma_2, \Gamma_3)$ is in general difficult (You have to solve for the motion and its cause simultaneously).

This is why we focused on the torque free case last lecture.

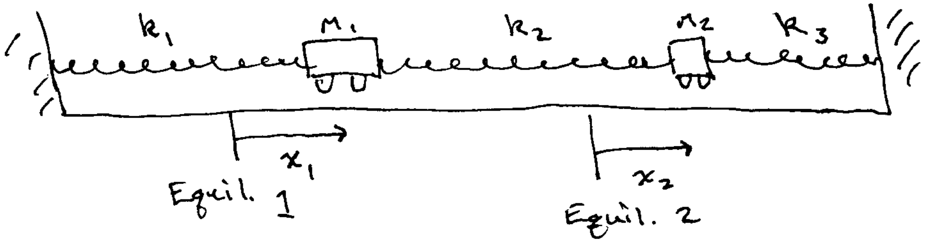
As with the limitations of Newton's laws, the limitations of Euler's equations can be overcome by going to the Lagrangian formalism. (see textbook for an intro). For example you can explain the very cool phenomenon of nutation that we saw in the gyroscope demo. We'll forgoe this material in favor of special topics but do take a look

II Intro to coupled oscillations

In the first lecture of the semester we saw that the potential surrounding a generic stable equilibrium gave rise to harmonic oscillations (in other words, SHO is everywhere).

In our next step into the real world we see that these oscillations are (almost) always coupled.

Great toy model:



F_1 = (net force on cart 1)

$$= -k_1 x_1 + k_2(x_2 - x_1)$$

$$= -(k_1+k_2)x_1 + k_2 x_2$$

F_2 = (net force on cart 2)

$$= -k_2(x_2 - x_1) - k_3 x_2$$

$$\Rightarrow \quad F_2 = -(k_2+k_3)x_2 + k_2 x_1$$

Then

$$m_1 \ddot{x}_1 = -(k_1+k_2)x_1 + k_2 x_2$$

$$m_2 \ddot{x}_2 = k_2 x_1 - (k_2+k_3)x_2$$

These are coupled and we will find that a good strategy is to collect them into a single matrix equation:

$$\overleftrightarrow{M}\ddot{\vec{x}} = -\overleftrightarrow{K}\vec{x}$$

with $\vec{x} = (x_1, x_2)$ and

$$\overleftrightarrow{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad \overleftrightarrow{K} = \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}$$

A matrix version of Hooke's law.

We make a reasonable physical guess: there should be solutions with both carts oscillating at the same frequency. Turns out to be true. Familiar differential eq.s and we pull out the "standard guess"

$$z_1(t) = \alpha_1 e^{i(\omega t - \delta_1)} = \alpha_1 e^{-i\delta_1} e^{i\omega t} = a_1 e^{i\omega t}$$

real constant standard guess complex constant

$$z_2(t) = \alpha_2 e^{i(\omega t - \delta_2)} = \alpha_2 e^{-i\delta_2} e^{i\omega t} = a_2 e^{i\omega t}$$

~~A~~ Notice that our standard guess builds in the same ω for both carts. As always these complex solutions are convenient mathematically but the physical solutions are $\mathrm{Re}(z_1)$ and $\mathrm{Re}(z_2)$.

We collect them into a vector,

$$\vec{z}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t} = \vec{a}\, e^{i\omega t}$$

Putting this into the matrix E.O.M., gives

$$\overleftrightarrow{M}\,\vec{a}\,(-\omega^2 \cancel{e^{i\omega t}}) = -\overleftrightarrow{K}\,\vec{a}\,\cancel{e^{i\omega t}}$$

$$\Rightarrow \boxed{(\overleftrightarrow{K} - \omega^2 \overleftrightarrow{M})\,\vec{a} = 0.}$$

A generalized eigenvalue equation

where $\overleftrightarrow{M}$ takes the role of the identity matrix and ω^2 is the eigenvalue we seek; here ω is called a "normal frequency" and the motion with both carts oscillating at this shared frequency is called a "normal mode".

Special case of toy example: The algebra is easier if we take $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. Then

$$\overleftrightarrow{M} = \mathrm{diag}(m, m) = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

and

$$\overleftrightarrow{K} = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}$$

with

$$(\overleftrightarrow{K} - \omega^2 \overleftrightarrow{M}) = \begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix}.$$

As always, to find a solution of

$$(\overleftrightarrow{K} - \omega^2 \overleftrightarrow{M})\,\vec{a} = 0$$

with $\vec{a} \neq 0$ we require that

$$\det\left(\overleftrightarrow{K} - \omega^2 \overleftrightarrow{M}\right) = 0$$

In our example this is,

$$(2k - m\omega^2)^2 - k^2 = 0$$

$$\Rightarrow \quad m^2\omega^4 - 4km\omega^2 + 3k^2 = 0$$

$$\Rightarrow \quad (k - m\omega^2)(3k - m\omega^2) = 0$$

$$\Rightarrow \quad \omega = \sqrt{\frac{k}{m}} \equiv \omega_1 \quad \text{or} \quad \omega = \sqrt{\frac{3k}{m}} \equiv \omega_2$$

which yields

$$\vec{z}(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega_1 t} = \begin{bmatrix} A \\ A \end{bmatrix} e^{i(\omega_1 t - \delta)}$$

Physical Solution is

$$\vec{x}(t) = \mathrm{Re}(\vec{z}(t)) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} A \\ A \end{pmatrix} \cos(\omega_1 t - \delta)$$

or

$$x_1(t) = A\cos(\omega_1 t - \delta)$$

$$x_2(t) = A\cos(\omega_1 t - \delta)$$

[sloshing mode]

Next we solve for the eigenvectors $\vec{a}$, to find the normal mode motion.

Normal Modes

$$(\overleftrightarrow{K} - \omega_1^2 \overleftrightarrow{M})\vec{a} = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

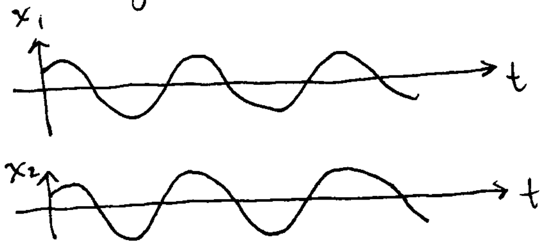
$$\Rightarrow \quad \left.\begin{matrix} a_1 - a_2 = 0 \\ -a_1 + a_2 = 0 \end{matrix}\right\} \Rightarrow \quad a_1 = a_2$$

Then we can take this single complex number to be,

$$a_1 = a_2 = A e^{-i\delta}$$

just the amp×phase form of any complex #.

In this mode the two carts move in the same direction with the same frequency and amplitude; they slosh.



Notice that k_2 is irrelevant and so the two carts only feel the left most and right most springs respectively: hence $\omega_1 = \sqrt{k/m}$ for both carts.