Lecture 32

November 16th, 2011

Today's outline:

I. Last Lecture

III. Chaos: Beyond $\gamma = 1$

II Sensitivity to initial conditions & Liapunov exponent

I. Last Lecture

We examined the period doubling cascade for γ in the range

$$\gamma_1 = 1.0663 < \gamma < 1.0829 = \gamma_c$$

Following Feigenbaum we noted that

$$(\gamma_{n+1} - \gamma_n) \approx \frac{1}{\delta}(\gamma_n - \gamma_{n-1})$$

with $\delta = 4.6692016$.

We noted the erratic, non-periodic motion of the DDP for a single $\gamma > \gamma_c$; $\gamma = 1.105$. This motion is characteristic of chaos. Before examing more ~~fa~~ of control parameter space we turn to picking up another quantitative aspect of chaos: sensitivity to initial conditions

II Sensitivity & Liapunov exponent

We touched on this last Wednesday.

We can only measure initial conditions with finite precision; What are the consequences of this limitation?

Let $\phi_1(t)$ and $\phi_2(t)$ describe the motion of two pendula with identical parameters and E.O.M. but slightly different initial conditions.

Further let

$$\Delta\phi(t) = \phi_2(t) - \phi_1(t)$$

What is the behavior of $\Delta\phi$?

From chapter 5 for DDHO,

$$\phi(t) = A\cos(\omega t - \delta) + C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

and so,

$$\Delta\phi = B_1 e^{r_1 t} + B_2 e^{r_2 t}$$
$$= D e^{-\beta t}\cos(\omega_1 t - \delta)$$

↑ true for underdamped

where r_1, r_2 have form $-\beta \pm i\omega_1$

The difference approaches zero exponentially!

$$\ln|\Delta\phi| = \underbrace{\ln D}_{\text{constant}} \underbrace{- \beta t}_{\text{linear, slope } -\beta} + \ln|\cos(\omega_1 t - \delta)|$$

↑ diverges $\to -\infty$ whenever $\cos(\omega_1 t - \delta) = 0$.

See plot (slide 3). Conclusion: DDHO is insensitive to initial conditions.

This remains true for $\gamma < \gamma_c$ (slide 4).

For $\gamma > \gamma_c$ no longer true; solutions with nearby initial conditions have exponentially diverging solutions for some time: "extreme sensitivity to initial conditions." In summary a commonly used signature of chaos is that

$$\Delta\phi \sim K e^{\lambda t}$$

(λ ← largest Liapunov exponent)

the largest Liapunov exponent is positive. (The "goes like" symbol $\sim$ is used because the exponential may be the envelope of an oscillatory motion).

III Chaos: Beyond $\gamma = 1$

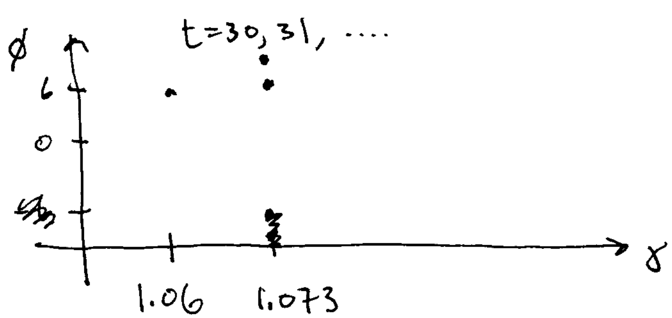
The natural guess that chaos continues for all $\gamma > \gamma_c$ turns out to be false!

At $\gamma = 1.13$ we get non-chaotic period 3 motion.

At $\gamma = 1.503$ chaos again.

