Today's Outline:

I Last Lecture

II Legendre Transform

III Hamiltonian Mechanics

Lecture 34

November 21st, 2011

I Last Lecture

We saw that 1st order E.O.M. have geometrical and technical advantages: Existance & Uniqueness of solutions, vector fields and non-crossing orbits.

We also saw that the Lagrangian formalism is clumsy when it comes to formulating 1st order E.O.M.

Today we will see that the E.O.M. in the Hamiltonian formulation are naturally 1st order; this is the major advantage of this formulation. We begin with the

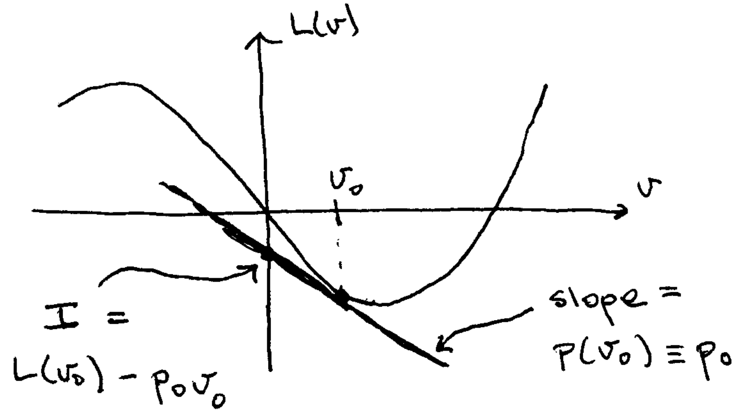
II Legendre Transform

The Legendre transform is like the Fourier transform in that it:

- It takes you from a given function to a totally <u>different function</u>. (e.g. $f \mapsto \tilde{f}$)
- The new function depends on different variables ($f(v) \mapsto \tilde{f}(p)$)
- It is its own inverse, that is, apply it twice and you get the function you started with ($f \mapsto \tilde{f} \mapsto f$)
- It is very symmetrical when properly formulated

Let's see how it works for 1 variable, more vars works similarly.

Let $L = L(v)$, the graph of L is a collection of pairs $(v, L(v))$



Key assumption: $p(v)$ is invertible, that is, we can solve $p = p(v)$ for $v = v(p)$. Geometrically, each slope occurs only once.

With this assumption we can define

$$H(p) \equiv -I(v(p), p) = p\,v(p) - L(v(p))$$

this is the Legendre transform.

As the drawing from last lecture demonstrates we can reconstruct

Define,

$$p(v) \equiv \frac{dL(v)}{dv}$$

the Legendre transform is a way of describing the curve using p instead of v; but to get the curve we need $(v, L(v))$, similarly we will need $(p, H(p))$, with H a new function.

Introduce, the intercept

$$I(v, p) = L(v) - pv$$

L from H also (see Fig 1 of Lect. 33, on p4); let's prove it:

$$\frac{dH}{dp} = v(p) + p\frac{dv}{dp} - \overbrace{\frac{dL}{dv}}^{=p}\frac{dv}{dp} = v(p)$$

If $v(p)$ is invertible we get $p = p(v)$ and

$$L(v) = p(v)\,v - H(p(v))$$

(Aside: the identical form of the equations for L and H is why we defined $H = -I$ above).

III Hamiltonian Mechanics

To get from Lagrangian to Hamiltonian dynamics we perform a Legendre transform on each of the velocity coordinates

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$$

and

$$\mathcal{H} \equiv \sum_i p_i \dot{q}^i - \mathcal{L}$$

but leave the positions q^i alone.

