Lecture 30

November 9th, 2011

Today's Outline

I Last Lecture:

Completed discussion of Normal modes

II Introduction to Chaos & Nonlinear Mechanics

I Last Lecture

Found normal mode E.O.M.

$$\overleftrightarrow{M}\,\ddot{\vec{q}} = -\overleftrightarrow{K}\,\vec{q}$$

with

$$M_{jk} = A_{jk}(\vec{0}) = \sum_\alpha m_\alpha \left(\frac{\partial \vec{r}_\alpha}{\partial q_j}\right)_{\vec{0}} \cdot \left(\frac{\partial \vec{r}_\alpha}{\partial q_k}\right)_{\vec{0}}$$

Evaluated at $\vec{0}$

and

$$K_{jk} = \left.\frac{\partial^2 U}{\partial q_j \partial q_k}\right|_{\vec{q}=\vec{0}}$$

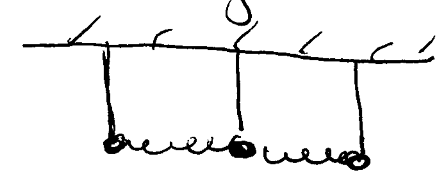
Notice that you, in fact, needn't write down the Lagrangian at all for these problems; you can just find $\overleftrightarrow{M}$ and $\overleftrightarrow{K}$ and proceed as before.

Guess: $\vec{q}(t) = \mathrm{Re}\,\vec{z}(t)$, $\vec{z}(t) = \vec{a}e^{i\omega t}$

and

Solve: $\begin{cases} \det(\overleftrightarrow{K} - \omega^2\overleftrightarrow{M}) = 0 & \text{Normal freq.s} \\ (\overleftrightarrow{K} - \omega^2\overleftrightarrow{M})\,\vec{a} = 0 & \text{Normal modes} \end{cases}$

Your text goes through another example:



Check it out!

II Chaos & Nonlinear Mechanics

A system is said to be chaotic if it obeys deterministic E.O.M. but still the motion cannot be predicted on long time scales.

Most realistic systems are nonlinear:

e.g. $m\ddot{\vec{r}} = -\dfrac{GmM\hat{r}}{r^2}$ or $\dfrac{d^2x}{dt^2} + \mu(x^2-1)\dfrac{dx}{dt} + x = 0$

Often, but not always, nonlinear systems exhibit chaos.

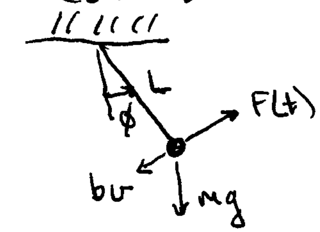
Nonlinearity changes many things but important amongst these is that nonlinear systems do not allow superposition

$$D(a_1 x_1(t) + a_2 x_2(t)) = a_1 D x_1 + a_2 D x_2 \quad [D \text{ linear}]$$

$$D(a_1 x_1 + a_2 x_2) \neq a_1 D x_1 + a_2 D x_2$$

$[D$ nonlinear$]$

We'll explore all of this with a particular nonlinear system: the damped driven pendulum (DDP)



The E.O.M is:

$$I\ddot{\phi} = \Gamma$$

$$\Rightarrow mL^2\ddot{\phi} = -mgL\sin\phi - \underbrace{bvL}_{\text{assumed damping}} + LF(t)$$

$$= -mgL\sin\phi - bL^2\dot{\phi} + LF_0\cos(\omega t)$$

Assume sinusoidal drive at drive frequency ω.

$$\Rightarrow \ddot{\phi} + \frac{b}{m}\dot{\phi} + \frac{g}{L}\sin\phi = \frac{F_0}{mL}\cos(\omega t)$$

Let $2\beta = \dfrac{b}{m}$, β the "damping constant"

$\omega_0 = \sqrt{g/L}$ the "natural frequency"

and $\gamma = \dfrac{F_0}{mL\omega_0^2}$ the "drive strength"

γ is dimensionless (by design) since F_0/mL has dimension $\dfrac{1}{(\text{time})^2}$

The name makes sense because

$$\gamma = \frac{F_0}{mL\omega_0^2} = \frac{F_0}{mg}$$

for $\gamma < 1$, $F_0 < mg$ and has a small effect, while for $\gamma \geq 1$, $F_0 > mg$ and has a large effect.

Finally then,

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2 \sin\phi = \gamma\omega_0^2 \cos\omega t$$

To appreciate how wild chaos is we first need to remind ourselves of what we would naively expect.

For small oscillations,

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2\phi = \gamma\omega_0^2 \cos\omega t$$

we get a damped driven oscillator!

If we now increase the drive strength the amplitude will increase and more terms will matter: $\sin\phi \approx \phi - \frac{1}{6}\phi^3 + \cdots$

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2\left(\phi - \tfrac{1}{6}\phi^3\right) = \gamma\omega_0^2\cos\omega t$$

This is a small perturbation, so guess

$$\phi(A) \approx A\cos(\omega t - \delta)$$

When we put this guess into the E.O.M. we get a term $\phi^3 = A^3\cos^3(\omega t - \delta)$. A useful trig identity is

We expect that for weak driving and small oscillations: The initial behavior of the pendulum depends on the initial conditions, but this transient motion dies out rapidly, and the motion approaches a unique attractor:

drive frequency

$$\phi(t) = A\cos(\omega t - \delta)$$

see plot (Fig 1).

$$\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x).$$

Then a simplification of the ϕ^3 term will include $\cos(3[\omega t - \delta])$ but the drive has a prescribed frequency ω and so this tripled frequency must be cancelled by one of $\phi, \dot{\phi}, \ddot{\phi}$ (in fact all three), and a better guess is

$$\phi(t) = A\cos(\omega t - \delta) + B\cos 3(\omega t - \delta)$$

As the drive increases this pattern continues and we get higher harmonics (multiples n of $(\omega t-\delta)$). This slowly modifies the attractor solution (see Fig. 2) as we approach $\gamma = 1$. Near $\gamma = 1$ we get dramatically different results!

III Midterm 2 Histogram & Results.

New plan for the final exam

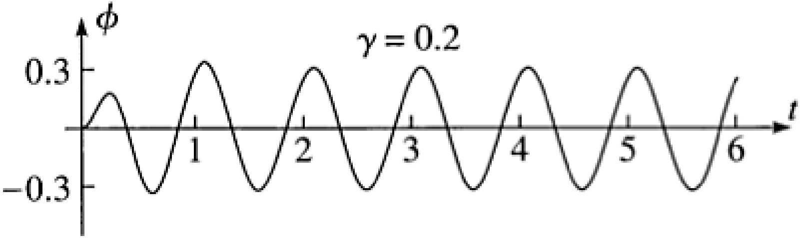


Figure: 1) Taken from Taylor p464

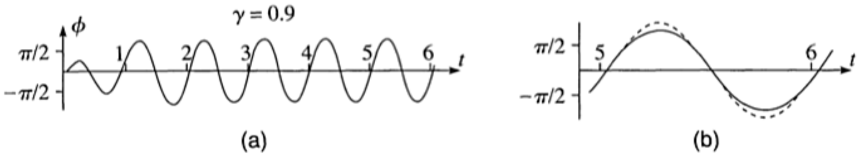


Figure: 2) Taken from Taylor p466