We would like to get a more global picture of what's going on: this is the purpose of a

Bifurcation diagram:

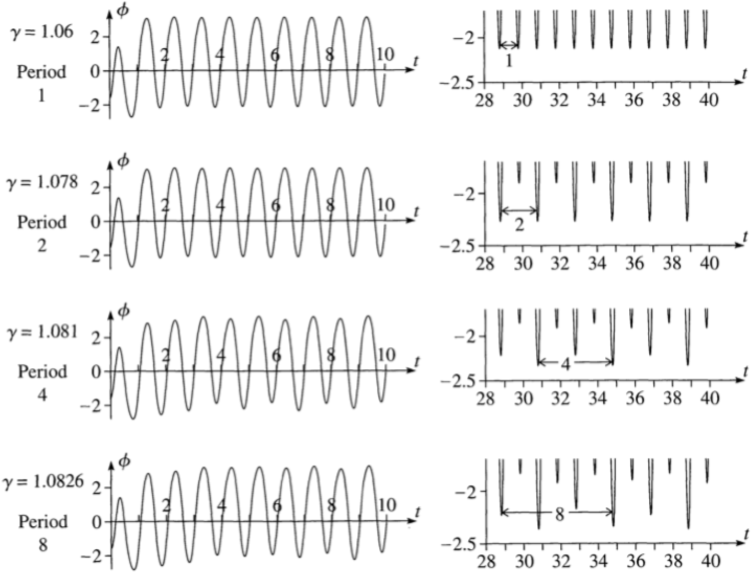


and so on. See slides 11, 12.

Note the period 3 "window" near $\gamma = 1.0845$. A zoom in of this window shows another period doubling cascade!

How common are stable periodic windows? It turns out that, for the logistic map, they are dense.

But then what about the chaotic regions? If you randomly choose a control parameter value the probability of getting chaos is nonzero. In fact, the chaotic set, that is, the set of control parameters that gives chaos is like a Cantor set!



γ = 1.06
Period 1
γ = 1.078
Period 2
γ = 1.081
Period 4
γ = 1.0826
Period 8

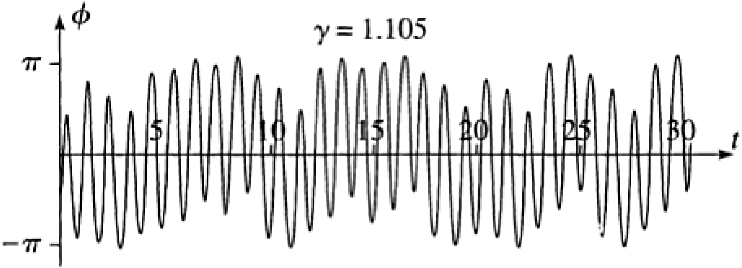


Figure: $\phi(0) = -\pi/2$ and $\dot{\phi}(0) = 0$

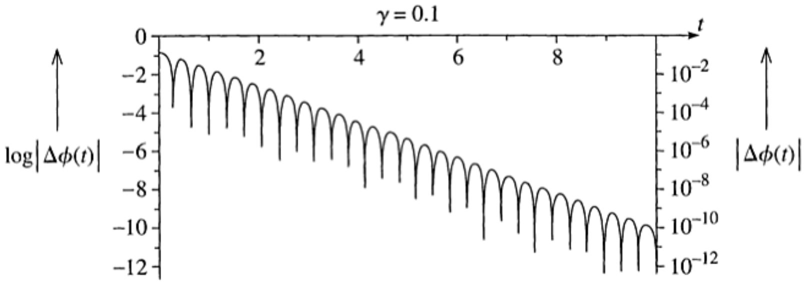
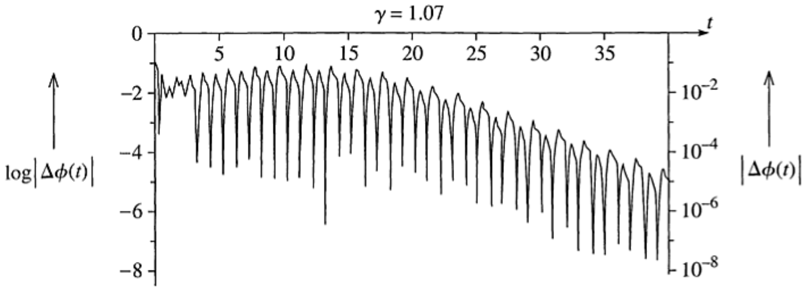


Figure: Logarithmic plot; log is $\log_{10}$. $\Delta\phi(0) = .1\text{rad} \approx 6^\circ, \Delta\dot{\phi}(0) = 0$.



