Today's Outline

Lecture 35

November 23rd, 2011

I Summary

II Symmetry & Reduction

III Liouville's Theorem

I Summary of the transition to Hamiltonian Mechanics

- Choose suitable generalized coord.s $q_1, \dots, q_n$
- Find T and U as functions of q's and $\dot{q}$'s.
- Find $p_1, \dots, p_n$ with $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$

(For conservative systems $U = U(q_i)$ and the p's simplify: $p_i = \partial T / \partial \dot{q}_i$.)

- Solve for the $\dot{q}$'s in terms of q's and p's.
- Find $\mathcal{H} = \mathcal{H}(q, p)$. In general

$$\mathcal{H} = \sum_i p_i \dot{q}^i - \mathcal{L}.$$

If our coord.s are "natural" (i.e. the relation between the generalized coord.s and the underlying Cartesians is independent of time) then $\mathcal{H} = T + U = E$.

- Finally write down Hamilton's equations:

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i}$$

II Symmetry & Reduction

A great advantage of the Hamiltonian formulation is that q and p are treated on an equal footing. This helps us to recognize more general symmetries; even symmetries that mix the q's and p's. We will only explore the tip of the symmetry iceberg. Assume that $\mathcal{H}$ is independent of q_i, we say q_i is "ignorable", then

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} = 0 \Rightarrow p_i = \text{const.}$$

and we've found a conserved quantity. Conversely, if $\mathcal{H}$ is independent of

$$\mathcal{H} = \mathcal{H}(q_1, \ldots, \overset{\text{no } q_i \text{ there}}{\hat{q}_i}, \ldots, q_n, p_1, \ldots, k, \ldots, p_n)$$

p_i replaced by its constant value k.

The Hamiltonian has effectively been reduced by one of its degrees of freedom. We couldn't accomplish this in the Lagrangian setup because even when p_i was constant $\dot{q}_i$ wasn't necessarily (e.g. $p_\phi = l = \text{const.} = \mu r^2 \dot{\phi}$ but $\dot{\phi} \neq \text{const}$

p_i then,

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} = 0 \Rightarrow q_i = \text{const}$$

(This example is contrived if we arrived at $\mathcal{H}$ by Legendre transform but otherwise realistic.)

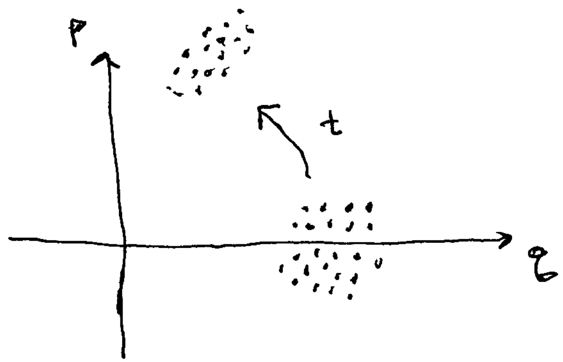
In the 1st order Hamiltonian setup this has a dramatic consequence: if q_i is ignorable then $p_i = \text{const} = k$ and q_{cauit}

from the Kepler problem.)

Reduction can be immensely useful for solving problems — again the Kepler problem is an excellent example.

III Liouville's Theorem

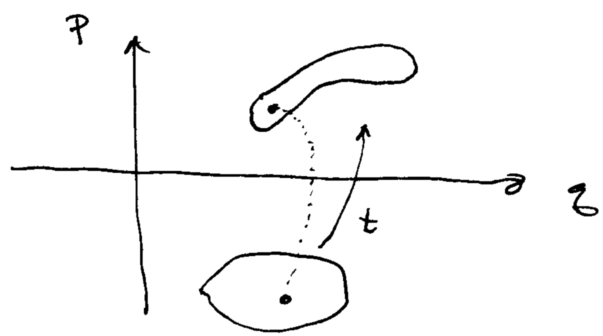
An example of the elegant geometrical results one can obtain in the Hamiltonian formalism is Liouville's theorem. For simplicity let's treat 1 D.O.F, the theorem works the same way for more D.O.F. Consider a cloud of initial conditions in phase space



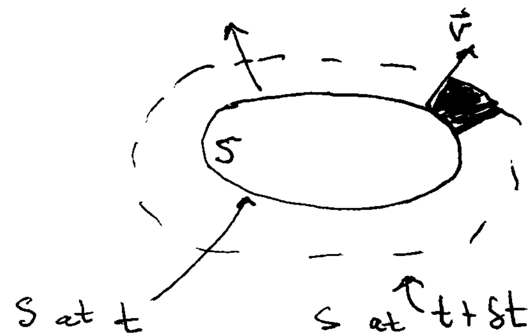
over time each initial condition evolves into another point of phase space via Hamilton's equations and the shape of the cloud changes. However Liouville's theorem gaurentees that the phase space volume (in our picture area) of the cloud is constant.

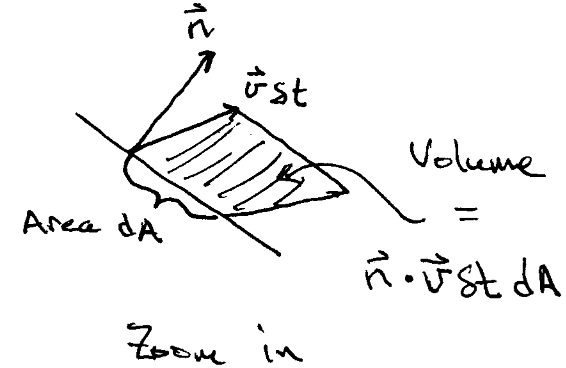
Let's prove it:

We know that phase space trajectories can't cross and this means that any initial condition originally inside a bubble stays inside that bubble; this means



that we can focus on the boundary. Suppose a bubble could expand (e.g. a fluid bubble instead of a bubble of initial conditions in phase space) then





So,

$$\delta V = \int_S \vec{n}\cdot\vec{v}\,\delta t\, dA$$

or

$$\frac{dV}{dt} = \int_S \vec{n}\cdot\vec{v}\, dA$$

This is true in any number of dimensions (with the proper interpretations of V, A, ~~S~~ S etc).

Recall a mathematical result from advanced calculus course:

Divergence theorem

$$\int_S \vec{n}\cdot\vec{v}\, dA = \int_V \vec{\nabla}\cdot\vec{v}\, dV$$

So,

$$\frac{dV}{dt} = \int_S \vec{n}\cdot\vec{v}\, dA = \int_V \vec{\nabla}\cdot\vec{v}\, dV$$

What is $\vec{v}$ in our case? The velocity of the surface is given by Hamilton's eqn.s:

Let $\vec{z} = (q, p)$ then $\dot{\vec{z}} = (\dot{q}, \dot{p}) \doteq \vec{v}$

and

$$\dot{\vec{z}} = (\dot{q}, \dot{p}) = \left(\frac{\partial \mathcal{H}}{\partial p}, -\frac{\partial \mathcal{H}}{\partial q}\right).$$

Then

$$\vec{\nabla}\cdot\vec{v} = \vec{\nabla}\cdot\dot{\vec{z}} = \frac{\partial}{\partial q}\left(\frac{\partial \mathcal{H}}{\partial p}\right) + \frac{\partial}{\partial p}\left(-\frac{\partial \mathcal{H}}{\partial q}\right)$$

$$= 0 \quad (!)$$

So,

$$\frac{dV}{dt} = \int_V \vec{\nabla}\cdot\vec{v}\, dV = 0 \quad \text{(Liouville's thm).}$$