Lecture 33

November 18th, 2011

Today's Outline

I Prelude

II Interlude: Lagrangian & First order?

0. Announcements: (i) For this material I depart from Taylor. (ii) Last HW due last day of class (iii) Brunch at my house Dec 4th, 10:30 AM

I Prelude

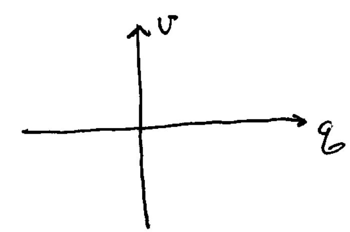
Hamiltonian Mechanics is a third approach, distinct from both the Lagrangian and Newtonian approaches we have been studying so hard. A question suggests itself: Why consider another approach?

This prelude begins to answer this question.

Consider a harmonic oscillator; we usually write the E.O.M. as

$$m\ddot{q} = -kq \quad \text{or} \quad \ddot{q} = -\omega_0^2 q,$$

a second order, ordinary differential equation. Equivalently we could write this as two first order ordinary differential equations: Define a new coordinate v, the velocity. With these two coordinates (q, v) the arena for our dynamics becomes



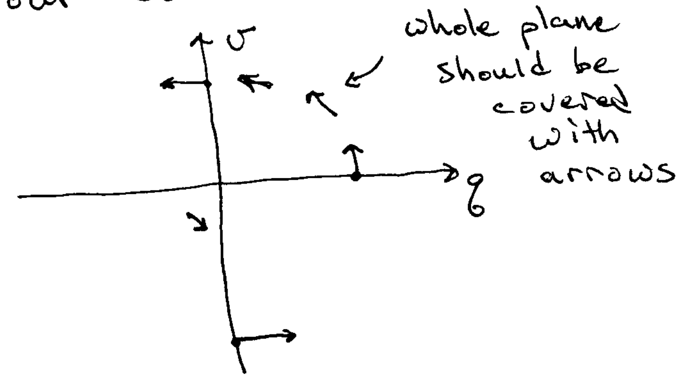
and we call it "velocity phase space" or "state space". The equations of motion are

$$\dot{q} \equiv \frac{dq}{dt} = v$$

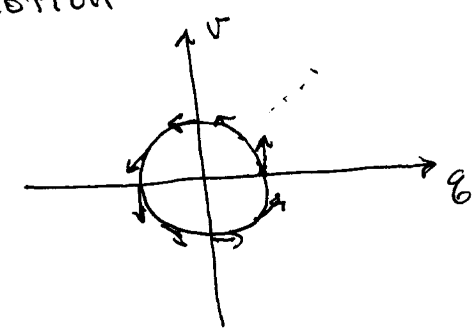
$$\dot{v} = -\omega_0^2 q$$

As far as solving the E.O.M goes this does not represent major progress. However, there are technical and geometrical reasons that it is advantageous. Let's begin with geometry: In this form the E.O.M. represent a vector field, that is, a set of equations that tell us how our coord.s change (in time) at every point of our velocity phase space.

For our oscillator,



An example of a vector field, familiar to many of you, is the electric field: $\vec{E}(x,y,z)$. Instead of drawing all of the little arrows we usually visualize the field lines. The vector field is tangent to the field lines at every point of space, and in this context we usually call the field lines "integral curves" of the vector field. Similarly our first order equations have integral curves and these are precisely the orbits of the motion



This begins to introduce the geometry.
Now for some technical resuts:

Under appropriate mathematical conditions,
(e.g. the R.H.S.s are continuous and differentiable)
Solutions

- Exist and
- Are unique for given initial conditions q_0, v_0.

These are things physicists often don't worry about, but it's nice to have a proof

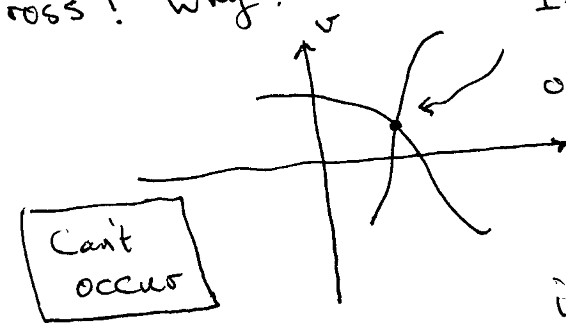
Hopefully you are (partially) convinced that there is much of interest when you write the E.O.M. in first order form.

II Interlude

Writing the E.O.M. in first order form turns out to be clumsy in the Lagrangian formulation. Let's show this. To get E.O.M. in first order form we need

$$\dot{\vec{q}} = \vec{v} \quad \text{and} \quad \dot{\vec{v}} = \ddot{\vec{q}}$$

here (see Arnold, Hirsch & Smale, or Hale). The second point has a nice geometrical interpretation: orbits in velocity phase space do not cross! Why?



If we took this as our initial condition which orbit would it follow?

the tricky part of this is the acceleration

$$\ddot{q} = \ddot{q}(q, v, t).$$

The acceleration is hiding in the Euler-Lagrange equation.
Let $L = L(\vec{q}, \vec{v} = \dot{\vec{q}}, t)$ then, we can evaluate the d/dt term by the chain rule

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \quad \Rightarrow$$

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \ddot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} \dot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \partial t} = \frac{\partial L}{\partial q_i}$$

To isolate $\ddot{q}_j$ we need to invert the matrix

$$M_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$$

and the resulting equation is complicated. In the Hamiltonian formulation we will find first order E.O.M. expressed in a very simple form.

Preview: Next week we will mathematically capture how Fig 1 encodes a curved line even though it was constructed solely by drawing straight lines:

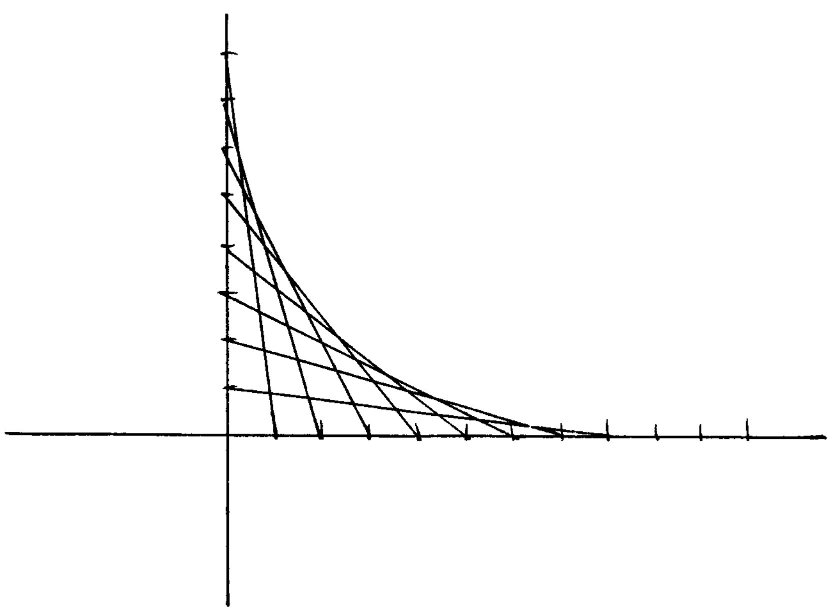


Fig.1: Asteroid exemplifying the Legendre Transform.