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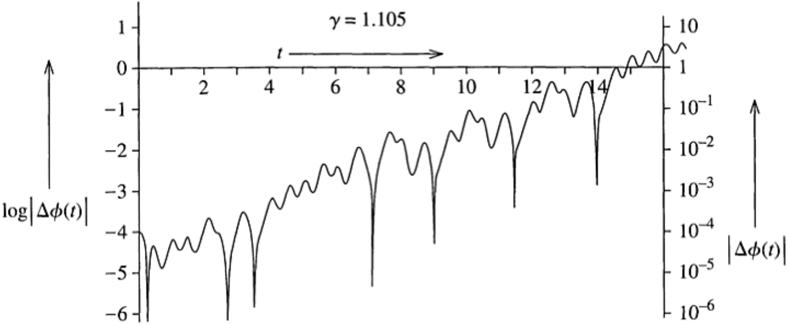


Figure: $\Delta\phi(0) = .00001\text{rad} \approx 0.0006^\circ, \Delta\dot{\phi}(0) = 0$. Exponential growth!

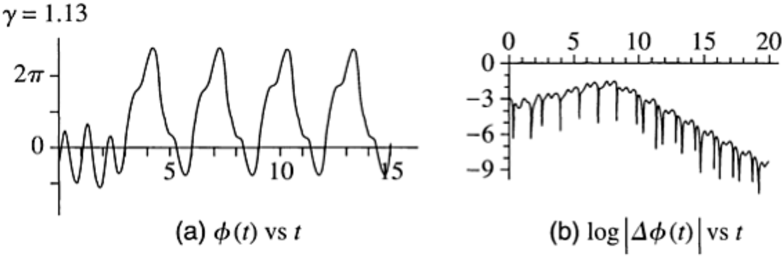
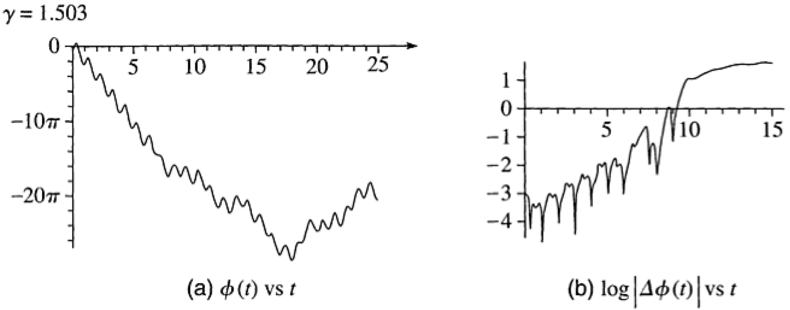


Figure: $\phi(0) = -\pi/2, \dot{\phi}(0) = 0; \Delta\phi(0) = 0.001\text{rad}, \Delta\dot{\phi}(0) = 0.$



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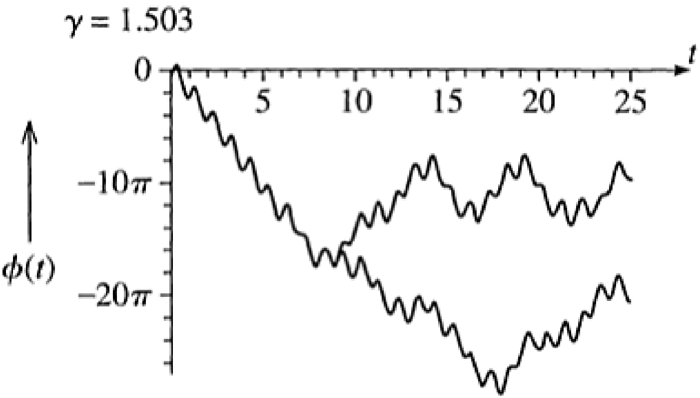

