Lecture 29

November 7th, 2011

Today's Outline

I Last Lecture
& finish double pendulum

II Lagrangian Approach:
the general case

I Last Lecture

We studied the double pendulum and found the E.O.M

$$(m_1+m_2)L_1^2\ddot{\phi}_1 + m_2 L_1 L_2 \ddot{\phi}_2 = -(m_1+m_2)gL_1\phi_1$$

$$m_2 L_1 L_2 \ddot{\phi}_1 + m_2 L_2^2 \ddot{\phi}_2 = -m_2 g L_2 \phi_2$$

These E.O.M are aquivalent to

$$\overleftrightarrow{M}\ddot{\vec{\phi}} = -\overleftrightarrow{K}\vec{\phi}$$

with $\vec{\phi} = (\phi_1, \phi_2)$ and

$$\overleftrightarrow{M} = \begin{pmatrix} (m_1+m_2)L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{pmatrix}, \quad \overleftrightarrow{K} = \begin{pmatrix} (m_1+m_2)gL_1 & 0 \\ 0 & m_2 g L_2 \end{pmatrix}$$

In this ~~form~~ ~~the~~ problem is exactly like our toy model:

Guess: $\vec{\phi}(t) = \mathrm{Re}\,\vec{z}(t)$, $\vec{z}(t) = \vec{a}e^{i\omega t} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}$

and

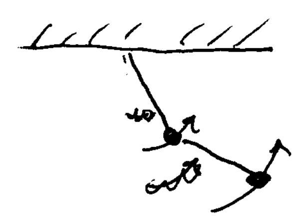
Solve: $\begin{cases} \det(\overleftrightarrow{K} - \omega^2 \overleftrightarrow{M}) = 0 & \text{for normal freq.s} \\ (\overleftrightarrow{K} - \omega^2 \overleftrightarrow{M})\vec{a} = 0 & \text{for normal modes} \end{cases}$

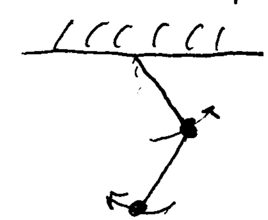
For example when $m_1 = m_2 = m$ and $L_1 = L_2 = L$ and we let $\omega_0 = \sqrt{g/L}$ then

$$\omega_1 = \sqrt{2-\sqrt{2}}\,\omega_0 \approx .77\,\omega_0$$

$$\omega_2 = \sqrt{2+\sqrt{2}}\,\omega_0 \approx 1.85\,\omega_0$$

$$\vec{a}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} A_1 e^{-i\delta_1} \qquad \vec{a}_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} A_2 e^{-i\delta_2}$$





II Lagrangian Approach: the general case

A system of n degrees of freedom oscillating about a point of stable equilibrium. We will describe the system (assumed holonomic) by n generalized coordinates: $q_1, \dots, q_n$ or $\vec{q} = (q_1, \dots, q_n)$ (e.g. $\vec{q} = (x_1, x_2)$ for toy model or $\vec{q} = (\phi_1, \phi_2)$ for double pendulum).

As always the K.E. is

$$T = \cancel{\tfrac{1}{2}} \sum_\alpha \tfrac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2} \qquad \alpha = (1, \dots, N)$$

N: # of particles

We write this in terms of the generalized coordinates by using the transformation

$$\vec{r}_\alpha = \vec{r}_\alpha(q_1, \dots, q_n)$$

By the chain rule

$$\dot{\vec{r}}_\alpha = \sum_{i=1}^{n} \frac{\partial \vec{r}_\alpha}{\partial q_i} \dot{q}_i$$

Then

$$T = \frac{1}{2} \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha \cdot \dot{\vec{r}}_\alpha$$

$$= \frac{1}{2} \sum_\alpha m_\alpha \left(\sum_j \frac{\partial \vec{r}_\alpha}{\partial q_j} \dot{q}_j \right) \cdot \left(\sum_k \frac{\partial \vec{r}_\alpha}{\partial q_k} \dot{q}_k \right)$$

$$= \frac{1}{2} \sum_{j,k} A_{jk} \dot{q}_j \dot{q}_k$$

with $A_{jk} \equiv \sum_\alpha m_\alpha \left(\frac{\partial \vec{r}_\alpha}{\partial q_j} \right) \cdot \left(\frac{\partial \vec{r}_\alpha}{\partial q_k} \right)$ and

