Lecture 36

November 28^{th}, 2011

Today's Outline

I Brief comments on the relationship between Hamiltonian Mechanics and Quantum Mechanics

II Continuum Mechanics in 1D: strings

III Continuum Mechanics in 3D: Some generalities

I Hamiltonian Mech. & Q.M.

Remind me, what is the advantage of Hamiltonian Mechanics?

- Eqns of motion are 1^{st} order (in time).

This plays a key role in quantum theory. The Schrodinger eqn,

$$i\hbar \frac{\partial}{\partial t} \Psi = \hat{H} \Psi$$

is also 1^{st} order in time. This makes it possible for the probability continuity equation to be satisfied:

$$\rho = \Psi^* \Psi$$

$$\vec{J} = \frac{i\hbar}{2m}\left(\Psi \vec{\nabla} \Psi^* - \Psi^* \vec{\nabla} \Psi \right)$$

then

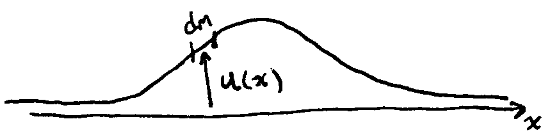
$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

This says that probability is neither created or destroyed, it simply moves around (as it must). If this reminds you of Liouville's theorem you're exactly right.

This is one of the aspects of quantum mechanics that is difficult to generalize to the ~~field~~ relativistic context — and why Dirac went beyond the Klein-Gordan equation.

II Continuum Mechanics in 1D: strings

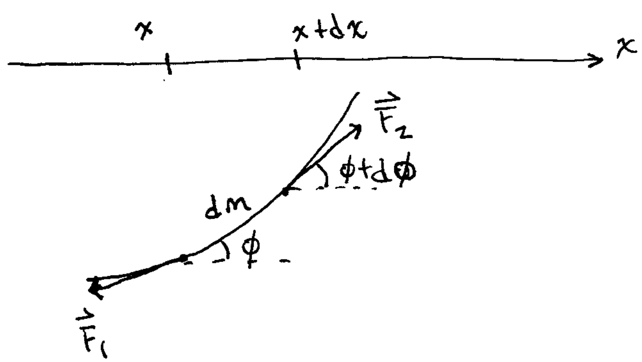
Suppose a transverse wave moves down a taut string:



Model string

Fundamental string

Comments: • In many applications the continuum hypothesis is a useful approximation but not fundamental. In others (e.g. QFT) it is fundamental.



Assume $u(x)$ small so that tension is roughly constant, $|\vec{F}_1| = |\vec{F}_2| = T$, and ϕ small. Then

$$F_x^{net} = T\cos(\phi + d\phi) - T\cos\phi \approx T - T = 0$$

and

$$F_y^{net} = T\sin(\phi + d\phi) - T\sin\phi \Rightarrow$$

• Fundamental string has a finite number of D.O.F. indexed by the label i. Model string has ∞ number of D.O.F. indexed by the continuous label x

• Fundamental string: $i = 1, \dots, n$ leads to n (coupled) ODEs for n masses

Model string: $u = u(x,t)$ leads to a single P.D.E. = partial diff. eq. for string.

Let's derive this E.O.M:

$$F_y^{net} = T\Big[\cos(\phi)\underbrace{\sin(d\phi)}_{\approx d\phi} + \sin\phi\underbrace{\cos d\phi}_{\approx 1}\Big] - T\sin\phi$$

$$\approx T\underbrace{\cos(\phi)}_{\approx 1} d\phi \approx T\,d\phi$$

But $\phi = \phi(x,t)$, so at fixed t,

$$d\phi = \frac{\partial\phi}{\partial x}dx$$

From $\tan\phi \approx \phi$ we have

$$\phi \approx \frac{\partial u}{\partial x}.$$

Put it all together

$$F_y^{net} = T\,d\phi = T\frac{\partial \phi}{\partial x}dx = T\frac{\partial^2 u}{\partial x^2}dx$$

and $F_y^{net} = dm\,\frac{\partial^2 u}{\partial t^2}$ ← acceleration at x

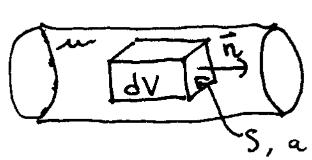
$$= \mu \frac{\partial^2 u}{\partial t^2} dx$$

(μ ← mass per unit length)

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 u}{\partial x^2}}$$ Wave Equation

The remaining lectures will setup how we derive E.O.M for 3D continuous media

III 3D Generalities



In general dV can have any shape we like; a rectangular brick will often be useful. The vector $\vec{n}$ is a unit, outward pointing normal, and we will often denote all or part of the bounding surface of dV by S.

The speed of the wave is

$$c = \sqrt{\frac{T}{\mu}}$$

as you can check by looking for solutions of the wave equation.

We'll skip exploring the solutions — but learn it. It's great stuff; you see how the PDE captures info about all those coupled ODES.

I wanted to present this derivation because it shows how PDEs come into the picture.

We distinguish two types of forces on dV, volume forces and surface forces.

As an example

$$\vec{F} = \rho \vec{g}\, dV$$

is a volume force, it is the gravitational force on dV. Generally

Volume Force = A force is proportional to dV

An example of a surface force is pressure, or rather force $= p\,dA$.

Generally,

Surface Force = A force proportional to dA

Next time: Stress & Strain

Material Moduli