<u>Example</u>: pendulum, length L, mass M,

$$\mathcal{L} = \frac{1}{2} M L^2 \dot{\phi}^2 - (L(1-\cos\phi))$$

so,

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = M L^2 \dot{\phi}$$

and

$$\mathcal{H} = p_\phi \dot{\phi} - \mathcal{L}$$

$$= \frac{p_\phi^2}{ML^2} - \frac{1}{2}\frac{p_\phi^2}{ML^2} + L(1-\cos\phi)$$

$$= T(p_\phi) + U(\phi) = \text{total energy.}$$

This means that

$$\mathcal{H} = \mathcal{H}(q_1, \dots, q_n, p_1, \dots, p_n)$$

<u>Example</u>: Free particle, i.e. $U=0$,

$$\mathcal{L} = \frac{1}{2} m \dot{q}^2$$

So,

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q} \Rightarrow \dot{q} = \frac{p}{m}$$

and

$$\mathcal{H} = p\dot{q} - \mathcal{L} = p \cdot \frac{p}{m} - \frac{m}{2}\left(\frac{p}{m}\right)^2 = \frac{p^2}{2m}$$

What are the E.O.M. in the Hamiltonian formulation?

For 1 D.O.F. we have, by the chain rule,

$$\frac{\partial \mathcal{H}}{\partial q} = p\frac{\partial \dot{q}}{\partial q} - \left[\frac{\partial \mathcal{L}}{\partial q} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q}\right]$$

$$= p\frac{\partial \dot{q}}{\partial q} - \frac{\partial \mathcal{L}}{\partial q} - p\frac{\partial \dot{q}}{\partial q}$$

$$= -\frac{\partial \mathcal{L}}{\partial q} \underset{\text{by E-L eq.'s}}{=} -\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = -\dot{p}$$

Similarly,

$$\frac{\partial \mathcal{H}}{\partial p} = \dot q + p\frac{\partial \dot q}{\partial p} - \frac{\partial \mathcal{L}}{\partial \dot q}\frac{\partial \dot q}{\partial p}$$

$$= \dot q$$

So, Hamilton's equations are

$$\dot q = \frac{\partial \mathcal{H}}{\partial p} \quad \text{and} \quad \dot p = -\frac{\partial \mathcal{H}}{\partial q}$$

These are 1st order and very

$$\Rightarrow \mathcal{H} = \frac{p^2}{2m} + \tfrac{1}{2}kx^2$$

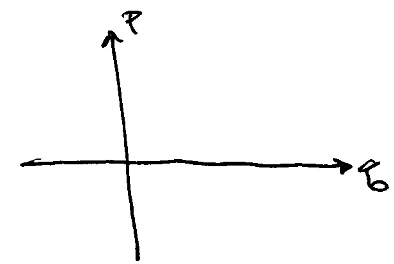
The E.O.M are

$$\dot x = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}$$

$$\dot p = -\frac{\partial \mathcal{H}}{\partial x} = -kx$$

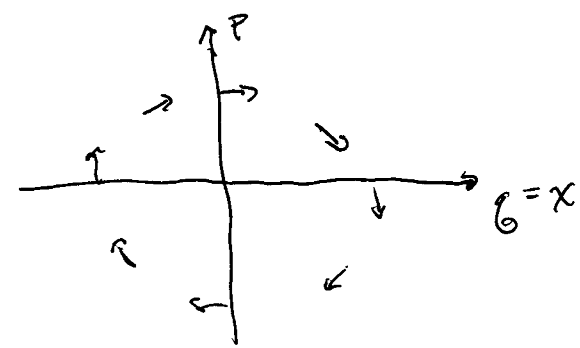
A very simple vector field with familiar orbits

simply expressed. They provide a vector field on "phase space":



<u>Example</u>: SHO again. $\mathcal{L} = \frac{1}{2}m\dot q^2 - \frac{1}{2}kx^2$

$$p = \frac{\partial \mathcal{L}}{\partial \dot q} = m\dot q \Rightarrow \quad \mathcal{H} = p\dot q - \mathcal{L}$$
$$= p^2/m - p^2/2m + \tfrac{1}{2}kx^2$$



Next time: When does $\mathcal{H} = T + U = E$?

Ignorable coordinates

Liouville's theorem

What's the connection with Quantum?