Today's Outline: Lecture 3
Aug. 31st, 2011

I ~~Last Lecture~~
Homogeneous linear Diff. Eq.s

II Last Lecture &
damped oscillator wrap up

III Driven damped oscillator

$$a_n \frac{d^n x}{dt^n} + \dots + a_1 \frac{dx}{dt} + a_0 x = 0.$$

Any equation ~~of~~ like this ~~form~~ can be solved by a solution of the form:

$$x(t) = e^{rt}$$

with r constant (our standard guess).

Why it works: $x = e^{rt} \Rightarrow \frac{dx}{dt} = r e^{rt}$

$\frac{d^2 x}{dt^2} = r^2 e^{rt}, \dots, d^n x/dt^n = r^n e^{rt}$

Homogeneous linear eqtns w/ Constant Coeff.:

Linear: $a_n(t) \frac{d^n x}{dt^n} + a_{n-1}(t) \frac{d^{n-1} x}{dt^{n-1}} + \dots$

$$+ a_1(t) \frac{dx}{dt} + a_0(t) x = f(t)$$

Homogeneous: $f(t) = 0$; constant coeff.:

~~$a_0(t), a_1(t)$~~ $a_0, a_1, \dots, a_{n-1}, a_n$ do not depend on t.

$$a_n r^n \cancel{e^{rt}} + a_{n-1} r^{n-1} \cancel{e^{rt}} + \dots + a_0 \cancel{e^{rt}} = 0$$

$$\Rightarrow a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0 \quad (*)$$

Fundamental Theorem of Algebra gaurentees that there are n solutions to this equation.

Clean up: Collect the a_i and the derivatives into a differential operator:

$$D \equiv a_n \frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1 \frac{d}{dt} + a_0$$

Example: damped oscillator

$$D \equiv \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$$

$$Dx = \ddot{x} + 2\beta\dot{x} + \omega_0^2 x$$

This operator is linear

$$D(ax) = aDx \qquad D(x_1+x_2) = Dx_1 + Dx_2$$

For n^{th} order equations the linear combination with n constants is the general solution

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}$$

with r_i solutions of $(*)$.

Except: if the algebraic equation gives multiple roots, then I'm missing solutions.

~~Last time we recognized that the damped oscillator EOM could be solved by the standard guess, and found~~

For linear/homogeneous diff. eq.s any linear combination of solutions is a solution because

$$D(ax_1 + bx_2) = aDx_1 + bDx_2 = 0 + 0 = 0. \checkmark$$

In this case the "extra" solution is te^{rt}. In general, for multiple roots (repeated m times)

$$e^{rt},\ te^{rt},\ t^2e^{rt},\ \ldots,\ t^{m-1}e^{rt}$$

Last time we recognized that the damped oscillator EOM could be solved by the standard guess, and found

$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

with $r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}$ $\quad r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$

$\beta = \dfrac{b}{2m}$ ← damping coeff. $\qquad \omega_0 = \sqrt{\dfrac{k}{m}}$. $\quad B \equiv \sqrt{\beta^2 - \omega_0^2}$

$\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$

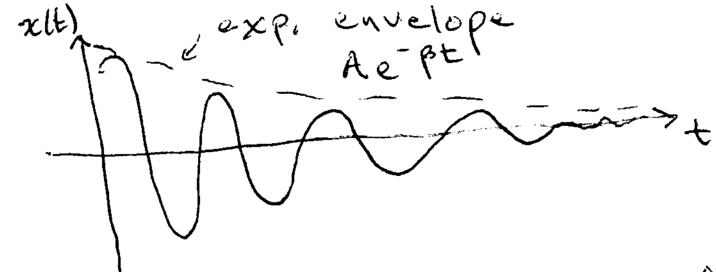
Started to explore three regimes

(1) Overdamped: $\beta > \omega_0$. Exponentially damped with two characteristic times $\tau_1 = \dfrac{1}{-\beta + B}$, $\tau_2 = \dfrac{1}{\beta + B}$.

(2) Underdamped: $\omega_0 > \beta$

Showed that

$x(t) = A e^{-\beta t} \cos(\omega_1 t - \delta)$



Characteristic time for damping is $\tau_1 = 1/\beta$.

