Lecture 31

November 14th, 2011

Today's Outline

I. Last Lecture

II. Approach to Chaos: near $\gamma = 1$

I Last Lecture

We found the E.O.M for a damped driven pendulum (DDP):

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2 \sin\phi = \gamma\omega_0^2 \cos\omega t$$

- damping constant $\beta = b/2m$
- natural freq. $\omega_0 = \sqrt{g/L}$
- drive strength $\gamma = F_0/mg$
- drive frequency ω

We expected and confirmed that for small γ ($\gamma < 1$) ~~the DDP~~ and small oscillations ($\phi, \dot{\phi}$ both small) the DDP exhibits motion similar to DD~~H~~O:

$$\phi(t) \approx A\cos(\omega t - \delta)$$

(ω: drive freq.)

This becomes modified by harmonic contributions $(n(\omega t - \delta))$ as the drive γ and amplitudes $\phi, \dot{\phi}$ increase.

II Approach to chaos

See attached figures from Taylor's chapter 12.

- Contribution of subharmonics, ω/n ~~period doubling~~ — Initial conditions $\phi(0) = 0$, $\dot{\phi}(0)$ unless otherwise noted.
- Dominant subharmonics
- Non-uniqueness of attractors: also $\phi(0) = -\frac{\pi}{2}$, $\dot{\phi}(0) = 0$
- Period doubling cascade
  - call γ a control parameter

Period doubling is ubiquitous: electrical circuits, chemical reactions, balls bouncing on oscillating surfaces, etc. Our 8th slide depicts experimental observations of period doubling in convection of Mercury.

How can we quantify the period doubling? The idea is to track exactly where, that is, at what value of the control parameter the period doubles, the "threshold values" or "bifurcation point".

For the DDP Taylor finds

n	period	γ_n	interval
1	$1 \to 2$	1.0663	
			0.0130
2	$2 \to 4$	1.0793	
			0.0028
3	$4 \to 8$	1.0821	
			0.0006
4	$8 \to 16$	1.0827	

Let γ_n be threshold for period transition $2^{n-1} \to 2^n$ and $\gamma_n - \gamma_{n-1}$ be the interval between successive thresholds. Remarkably it turns out that these intervals are patterned

$$(\gamma_{n+1} - \gamma_n) \approx \frac{1}{\delta}(\gamma_n - \gamma_{n-1})$$

with $\delta = 4.6692016$, the "Feigenbaum delta".

Even more incredibly this relationship (with the same δ!) is not unique to the DDP but common to <u>many</u> systems which exhibit period doubling. Physicists call this commonality across diverse systems "universality"! (= not truly universal but much more common than you would expect.) [gloss]

Because $\delta > 1$ this process rapidly approaches a limiting value, called γ_c,

$$\gamma_n \to \gamma_c \quad (\text{as } n \to \infty)$$

$$\gamma_1 < \gamma_2 < \cdots < \gamma_n < \cdots < \gamma_c$$

For DDP (with Taylor's parameters)

$$\gamma_c = 1.0829$$

Beyond this value chaos begins.

What we have been studying is called the period doubling route to chaos but this is not the only route (not all chaotic systems exhibit period doubling first).

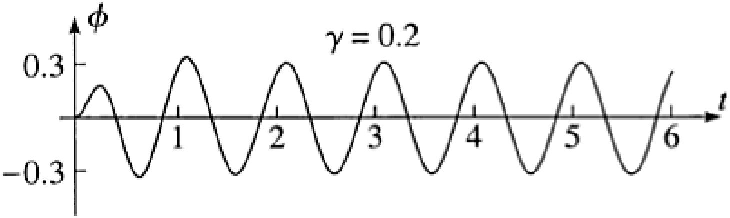
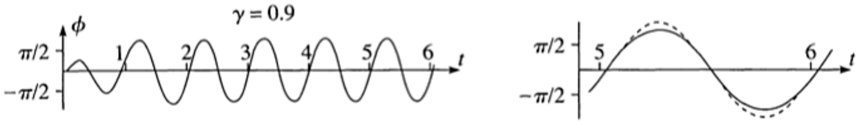
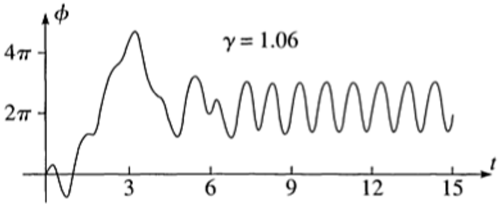


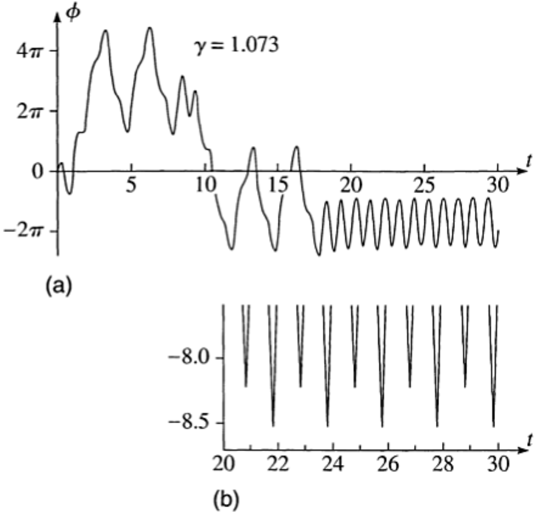
Figure: All Figures taken from Taylor's Chapter 12.