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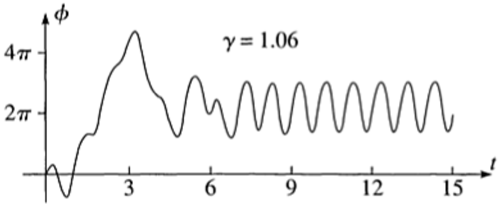
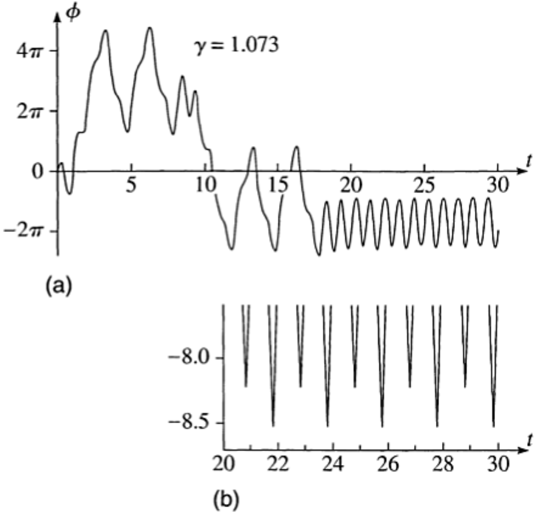
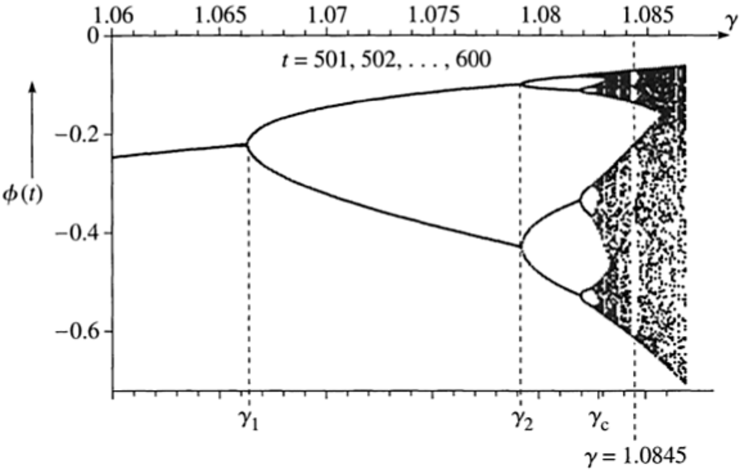
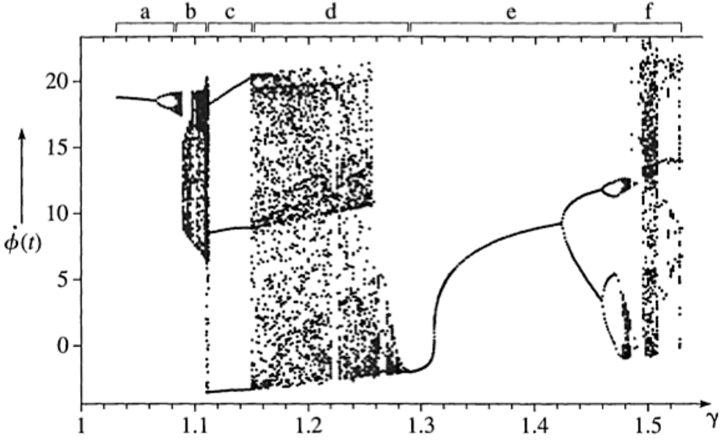


t	$\phi(t)$
34	6.0366
35	6.0367
36	6.0366
37	6.0366
38	6.0366
39	6.0366



t	$\phi(t)$
30	−6.6438
31	−6.4090
32	−6.6438
33	−6.4090
34	−6.6438
35	−6.4090


1.06
1.065
1.07
1.075
1.08
1.085
γ
0
t = 501, 502, . . . , 600
−0.2
−0.4
−0.6
φ (t)
γ1
γ2
γc
γ = 1.0845



a
b
c
d
e
f
20
15
10
5
0
$\dot{\phi}(t)$
1
1.1
1.2
1.3
1.4
1.5
γ