$$A_{jk} = A_{jk}(q_1, \dots q_n) = A_{jk}(\vec{q})$$

We assume the forces are conservative, so that

$$U = U(q_1, \dots, q_n) = U(\vec{q})$$

Next, we assumed $\vec{q}_0$ is a stable equilibrium and by translating ~~take~~ our origin take $\vec{q}_0 = \vec{0}$. For <u>small</u> oscillations we can Taylor expand

$$U(\vec{q}) = \underbrace{U(0)}_{\text{constant which can be dropped.}} + \sum_j \overbrace{\frac{\partial U}{\partial q_j}}^{=0 \;\leftarrow\; \text{(equilibrium)}} q_j + \frac{1}{2} \sum_{j,k} \frac{\partial^2 U}{\partial q_j \partial q_k} q_j q_k + \dots$$

If we let $K_{jk} \equiv \left.\frac{\partial^2 U}{\partial q_j \partial q_k}\right|_{\vec{q}=\vec{0}}$ (note that $K_{jk} = K_{kj}$) and neglect higher order terms, we have,

$$U = U(\vec{q}) = \frac{1}{2}\sum_{j,k} K_{jk} q_j q_k$$

Actually since we're dropping higher order terms

$$T = \frac{1}{2}\sum_{j,k} A_{jk}(\vec{q}) \overbrace{\dot{q}_j \dot{q}_k}^{\text{already of 2nd order}} \approx \frac{1}{2}\sum_{j,k} M_{jk}\dot{q}_j\dot{q}_k$$

where

$$M_{jk} = A_{jk}(\vec{0})$$

is a matrix of constants.

Note that this simplifies T so that $T = T(\vec{q}, \dot{\vec{q}})$ reduces to $T = T(\dot{\vec{q}})$.

What have we achieved? Once again T and U are homogeneous quadratic functions of the $\dot{q}$s and qs respectively! Again we'll get solvable linear E.O.M out of 'em.

Lets do it:

$$\mathcal{L} = T - U$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}\sum_{j,k} M_{jk}\dot{q}_j\dot{q}_k\right) - \cancelto{0}{\frac{\partial U}{\partial \dot{q}_i}}$$

$$= \frac{1}{2}\left(\sum_{j,k} M_{jk}\delta_{ij}\dot{q}_k + \sum_{j,k} M_{jk}\dot{q}_j \delta_{ik}\right)$$

$$= \frac{1}{2}\left(\sum_k M_{ik}\dot{q}_k + \sum_j \underset{\uparrow \text{symmetric}}{M_{ji}}\dot{q}_j\right) = \sum_j M_{ij}\dot{q}_j$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \sum_j M_{ij}\ddot{q}_j$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = \cancelto{0}{\frac{\partial T}{\partial q_i}} - \frac{\partial U}{\partial q_i}$$

$$= -\frac{1}{2}\frac{\partial}{\partial q_i}\left(\sum_{j,k} K_{jk} q_j q_k\right)$$

$$= -\sum_j K_{ij} q_j$$

$$\Rightarrow \overleftrightarrow{M}\ddot{\vec{q}} = -\overleftrightarrow{K}\vec{q}$$

Again we guess,

$\vec{q}(t) = \mathrm{Re}\,\vec{Z}(t)$ with $\vec{Z}(t) = \vec{a}\,e^{i\omega t}$

and

solve: $\begin{cases} \det(\overleftrightarrow{K} - \omega^2 \overleftrightarrow{M}) = 0 & \text{normal freq.s} \\ (\overleftrightarrow{K} - \omega^2 \overleftrightarrow{M})\,\vec{a} = 0 & \text{normal modes} \end{cases}$

The det equation is an nth degree polynomial in ω^2 which we solve for the n normal frequencies. Then we determine the n $\vec{a}$ vectors. And finally the general solution is a linear combination of the n normal modes.

Notice that in fact you needn't write down the Lagrangian at all for these problems anymore; you can just find $\overleftrightarrow{M}$ and $\overleftrightarrow{K}$ from the general formulae we wrote down.

Your text goes through another example in detail, take a look at it.

We've completed the core of the course:

Oscillations
Calculus of variations
Lagrangian Mechanics
Central Forces
Noninertial Frames
Rigid Bodies
Normal Modes

Special Topics:
(to come)
Nonlinear mechanics & chaos
Hamiltonian Mechanics
Continuum Mechanics