Lecture 28

November 4^{th}, 2011

Today's Outline

I Last lecture

II Lagrangian approach: an example

I Last Lecture

- Found the two normal modes of our toy model: sloshing and breathing.
- Discovered that the general solution was a linear combination of the two normal modes.
- Discovered normal coordinates: ξ_1 and ξ_2.

We saw that for weak coupling we got oscillations that exhibited beating. What's doing the beating?

It's the two normal coordinates:

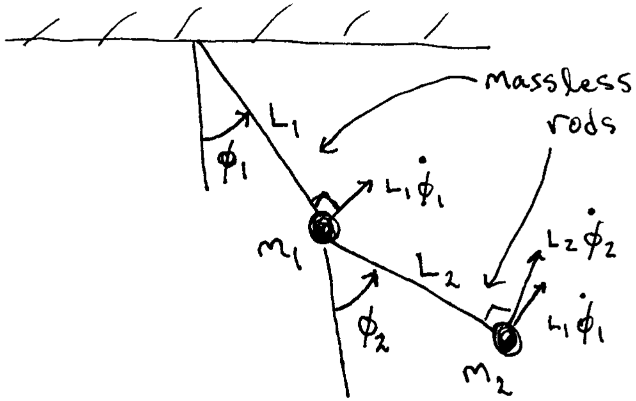
$$\xi_1(t) = \frac{1}{2} A \cos[(\omega_0 - \epsilon)t] = \frac{1}{2} A \cos(\omega_1 t)$$
$$\xi_2(t) = \frac{1}{2} A \cos[(\omega_0 + \epsilon)t] = \frac{1}{2} A \cos(\omega_2 t)$$

ω_1 and ω_2 are the two nearby frequencies beating against each other.

II Lagrangian approach: an example

At this point we've seen that coupled oscillations are physically interesting. In this section we see how to treat them in the Lagrangian formalism. Our example is the double pendulum (see next page for diagram). The potential energy of the first mass is

$$U_1 = m_1 g L_1 (1 - \cos\phi_1)$$



$\hat\phi_1$ = direction perp. to rod 1 towards positive ϕ_1.

$\hat r_1$ = direction along rod 1 towards mass 1

while its kinetic energy is

$$T_1 = \tfrac{1}{2} m_1 L_1^2 \dot\phi_1^2$$

The second mass is slightly more involved. Its potential is,

$$U_2 = m_2 g \left[L_1(1-\cos\phi_1) + L_2(1-\cos\phi_2) \right]$$

The velocity of the second mass is

$$\vec v_2 = L_1 \dot\phi_1 \hat\phi_1 + L_2 \dot\phi_2 \hat\phi_2$$

and so its kinetic energy is,

$$T_2 = \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_2 \vec v_2 \cdot \vec v_2$$

$$= \tfrac{1}{2} m_2 \left[L_1^2 \dot\phi_1^2 + 2 L_1 L_2 \dot\phi_1 \dot\phi_2 \hat\phi_1 \cdot \hat\phi_2 + L_2^2 \dot\phi_2^2 \right].$$

But $\hat\phi_1 \cdot \hat\phi_2 = \hat r_1 \cdot \hat r_2 = \cancel{(\phi_2 - \phi_1)} \cos(\phi_2 - \phi_1)$

Putting this all together we could find

$$\mathcal{L} = T - U$$
$$= T_1 + T_2 - U_1 - U_2$$

and write out the E-L equations for ϕ_1 and ϕ_2. However, just like a single pendulum, whose E.O.M. is $L\ddot\phi = -g\sin\phi$, these equations are ~~too~~ difficult to solve analytically. Instead we focus on small oscillations and make the small angle approximation: $\sin\phi \approx \phi$ and $\cos\phi \approx 1 - \frac{1}{2}\phi^2$. Thus we Taylor expand T and U and retain only up to second order:

$$T \approx \tfrac{1}{2} m_1 L_1^2 \dot\phi_1^2 + \tfrac{1}{2} m_2 L_1^2 \dot\phi_1^2 + m_2 L_1 L_2 \dot\phi_1 \dot\phi_2 (1) + \tfrac{1}{2} m_2 L_2^2 \dot\phi_2^2$$

and

$$U \approx \tfrac{1}{2} m_1 g L_1 \phi_1^2 + \tfrac{1}{2} m_2 g L_2 \phi_1^2 + \tfrac{1}{2} m_2 g L_2 \phi_2^2$$

What have we accomplished?

Before expansion:

$T = T(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2)$ and its dependence on ϕ_1 and ϕ_2 was transcendental = depend on every power of variables (not just a finite set of powers, like a polynomial would).
(roughly)

$U = U(\phi_1, \phi_2)$ and its depend. on ϕ_1 and ϕ_2 was transcendental.

After expansion:

$T = T(\dot{\phi}_1, \dot{\phi}_2)$ and it depends on $\dot{\phi}_1$ and $\dot{\phi}_2$ in a homogeneous quadratic manner. Homogeneous means that each term has the same power.

$U = U(\phi_1, \phi_2)$ and is also homogeneous quadratic.

This before/after situation is the same as what we'll find in the general case. In fact it is this nice situation that leads to nice, homogeneous linear E.O.M. that can always be solved.

Let's do our example:

$$\mathcal{L} = T - U = \tfrac{1}{2}(m_1+m_2)L_1^2\dot{\phi}_1^2 + m_2 L_1 L_2 \dot{\phi}_1\dot{\phi}_2 + \tfrac{1}{2} m_2 L_2^2 \dot{\phi}_2^2 - \tfrac{1}{2}(m_1+m_2) g L_1 \phi_1^2 - \tfrac{1}{2} m_2 g L_2 \phi_2^2$$

So, $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1}\right) = \frac{\partial \mathcal{L}}{\partial \phi_1}$ becomes

$$(m_1+m_2)L_1^2\ddot{\phi}_1 + m_2 L_1 L_2 \ddot{\phi}_2 = -(m_1+m_2) g L_1 \phi_1$$

and $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_2}\right) = \frac{\partial \mathcal{L}}{\partial \phi_2}$ is

$$m_2 L_1 L_2 \ddot{\phi}_1 + m_2 L_2^2 \ddot{\phi}_2 = -m_2 g L_2 \phi_2.$$

This is $\overleftrightarrow{M}\ddot{\vec{\phi}} = -\overleftrightarrow{K}\vec{\phi}$ with

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \overleftrightarrow{M} = \begin{pmatrix} (m_1+m_2)L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{pmatrix}$$