Also two times at work here: damping $\tau_1 = 1/\beta$; and the period of the oscillations

$$\tau_2 = \frac{2\pi}{\omega_1}$$

Question: How many oscillations in one characteristic time?

$$\# = \frac{\tau_1}{\tau_2} = \frac{1}{\beta \cdot \frac{2\pi}{\omega_1}} = \frac{1}{\pi}\frac{\omega_1}{2\beta} \approx \frac{1}{\pi}\frac{\omega_0}{2\beta}$$

small damping or large mass

$$Q \text{ (quality factor)} \equiv \frac{\omega_0}{2\beta}$$

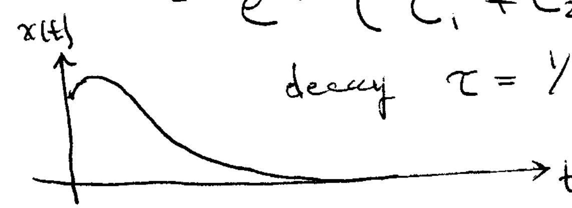
Tells you how many times it rings.

(3) Critical damping: $\beta = \omega_0$

$\Rightarrow r_1 = r_2$ multiple roots!

General solution is,

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t}$$
$$= e^{-\beta t}(C_1 + C_2 t)$$



decay $\tau = 1/\beta$.

That's it for damping now we add in driving.

III Driven damped oscillator:

Think of a child on a swing
oscillator (swing), friction (damping), driven (you pushing the child).

$$F_{net} = -kx - bv + F(t)$$

In general driving depends on time.

ω_0 the "natural frequency" and ω the "driving frequency".

This is an example of an inhomogeneous linear diff. eq. — stopped here

S'pose $x_h(t)$ is the general solution to the associated homogeneous equation then

diff operator

$$D x_h = 0$$

$$m\ddot{x} + kx + bv = F(t)$$

$$\Rightarrow \quad \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \equiv \frac{F(t)}{m}$$

f is the force per unit mass.
We'll be particularly interested in

$$f(t) = f_0 \cos(\omega t)$$

('cause it's physically reasonable and solvable; can build more complex driving out of cosines too!)

Note well the distinction btwn

$$D = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$$

S'pose further that $x_p(t)$ is any old ("particular") solution to the inhomog. eq.

$$D\, x_p(t) = f(t).$$

Then $x_h + x_p$ solves the inhomog. eq

$$D(x_h + x_p) = Dx_h + Dx_p = 0 + f(t)$$

and is, in fact, the general solution.

[Note: It doesn't matter what particular solution you use (!) — the difference can be sealed up in the constants in x_h. That is, $x_{p_1} - x_{p_2}$, is a solution of the homog. eq:

$$D(x_{p_1} - x_{p_2}) \overset{= D(x_{p_1}) - D(x_{p_2})}{=} f(t) - f(t) = 0$$

So, to solve driven case we need to focus on finding a particular solution to the inhomog. eq. Do this by hook or by crook!

Want to solve

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

The standard guess has shown how nice complex exponentials are. So we view this EOM as the real part of

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$$

with $z(t) \equiv x(t) + iy(t)$. The imaginary part is then

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f_0 \sin\omega t.$$

Try to guess a particular solution

$$z_p = C e^{i\omega t}$$

Physically this says that if we drive an oscillator at frequency ω we ~~will~~ might expect it to respond at the same frequency.

~~Putting~~ Then,

$$\ddot{z}_p = -C\omega^2 e^{i\omega t} \quad \text{and} \quad \dot{z}_p = i\omega C e^{i\omega t}$$

and

$$-C\omega^2 \cancel{e^{i\omega t}} + 2\beta i\omega C \cancel{e^{i\omega t}} + \omega_0^2 C \cancel{e^{i\omega t}} = f_0 \cancel{e^{i\omega t}}$$

$$\Rightarrow \quad C = \frac{f_0}{\omega_0^2 - \omega^2 + 2\beta i\omega}$$

That gives us a particular solution!
We will work on making z_p
look simpler next time.