Lecture 27

November 2nd, 2011

Today's Outline

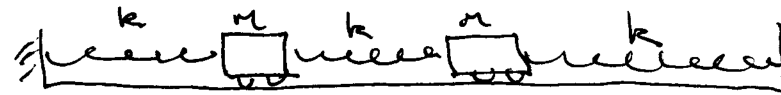
I. Last Lecture

II. Normal Modes for specialized toy model.

III Weak Coupling

I Last Lecture

Found E.O.M for our toy model and then specialized to



and found

$$(\overleftrightarrow{K} - \omega^2 \overleftrightarrow{M})\vec{a} = 0$$

with

$$\overleftrightarrow{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \quad \overleftrightarrow{K} = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}$$

and

$$\vec{Z}(t) = \vec{a}\, e^{i\omega t} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}$$

← complex constants

our trial solution. This led to the generalized eigenvalue equation

$$\det(\overleftrightarrow{K} - \omega^2 \overleftrightarrow{M}) = 0$$

and the two solutions

$$\omega_1 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{3k}{m}}$$

↑ "sloshing mode"

II Normal Modes for our Specialized toy model

$$(\overleftrightarrow{K} - \omega_1^2 \overleftrightarrow{M})\vec{a} = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} a_1 - a_2 &= 0 \\ -a_1 + a_2 &= 0 \end{aligned}\right\} \Rightarrow a_1 = a_2$$

This means the mode is determined by a single complex number,

$$a_1 = a_2 = A e^{-i\delta}$$

↑ ~~choose~~ call it

← just amp. & phase form of any complex #.

Now that we've set $a_1 = a_2 = Ae^{-i\delta}$ our trial solution becomes

$$\vec{z}(t) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega_1 t} = \begin{pmatrix} A \\ A \end{pmatrix} e^{i(\omega_1 t - \delta)}$$

and the physical solution is,

$$\vec{x}(t) = \mathrm{Re}(\vec{z}(t)) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} A \\ A \end{pmatrix} \cos(\omega_1 t - \delta)$$

or

$$x_1(t) = A\cos(\omega_1 t - \delta)$$
$$x_2(t) = A\cos(\omega_1 t - \delta)$$

[Sloshing mode]

At any given moment the carts have opposite amplitudes: we call this the "breathing mode".

All of this was based on the standard guess and the physical assumption that the carts oscillate at the same frequency.

These equations are linear and we can add our solutions to get:

This confirms our prediction about the sloshing and the irrelevance of the middle spring.

2nd mode: $\omega_2 = \sqrt{3k/m}$

$$(\overleftrightarrow{K} - \omega_2^2 \overleftrightarrow{M})\vec{a} = \begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$\Rightarrow a_1 + a_2 = 0 \quad \Rightarrow a_1 = -a_2 = Ae^{-i\delta}$$

$$\Rightarrow \vec{x}(t) = \begin{pmatrix} A \\ -A \end{pmatrix} \cos(\omega_2 t - \delta)$$

$$\vec{x}(t) = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2 t - \delta_2)$$

But wait, this solution has four arbitrary constants, $\Rightarrow$ It's the general solution! (two 2nd order linear ODES $\Rightarrow$ 4 arbitrary constants). As always we determine A_1, A_2, δ_1 and δ_2 from the initial conditions $x_1(0)$, $\dot{x}_1(0)$, $x_2(0)$, $\dot{x}_2(0)$.

Normal coordinates: When we began we found

$$m\ddot{x}_1 = -(2k)x_1 + kx_2$$
$$m\ddot{x}_2 = kx_1 - 2kx_2$$

Are there coordinates in which these two equations decouple? Yes, and they are called "normal coordinates":

$$\xi_1 = \tfrac{1}{2}(x_1 + x_2)$$
$$\xi_2 = \tfrac{1}{2}(x_1 - x_2).$$

In normal coordinates our two modes simplify

$$\left.\begin{aligned}\xi_1(t) &= A\cos(\omega_1 t - \delta)\\ \xi_2(t) &= 0\end{aligned}\right\}\ \text{sloshing}$$

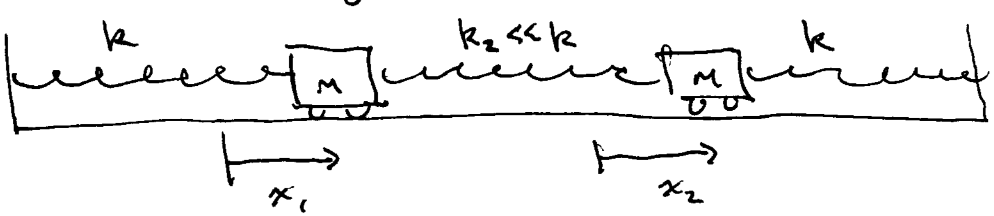
and

$$\left.\begin{aligned}\xi_1(t) &= 0\\ \xi_2(t) &= A\cos(\omega_2 t - \delta)\end{aligned}\right\}\ \text{breathing}$$

Again the general solution is a combination but now the frequencies don't mix.

III Weak Coupling

This is a different specialization of our original toy model:



We have $\overleftrightarrow{M} = \text{diag}(m, m)$ and

$$\overleftrightarrow{K} = \begin{pmatrix} k+k_2 & -k_2 \\ -k_2 & k+k_2 \end{pmatrix} \qquad (\overleftrightarrow{K} - \omega^2\overleftrightarrow{M}) = \begin{pmatrix} k+k_2-\omega^2 m & -k_2 \\ -k_2 & k+k_2-\omega^2 m \end{pmatrix}$$

Then

skip some algebra

$$\det(\overleftrightarrow{K} - \omega^2\overleftrightarrow{M}) = (k - m\omega^2)(k + 2k_2 - m\omega^2)$$

$$\Rightarrow \omega_1 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{k+2k_2}{m}}$$

You won't be surprised to hear that the modes are the same: sloshing and breathing. <u>But</u>, there's still some interesting new physics.

For weak coupling ω_2 is only slightly larger than ω_1 (k_2 is small). Let

$$\omega_0 = \frac{\omega_1 + \omega_2}{2}$$

and define

$$\omega_1 = \omega_0 - \epsilon \quad \text{and} \quad \omega_2 = \omega_0 + \epsilon$$

i.e. $\epsilon = \frac{\omega_2 - \omega_1}{2}$. Lets also pick initial conditions s.t. $A_1 = A_2 = A/2$ and $\delta_1 = \delta_2 = 0$

then

$$x_1(t) = A \cos(\epsilon t) \cos \omega_0 t$$

$$x_2(t) = A \sin(\epsilon t) \sin \omega_0 t$$

These are enveloped oscillations at frequency ω_0, with the envelope period $2\pi/\epsilon$. This is beating!

What's doing the beating? It's the two normal coordinates

$$\xi_1(t) = \tfrac{1}{2} A \cos[(\omega_0 - \epsilon)t] = \tfrac{1}{2} A \cos(\omega_1 t)$$

$$\xi_2(t) = \tfrac{1}{2} A \cos[(\omega_0 + \epsilon)t] = \tfrac{1}{2} A \cos(\omega_2 t).$$