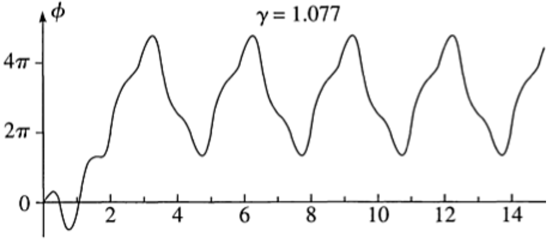
(a) (b)



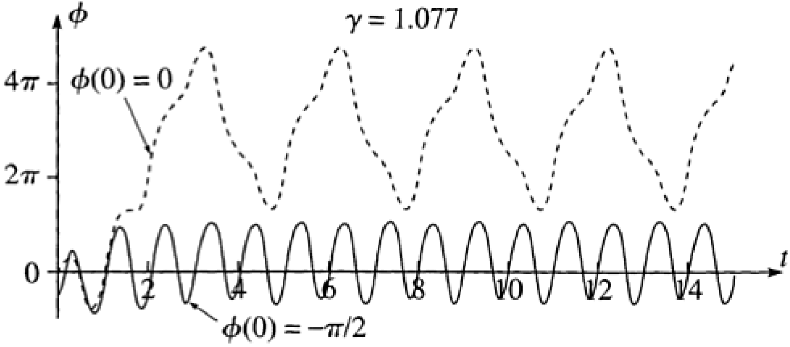
t	$\phi(t)$
34	6.0366
35	6.0367
36	6.0366
37	6.0366
38	6.0366
39	6.0366



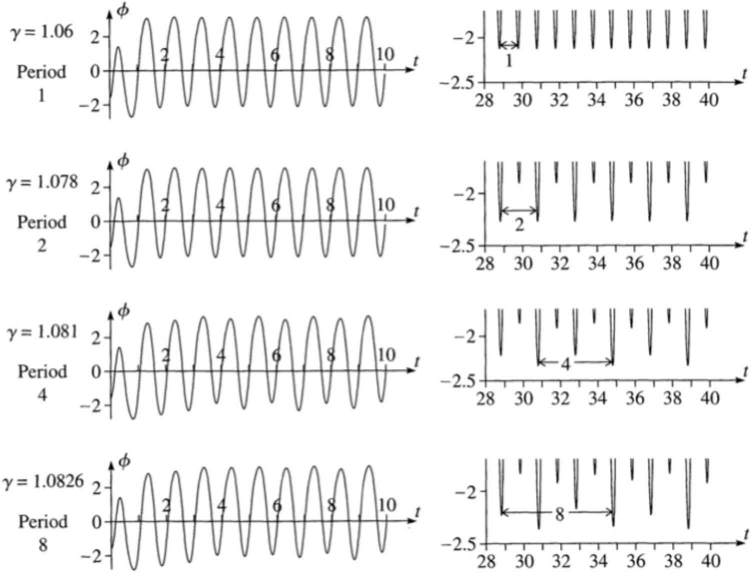
t	$\phi(t)$
30	−6.6438
31	−6.4090
32	−6.6438
33	−6.4090
34	−6.6438
35	−6.4090



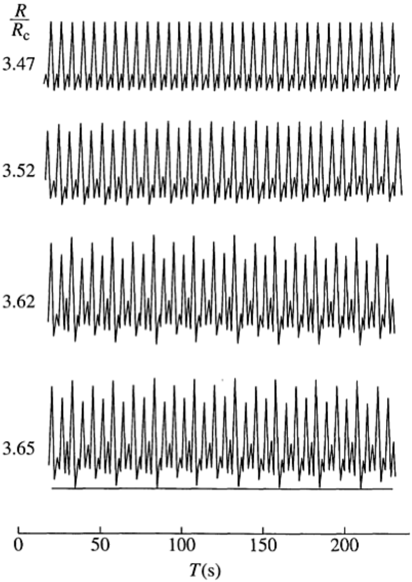
t	$\phi(t)$
30	13.81225
31	7.75854
32	6.87265
33	13.81225
34	7.75854
35	6.87265
36	13.81225
37	7.75854
38	6.87265



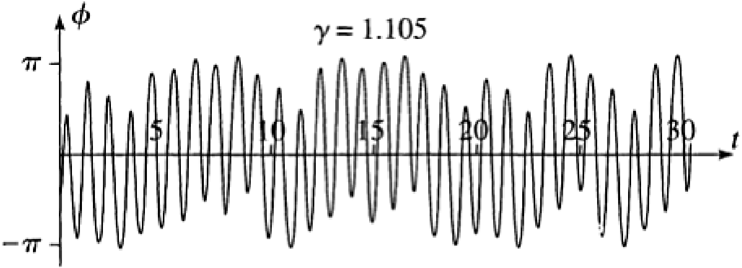
ϕ
$\gamma = 1.077$
4π
2π
0
$\phi(0) = 0$
t
2
4
6
8
10
12
14
$\phi(0) = -\pi/2$



γ = 1.06
Period 1
γ = 1.078
Period 2
γ = 1.081
Period 4
γ = 1.0826
Period 8
ϕ
t



$\frac{R}{R_c}$
3.47
3.52
3.62
3.65
0
50
100
150
200
T(s)



ϕ
$\gamma = 1.105$
π
$-\pi$
5
10
15
20
25
30
t